
TOPOLOGY PROCEEDINGS



Volume 10, 1985

Pages 151–158

<http://topology.auburn.edu/tp/>

A CHARACTERIZATION OF T_3 SPACES OF COUNTABLE TYPE

by

A. GARCIA-MÁYNEZ

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

A CHARACTERIZATION OF T_3 SPACES OF COUNTABLE TYPE

A. García-Máynez

1. Introduction

The class of *p-spaces* was introduced by A. V. Arhangel'skii in 1963 [1]. Metrizable and locally compact Hausdorff spaces are examples of *p-spaces*. Although in the original definition every *p-space* was supposed to be completely regular and T_1 (in fact, that definition depends on the existence of the Stone-Cech compactification of the space), D. Burke [2] gave a characterization of *p-spaces* which allows to generalize the concept to T_3 -spaces. Under this new definition, Moore spaces are *p-spaces* (see [2; Thm. 2.1]). R. Hodel introduced the concept of *plumbing degree* plX of a T_3 -space X [5]. According to his definition, a T_3 -space is a *p-space* if and only if $plX \leq \aleph_0$. A remarkable property of *p-spaces* is *countable typeness* [1], i.e., every compact subset of a *p-space* is contained in a compact subset which has a countable local basis for its neighborhood system. Chaber, Coban and Nagami [3] introduced the class of *monotonic p-spaces* which lies between the class of *p-spaces* and the class of T_3 -spaces of countable type. A common feature of these generalizations is their intrinsic character, i.e., the definitions concrete to the space in question.

In this paper we give an extrinsic characterization of *p-spaces* using the Wallman instead of the Stone-Cech

compactification. In a natural way we give an equivalent formulation of the inequality $plX \leq \lambda$, where λ is any infinite cardinal number. If we replace *countable typeness* by $\leq \lambda$ -*typeness* (a space X has $\leq \lambda$ -*type* if every compact subset lies in a compact set which has a local basis for its neighborhood system consisting of at most λ elements), it is natural to ask if $plX \leq \lambda$ implies $\leq \lambda$ -*typeness*. This seems to be a difficult problem and a possible solution may depend on a characterization of T_3 -spaces of $\leq \lambda$ -type using their Wallman compactification. We content ourselves with a characterization of T_3 -spaces of countable type which makes trivial the statement that a p -space has countable type.

2. Definitions and Preliminary Results

If X is a T_1 -space, we denote by \mathcal{J}_X the family of all closed subsets of X . The collection wX of all ultrafilters in \mathcal{J}_X may be topologized as follows: for each $A \subset X$, A^* denotes $\{\xi \in wX \mid F \subset A \text{ for some } F \in \xi\}$. The collection $\beta^* = \{U^* \mid U \subset X \text{ open}\}$ is closed under finite unions and finite intersections and it is obvious that $\phi^* = \phi$ and $X^* = wX$. Hence there is a topology τ of wX having β^* as a basis and the space (wX, τ) turns out to be compact and T_1 . If we identify each $x \in X$ with its fixed ultrafilter $\xi_x = \{F \in \mathcal{J}_X \mid x \in F\}$, we get an embedding of X into wX and hence wX is a T_1 -compactification of X , called the *Wallman compactification of X associated to \mathcal{J}_X* . In case X is a T_3 -space, wX is a *nearly T_2 -extension* of X , that is, if a, b are different points of wX and at least one of them belongs

to X , then a and b have disjoint neighborhoods. We need the following properties of wX (the reader may easily provide the proofs):

2.1. Let $F_1, \dots, F_n \in \mathcal{F}_X$, where X is a T_1 -space. Then $Cl_{wX}(F_1 \cap \dots \cap F_n) = \bigcap_{i=1}^n Cl_{wX}F_i$.

2.2. Let X be a T_3 -space. If A and B are disjoint closed subsets of wX and $A \subset X$, then A and B have disjoint neighborhoods.

Let A be a subset of a space X . A family \mathcal{G} of subsets of X containing A is a *local net* of A in X if for every open $U \supset A$, there exists an element $G \in \mathcal{G}$ contained in U . A local net of A in X consisting of open sets is called a *local basis* of A in X .

A space X is of *point countable type* if each point of X lies in a compact set having a countable local basis in X .

X is of *countable type* if each compact set in X lies in a compact set having a countable local basis in X .

A family $\{\mathcal{G}_\alpha \mid \alpha \in J\}$ of open covers of a space X is a *plumbing* of X if the following conditions are fulfilled:

1) If $G_\alpha \in \mathcal{G}_\alpha$, $\alpha \in J$ and $\bigcap_{\alpha \in J} G_\alpha \neq \emptyset$, then $\bigcap_{\alpha \in J} G_\alpha^-$ is compact;

2) If $G_\alpha \in \mathcal{G}_\alpha$, $\alpha \in J$ and $\bigcap_{\alpha \in J} G_\alpha \neq \emptyset$, then the family of finite intersections $\bigcap_{i=1}^n G_{\alpha_i}^-$, $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in J$ is a local net of $\bigcap_{\alpha \in J} G_\alpha^-$ in X .

Hodel proves in [5] that every T_3 -space has a plumbing. The least infinite cardinal number λ such that X has a

pluming $\{\mathcal{G}_\alpha \mid \alpha \in J\}$ with $|J| \leq \lambda$ is called the *pluming degree* of X . A T_3 -space X is a *p-space* if $plX = \aleph_0$.

3. Main Results

For each space X , $\Delta(X)$ denotes its diagonal $\{(x,x) \mid x \in X\}$.

3.1. *Let X be a T_3 -space and let λ be an infinite cardinal number. Then $plX \leq \lambda$ iff there exists a set $A \subset X \times wX$ such that $\Delta(X) \subset A \subset X \times X$ and such that A is the intersection of at most λ open subsets of $X \times wX$.*

Proof (Necessity). Let $\{\mathcal{G}_i \mid i < \lambda\}$ be a pluming of X . For each open G in X , let $G' = wX - Cl_{wX}(X - G)$. Define $U_i = \cup\{G \times G' \mid G \in \mathcal{G}_i\}$ and $A = \cap\{U_i \mid i < \lambda\}$. We have only to prove that $A \subset X \times X$. Assume, on the contrary, that there exists a point $(x,z) \in A \cap (X \times (wX - X))$. For each $i < \lambda$, obtain a set $G_i \in \mathcal{G}_i$ such that $(x,z) \in G_i \times G'_i$. By the definition of pluming, $L = \cap\{G_i^- \mid i < \lambda\}$ is compact and

$$\{G_{i_1}^- \cap \dots \cap G_{i_k}^- \mid k \in \mathbb{N}, i_1, \dots, i_k < \lambda\}$$

is a local net of L in X . By 2.2, there exists an open set T in wX such that

$$L \subset T \subset Cl_{wX}T \subset wX - \{z\}.$$

Hence there exist $i_1, \dots, i_k < \lambda$ such that

$$G_{i_1}^- \cap \dots \cap G_{i_k}^- \subset T.$$

But by 2.1,

$$\cap_{j=1}^k Cl_{wX}G_{i_j} = \cap_{j=1}^k Cl_{wX}G_{i_j}^- = Cl_{wX} \cap_{j=1}^k G_{i_j}^- \subset Cl_{wX}T$$

Therefore, for some j , $z \notin Cl_{wX}G_{i_j}$. Since X is dense in wX ,

we have $G_{i_j}' \subset Cl_{wX}G_{i_j}$. Hence, $z \notin G_{i_j}'$, a contradiction.

(Sufficiency). By hypothesis, there exist open sets $\{U_i | i < \lambda\}$ in $X \times wX$ such that $A = \cap\{U_i | i < \lambda\}$ lies between $\Delta(X)$ and $X \times X$. For each $i < \lambda$, let

$$\mathcal{G}_i = \{G | G \text{ open in } X, G^- \times Cl_{wX} G \subset U_i\}$$

Clearly each \mathcal{G}_i is an open cover of X . To prove

$\{\mathcal{G}_i | i < \lambda\}$ is a plumbing of X , take $G_i \in \mathcal{G}_i (i < \lambda)$

and assume there is a point $x \in \cap\{G_i | i < \lambda\}$. We must

prove $L = \cap\{G_i^- | i < \lambda\}$ is compact and

$$\{G_{i_1}^- \cap \dots \cap G_{i_k}^- | k \in \mathbb{N}, i_1, \dots, i_k < \lambda\}$$

is a local net of L in X . Observe

$$\cap\{Cl_{wX} G_i | i < \lambda\} \subset X:$$

if $z \in \cap\{Cl_{wX} G_i | i < \lambda\}$ and $z \notin X$, then $(x, z) \notin A$ and hence

$(x, z) \notin U_i$ for some $i < \lambda$. But $(x, z) \in G_i^- \times Cl_{wX} G_i \subset U_i$,

a contradiction. Therefore, $L = \cap\{Cl_{wX} G_i | i < \lambda\}$ is a compact subset of X . Let $T \subset X$ be open and assume $L \subset T$.

Let T' be any open set in wX such that $T = X \cap T'$. Accord-

ing to [4; 2.27], there exist $i_1, \dots, i_k < \lambda$ such that

$\cap_{j=1}^k Cl_{wX} G_{i_j} \subset T'$. Therefore,

$$\cap_{j=1}^k Cl_{wX} G_{i_j} \subset T' \cap X = T$$

$$\cap_{j=1}^k G_{i_j}^- \subset T' \cap X = T$$

and the proof is complete.

3.2. Lemma. Let Q be a compact subset of a T_3 -space

X and let V_1, V_2, \dots be open sets in $X \times wX$ such that

$\Delta(Q) \subset A = \cap_{i=1}^\infty V_i \subset X \times X$. Then there exists a compact

set $K \subset X$ such that K has a countable local basis in X and

$Q \subset K \subset X$.

Proof. For each $i = 1, 2, \dots$, let \mathcal{G}_i be the family of

open non-empty subsets of $X \times wX$ which may be written in the

form $(V \cap X) \times V$, with V open in wX , and such that $Cl_{wX}[(V \cap X) \times V] \subset V_i$. 2.2 implies that each \mathcal{G}_i is a cover of $\Delta(X)$. Let $\mathcal{U}_1 \subset \mathcal{G}_1$ be a finite subfamily such that \mathcal{U}_1 covers $\Delta(Q)$ irreducibly. Proceeding by induction, assume we have finite families $\mathcal{U}_1, \dots, \mathcal{U}_n$ such that $\mathcal{U}_i \subset \mathcal{G}_i$, \mathcal{U}_i covers $\Delta(Q)$ irreducibly ($i = 1, \dots, n$) and such that $\{Cl_{X \times wX} L \mid L \in \mathcal{U}_{i+1}\}$ refines \mathcal{U}_i for $i < n$. We find then a finite subfamily \mathcal{U}_{n+1} of \mathcal{G}_{n+1} such that \mathcal{U}_{n+1} covers $\Delta(Q)$ irreducibly and $\{Cl_{X \times wX} L \mid L \in \mathcal{U}_{n+1}\}$ refines \mathcal{U}_n . Let $q \in Q$ be arbitrary. Let $(T \cap X) \times T$ be the intersection of members of \mathcal{U}_n containing (q, q) and let W_q be an open set in wX such that

$$(q, q) \in (W_q \cap X) \times W_q \subset Cl_{X \times wX} [(W_q \cap X) \times W_q] \subset V_{n+1} \cap [(T \cap X) \times T].$$

By definition, $(W_q \cap X) \times W_q \subset \mathcal{G}_{n+1}$. Since $\Delta(Q)$ is compact, a finite subcollection \mathcal{U}_{n+1} of the family $\{(W_q \cap X) \times W_q \mid q \in Q\}$ covers $\Delta(Q)$ irreducibly and \mathcal{U}_{n+1} fulfills the required properties.

Define now $K = \bigcap_{n=1}^{\infty} S_n$, where

$$S_n = \cup \{V \mid (V \cap X) \times V \in \mathcal{U}_n\}.$$

By construction, $Cl_{wX} S_{n+1} \subset S_n$ for each $n = 1, 2, \dots$. Hence, K is a compact G_δ in wX . To complete the proof, it will be enough to show that $K \subset X$. Assume, on the contrary, that there exists a point $z \in K - X$. For each $n = 1, 2, \dots$, pick a sequence $W_1^{(n)}, W_2^{(n)}, \dots, W_n^{(n)}$ of open sets in wX such that $(W_i^{(n)} \cap X) \times W_i^{(n)} \in \mathcal{U}_i$, $z \in W_i^{(n)}$ for each $i = 1, \dots, n$ and such that $Cl_{wX} W_{i+1}^{(n)} \subset W_i^{(n)}$ for $i = 1, \dots, n-1$. Since the families $\mathcal{U}_1, \mathcal{U}_2, \dots$ are finite, there exists indices

$\lambda_1, \lambda_2, \dots$ such that $1 \leq \lambda_1 < \lambda_2 < \dots$ and $Cl_{wX} W_{n+1}^{(\lambda_{n+1})} \subset W_n^{(\lambda_n)}$ for $n = 1, 2, \dots$. For brevity, put $W_n = W_n^{(\lambda_n)}$. Therefore $Cl_{wX} W_{n+1} \subset W_n$ for $n = 1, 2, \dots$. Select a point $q^* \in Q \cap \bigcap_{n=1}^{\infty} (W_n \cap X)^-$. Then $(q^*, z) \in \bigcap_{n=1}^{\infty} [(W_n \cap X) \times W_n] \subset A \subset X \times X$, a contradiction.

3.3. *Theorem.* Let X be a T_3 -space and let \mathcal{G} be the family of G_δ subsets of $X \times wX$ which are contained in $X \times X$. Then:

- a) X is of point countable type iff every point of $\Delta(X)$ lies in an element of \mathcal{G} .
- b) X is of countable type iff every compact subset of $\Delta(X)$ lies in an element of \mathcal{G} .
- c) X is a p -space iff $\Delta(X)$ lies in an element of \mathcal{G} .

Proof. Lemma 3.2 takes care of the sufficiency conditions. c) is a direct consequence of 3.1. The proof will be complete if we show that whenever K is a compact subset of X having a countable local basis in X , then $X \times K \in \mathcal{G}$. Let $V_1 \supset V_2 \supset \dots$ be open sets in wX such that $V_1 \cap X, V_2 \cap X, \dots$ is a local basis of K in X . Since X is dense in wX and wX is compact, $V_1 \supset V_2 \supset \dots$ is a local basis of K in wX . Hence, $K = \bigcap_{n=1}^{\infty} V_n$ and K is a G_δ in wX . Clearly $X \times K$ is a G_δ in $X \times wX$ and $X \times K \subset X \times X$, that is $X \times K \in \mathcal{G}$.

3.3.1. *Corollary.* Every p -space is of countable type.

Bibliography

[1] A. V. Arhangel'skii, *On a class of spaces containing all metric and all locally bicomact spaces*, Soviet Math. Dokl. 4 (1963), 1051-1055. Amer. Math. Soc. Transl. 2 (1970), 1-39.

- [2] D. K. Burke, *On p-spaces and $w\Delta$ -spaces*, Pacific J. Math. 35 (1970), 285-296.
- [3] J. Chaber, M. M. Coban and K. Nagami, *On monotonic generalizations of Moore spaces, Cech complete spaces and p-spaces*, Fund. Math. 84 (1974), 107-119.
- [4] A. García-Máñez, *Introducción a la topología de conjuntos*, México, Trillas, 1971.
- [5] R. E. Hodel, *On the weight of a topological space*, Proc. of the Amer. Math. Soc. 43 (1974), 470-474.

Instituto de Matemáticas de la UNAM
Area de la Investigación Científica
México 04510, D.F. Mexico