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## OBSTRUCTING SETS FOR HYPERSPACE CONTRACTION

by

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## OBSTRUCTING SETS FOR HYPERSPACE CONTRACTION

C. J. Rhee

### 1. Introduction

Let  $X$  be a metric continuum. Denoted by  $2^X$  and  $C(X)$  the hyperspaces of nonempty closed subsets and subcontinua of  $X$  respectively and endow with the Hausdorff metric  $H$ .

In 1938 Wojkyslawski proved that  $2^X$  is contractible if  $X$  is locally connected [11]. In 1942 Kelly [2] proved that the contractibility of  $2^X$  is equivalent to the contractibility of  $C(X)$ . Furthermore, he introduced a sufficient condition, namely property (3.2), for the contractibility of the hyperspace of metric continua. In 1978 Nadler [3] called the Kelley's condition property  $K$  and raised a question. Find a necessary and/or sufficient condition in terms of  $X$  in order that  $2^X$  is contractible. In [6] a necessary condition, call it admissible condition, was given and introduced a notion of property  $C$  and proved that a space  $X$  with property  $C$  has a contractible hyperspace  $C(X)$  if and only if there is a continuous fiber map  $\alpha$  such that  $\alpha(x) \subset o(x)$  for each  $x \in X$ , where  $o(x)$  is the admissible fiber at  $x$ . Subsequently Curtis [1] proved that  $C(X)$  is contractible if and only if there exists a lower semicontinuous set-valued map  $\Phi: X \rightarrow C^2(X)$  such that for each  $x \in X$ , each element of  $\Phi(x)$  is an ordered arc in  $C(X)$  between  $\{x\}$  and  $X$ . The last two results do not fully provide the topological characterization of the space  $X$

having contractible hyperspaces. The obstruction lies on certain subsets of  $X$ , call it the  $M$ -set of  $X$ , which is our object to investigate and to prove a theorem characterizing the contractibility of  $C(X)$  and a theorem on the hyperspace contraction of the image of confluent maps.

Let  $\mu: C(X) \rightarrow I = [0,1]$  be a Whitney map [10] such that  $\mu(x) = 0$  for each  $x \in X$ , and  $\mu(X) = 1$ . For each  $x \in X$ , we define a total fiber map  $F: X \rightarrow 2^{C(X)}$  (not necessarily continuous) by  $F(x) = \{A \in C(X) \mid x \in A\}$ . An element  $A \in F(x)$  is admissible at  $x$  if, for each  $\epsilon > 0$ , there is  $\delta > 0$  such that each  $y$  in the  $\delta$ -neighborhood of  $x$  has an element  $B \in F(y)$  such that  $H(A,B) < \epsilon$ . For each  $x \in X$ , the collection  $\sigma(x) = \{A \in F(x) \mid A \text{ is admissible at } x\}$  is called the admissible fiber at  $x$ . We say that the space  $X$  is admissible if  $\sigma_t(x) = \sigma(x) \cap \mu^{-1}(t)$  is nonempty for each  $(x,t) \in X \times I$ . We define the  $M$ -set of  $X$  to be the set  $M = \{x \in X \mid F(x) \neq \sigma(x)\}$  and the points of  $X \setminus M$  as  $K$ -points of  $X$ . We state here some known results in [7] and [9].

*Theorem 1.0. Let  $X$  be a metric continuum.*

1. *For each  $x \in X$ ,  $\sigma(x)$  is closed in  $C(X)$ ,  $\{x\} \in \sigma(x)$ , and  $x \in \sigma(x)$ .*
2. *If  $A \in \sigma(\xi)$  and  $B \in A(x)$  and  $\xi \in A \cap B$  then  $A \cup B \in \sigma(x)$ .*
3. *For each  $B \in F(x)$ ,  $C = \bigcup \{A \in \sigma(x) \mid A \subset B\} \in \sigma(x)$ .*

*Theorem 1.1. If  $h: X \times I \rightarrow C(X)$  is a continuous increasing map such that  $\{x\} \in h(x,0)$  then  $h(x,t) \in \sigma(x)$  for  $(x,t) \in X \times I$ . Thus, if  $C(X)$  is contractible then  $X$  is an admissible space.*

*Theorem 1.2. For any metric continuum  $X$ , the following statements are equivalent:*

1.  $F(x) = a(x)$ ,
2.  $X$  has property  $K$  at  $x$ ,
3.  $F$  is continuous at  $x$ .

*Theorem 1.3. Let  $X$  be any metric continuum. If  $X$  is locally connected then  $X$  has property  $K$  at  $x$ .*

## 2. $\mathcal{M}$ -set

In this section, we investigate  $\mathcal{M}$ -sets of admissible spaces.

*Proposition 2.1. Let  $X$  be an admissible space. Then the components of its  $\mathcal{M}$ -set are nondegenerate.*

*Proof.* This proposition follows easily from the next proposition since  $\mu(A) > 0$  if and only if  $A$  is nondegenerate.

*Proposition 2.2. Let  $X$  be an admissible space and  $M$  be its  $\mathcal{M}$ -set. For each  $x \in M$ , let  $\mathcal{M}_x = \{A \in a(x) \mid A \subset M\}$ . Then there is a positive number  $t(x) \in I$  such that  $a_s(x) = a(x) \cap \mu^{-1}(s) \subset \mathcal{M}_x$  for  $0 \leq s < t(x)$ .*

*Proof.* Let  $x \in M$ . Since  $F(x) \neq a(x)$  there is  $A_0 \in F(x) \setminus a(x)$ . We show the nonexistence of  $t(x)$  implies  $A_0 \in a(x)$ . Suppose no such  $t(x)$  exists. Let  $\epsilon > 0$ . There is  $t_0 > 0$  such that the diameter of  $A$  is less than  $\epsilon/2$  for all  $A \in F(x) \cap \mu^{-1}(t)$  with  $0 \leq t \leq t_0$ . There is  $t$  with  $0 < t < t_0$  and  $B \in a_t(x)$  such that  $B \setminus M \neq \emptyset$ . One easily shows  $H(A_0, A_0 \cup B) < \epsilon/2$ . Let  $x_1 \in B \setminus M$ . Then  $F(x_1) = a(x_1)$ . Since  $A_0 \cup B$  is a continuum,  $A_0 \cup B \in a(x_1)$ . There is  $\delta_1 > 0$  such that for each  $z$  with  $d(z, x_1) < \delta_1$  there is

$C \in F(z)$  such that  $H(A_0 \cup B, C) < \varepsilon/2$ . Also, since  $B \in a(x)$ , there is  $\delta > 0$  such that for each  $y$  with  $d(x, y) < \delta$  there is  $D \in F(y)$  such that  $H(B, D) < \min\{\delta_1, \varepsilon/2\}$ . Consequently, let  $y$  be such that  $d(x, y) < \delta$  and  $D \in F(y)$  such that  $H(B, D) < \min\{\delta_1, \varepsilon/2\}$ . Then there is  $z \in D$  such that  $d(z, x_1) < \delta_1$ . Let  $C \in F(z)$  be such that  $H(A_0 \cup B, C) < \varepsilon/2$ . Since  $z \in C \cap D$ , we have  $C \cup D \in F(y)$ . By Lemma 1.4 [7]  $H(A_0 \cup B, C \cup D) = H((A_0 \cup B) \cup B, C \cup D) \leq \max\{H(A_0 \cup B, C), H(B, D)\} < \varepsilon/2$ . Therefore  $H(A_0, C \cup D) \leq H(A_0, A_0 \cup B) + H(A_0 \cup B, C \cup D) < \varepsilon$ . We conclude that  $A_0 \in a(x)$ , a contradiction. Hence a positive number exists and the proposition is proved.

*Corollary 2.3.* For each  $x \in M$ , let  $\tilde{M}_x = \{A \in a(x) \mid A \subset \bar{M}\}$ . Then there is a positive number  $\bar{t}(x)$  such that  $a_s(x) \subset \tilde{M}_x$  for  $0 \leq s \leq \bar{t}(x)$ .

*Proof.* Choose  $\bar{t}(x)$  with  $0 < \bar{t}(x) < t(x)$ .

We remark that since  $F(x) \neq a(x)$  for  $x \in M$  any increasing contraction  $h$  of  $X$  in  $C(X)$ , if it exists, must take admissible elements as its values, the above propositions and corollary provide some insight into the behavior of such map  $h$ .

### 3. T-admissibility

We introduce another condition on admissible fiber of  $X$  to give a characterization of contractibility of  $C(X)$  and a theorem on the contractibility of  $C(\hat{X})$  when  $\hat{X}$  is a confluent image of a T-admissible space  $X$ .

*Definition 3.1.* A metric continuum  $X$  is said to be top-admissible (abbreviated T-admissible) if, for each  $(x, s) \in X \times I$  the following condition is true:

For each  $A \in \sigma_s(x)$  and  $t \in [s, 1]$ , there is an element  $B \in \sigma_t(x)$  such that  $A \subset B$ .

Since  $\sigma_0(x) = \{x\}$  for each  $x \in X$ , we have that T-admissibility implies admissibility of a space  $X$ . We make a further remark that the contractibility of  $C(X)$  implies T-admissibility of  $X$ .

*Proposition 3.2.* Let  $X$  be T-admissible. Suppose  $M_\alpha$  is a component of the  $M$ -set  $M$  of  $X$ . Then, for each  $x \in M_\alpha$  and each  $t \in [0, \mu(\overline{M}_\alpha)]$ , there is an element  $A \in \sigma_t(x)$  such that  $A \subset \overline{M}_\alpha$ .

*Proof.* Let  $S = \{t \in [0, \mu(\overline{M}_\alpha)] \mid \exists A \in \sigma_t(x) \exists A \subset \overline{M}_\alpha\}$ . Obviously  $0 \in S$ . Since  $\sigma(x)$  is a compact set in  $C(X)$  by Theorem 1.0 and  $\mu$  is continuous we have that  $S$  is closed. Suppose  $[0, \mu(\overline{M}_\alpha)] \setminus S \neq \emptyset$ . Then there are  $t_0, t_1$  such that  $t_0 \in S$ ,  $0 < t_0 < t_1 \leq \mu(\overline{M}_\alpha)$  and  $(t_0, t_1) \cap S = \emptyset$ . Let  $t \in (t_0, t_1)$  and  $A_0 \in \sigma_{t_0}(x)$  with  $A_0 \subset \overline{M}_\alpha$ . Then there is a  $B \in F(x)$  such that  $A_0 \subset B \subset \overline{M}_\alpha$  and  $\mu(B) = t$ . Since  $t \in [0, \mu(\overline{M}_\alpha)] \setminus S$ , we conclude that  $B \notin \sigma_t(x)$ . By T-admissibility, there is for each positive integer  $n$  an element  $A_n \in \sigma(x)$  such that  $A_0 \subset A_n$ ,  $t_0 < \mu(A_n) < t_1$  and  $\lim_{n \rightarrow \infty} \mu(A_n) = t_0$ . Then, the sequence  $A_n$  converges to  $A_0$  in  $C(X)$ . Since  $\mu(A_n) \in (t_0, t_1)$  we have  $A_n \setminus \overline{M}_\alpha \neq \emptyset$ . Consequently  $A_n \setminus M \neq \emptyset$  because  $M_\alpha$  is a component of  $M$ . Let  $x_n \in A_n \setminus M$ . Then  $F(x_n) = \sigma(x_n)$ ,  $A_n \cup B \in \sigma(x_n)$  and

$x_n \in A_n \in \sigma(x)$ . By Theorem 1.0  $A_n \cup B = (A_n \cup B) \cup A_n \in \sigma(x)$ . Since  $A_n \cup B$  converges to  $A_0 \cup B = B$  in  $C(X)$ , we have by the compactness of  $\sigma(x)$  that  $B \in \sigma(x)$ . Since  $\mu(B) = t$  and  $B \subset \bar{M}_\alpha$ , we have  $t \in S$ , a contradiction. We conclude that  $S = [0, \mu(\bar{M}_\alpha)]$  and the proposition is proved.

We remark that there is an example of a  $T$ -admissible space  $X$  having contractible hyperspace  $C(X)$  and connected  $\mathcal{M}$ -set  $M$  in which there is an element  $A \in \sigma(x)$  for some  $x \in M$  such that  $0 < \mu(A) < \mu(\bar{M})$  and  $A \setminus \bar{M} \neq \emptyset$ .

*Proposition 3.3.* Let  $X$  be  $T$ -admissible. Suppose  $M_\alpha$  is a component of the  $\mathcal{M}$ -set  $M$  of  $X$ . Then for each  $x \in M_\alpha$  and  $B \in F(x)$  such that  $M_\alpha \subset B$  we have  $B \in \sigma(x)$ .

*Proof.* The proof is similar to that of Proposition 3.2. Let  $S = \{t \in [\mu(\bar{M}_\alpha), 1] \mid B \in F_t(x) \text{ and } B \supset M \Rightarrow B \in \sigma(x)\}$ . Since  $B \in C(X)$ ,  $B \supset M_\alpha$ ,  $\mu(B) = \mu(\bar{M}_\alpha)$  imply  $B = \bar{M}_\alpha$ , Proposition 3.2 yields  $\mu(\bar{M}_\alpha) \in S$ . Moreover,  $1 = \mu(X)$  implies  $1 \in S$ . Once  $S$  is proved to be closed, the connectedness of  $S$  is proved with an argument similar to that found in Proposition 3.2. We prove the closedness of  $S$  and leave the connectedness of  $S$  to the reader.

Let  $t$  be a limit point of  $S$  and let  $t_n \in S$  such that  $t_n \rightarrow t$  as  $n \rightarrow \infty$ . We may suppose  $t > \mu(\bar{M}_\alpha)$ . Let  $B \in F(x)$ ,  $\mu(B) = t$  and  $B \supset M_\alpha$ . If  $\mu(\bar{M}_\alpha) \leq t_n < t$ , there is  $A_n \in C(X)$  such that  $\bar{M}_\alpha \subset A_n \subset B$  and  $\mu(A_n) = t_n$ . Since  $t_n \in S$  and  $M_\alpha \subset A_n$  we have  $A_n \in \sigma(x)$ . If  $t < t_n$ , there is  $A_n \in C(X)$  such that  $B \subset A_n$  and  $\mu(A_n) = t_n$ . Since  $t_n \in S$  and  $M_\alpha \subset A_n$  we have  $A_n \in \sigma(x)$ . Because  $\mu(A_n) \rightarrow \mu(B)$  as  $n \rightarrow \infty$  and either  $B \subset A_n$  or  $A_n \subset B$ , we have  $A_n$  converging to  $B$  in  $C(X)$ .

Since  $a(x)$  is compact in  $C(X)$  we have  $B \in a(x)$ .  $S$  is now proved to be closed.

*Definition 3.4.* Let  $N$  and  $Z$  be subcontinua of  $X$  such that  $N \subset Z$ .

A set-valued function  $\alpha: N \rightarrow C(Z)$  is a fiber function if, for each  $x \in N$ , (1)  $\alpha(x) \subset a(x)$ , (2)  $\{\{x\}, Z\} \subset \alpha(x)$ , and (3)  $\alpha(x)$  is path-connected.  $\alpha$  is monotone-connected (4) if there is a path in  $\alpha(x) \cap C(A)$  between  $\{x\}$  and  $A$  for each  $A \in \alpha(x)$ . A monotone-connected, lower semicontinuous fiber function  $\alpha: X \rightarrow C(X)$  is called a  $c$ -function for  $X$ .

We rephrase Curtis' result [1] here in terms of  $c$ -function to prove the next theorem.  $C(X)$  is contractible if and only if there is a  $c$ -function  $\alpha: X \rightarrow C(X)$ .

*Theorem 3.5.* Let  $X$  be a  $T$ -admissible space with its  $\mathcal{M}$ -set  $M$ . Then  $C(X)$  is contractible if and only if there exists a subcontinuum  $Z$  of  $X$  containing  $M$  and a monotone-connected lower semicontinuous fiber function  $\alpha': \bar{M} \rightarrow C(Z)$ .

*Proof.* Suppose  $C(X)$  is contractible. Let  $h: X \times I \rightarrow C(X)$  be an increasing contraction map [7]. Then  $h(x, t) \in a(x)$  for each  $x \in X$  and the set-valued function  $\alpha$  defined by  $\alpha(x) = \{h(x, t) \mid t \in I\}$  is a  $c$ -function for  $X$ . The restriction of  $\alpha$  on  $\bar{M}$  is a monotone-connected continuous fiber map on  $\bar{M}$  into  $C(X)$ . For the converse, we let  $J$  be a monotone segment from  $Z$  to  $X$  which is provided by [2]. Since  $X$  is  $T$ -admissible and  $M \subset Z$ , by Proposition 3.3, each element of  $J$  is admissible at each point of  $M$ . If  $x \in \bar{M} \setminus M$ , then such a point is a  $K$ -point, thus, element of

$J$  containing  $x$  is admissible at  $x$ . Define a set-valued function  $\alpha: X \rightarrow C(X)$  by

$$\alpha(x) = \begin{cases} \sigma(x), & x \in X \setminus \bar{M} \\ \alpha'(x) \cup J, & x \in \bar{M}. \end{cases}$$

Let  $x \in X \setminus \bar{M}$ . Then  $x$  is a  $K$ -point and hence  $\sigma(x) = F(x)$ . The total fiber  $F(x)$  is always path-connected and monotone-connected by [2]. If  $x \in \bar{M}$  then  $\alpha'(x)$  is monotone-connected and  $A \subset Z$  for all  $A \in \alpha'(x)$  and  $J$  is a monotone segment from  $Z$  to  $X$ . Thus  $\alpha'(x) \cup J$  is monotone-connected.

To prove the lower semicontinuity of  $\alpha$ , let  $x \in X \setminus \bar{M}$ . Then  $x$  is a  $K$ -point and  $\alpha(x) = \sigma(x) = F(x)$ . Therefore  $\alpha$  is continuous at  $x$  by [9].

Suppose  $x \in \bar{M}$ ,  $A_0 \in \alpha'(x) \cup J$ , and  $\varepsilon > 0$ . Suppose  $A_0 \in \alpha'(x)$ . Since  $\alpha'$  is lower semicontinuous at  $x$  in  $\bar{M}$ , there exists  $\delta_1 > 0$  such that each point  $y$  in the  $\delta_1$ -neighborhood of  $x$  in  $\bar{M}$  has an element  $B \in \alpha'(y)$  such that  $H(A_0, B) < \varepsilon$ . Also, since  $A_0 \in \sigma(x)$ , there is  $\delta_2 > 0$  such that each point  $y$  in the  $\delta_2$ -neighborhood of  $x$  in  $X$  has an element  $B \in F(y)$ ,  $F(y) = \sigma(x)$  if  $y \in X \setminus \bar{M}$ , such that  $H(A_0, B) < \varepsilon$ . Combining the above two statements for  $\delta = \min\{\delta_1, \delta_2\}$ , each point  $y$  in the  $\delta$ -neighborhood of  $x$  in  $X$  has an element  $B \in \alpha(y)$  such that  $H(A_0, B) < \varepsilon$ . Thus we conclude that  $\alpha$  is a  $c$ -function for  $X$ . Hence by [1],  $C(X)$  is contractible.

Since it is rather easier to obtain a monotone-connected fiber function  $\alpha: \bar{M} \rightarrow C(\bar{M})$  and in view of Proposition 2.2 and Corollary 2.3, we state the following corollaries.

*Corollary 3.6.* Suppose  $X$  is a  $T$ -admissible space with a locally connected and connected subspace  $M$  as its  $\mathbb{M}$ -set such that each element  $A \in F(x) \cap C(\bar{M})$  is admissible at  $x$  in  $X$  for  $x \in \bar{M}$ . Then  $C(X)$  is contractible.

*Proof.* Let  $A \in F(x) \cap C(\bar{M})$ ,  $x \in M$ , and  $\epsilon > 0$ . Then there is an arbitrarily small connected neighborhood  $N$  of  $x$  in  $M$  such that  $H(A, A \cup \bar{N}) < \epsilon$  and the element  $A \cup \bar{N} \in F(x) \cap C(\bar{M})$ .

Define  $\alpha: \bar{M} \rightarrow C(\bar{M})$  by  $\alpha(x) = F(x) \cap C(\bar{M})$ . Then  $\alpha$  is a monotone-connected fiber function.

*Corollary 3.7.* Suppose the  $\mathbb{M}$ -set  $M$  of a  $T$ -admissible space  $X$  is the union of two components  $M_1$  and  $M_2$  with  $\bar{M}_1 \cap \bar{M}_2 = \emptyset$ . If there is a lower semi-continuous monotone-connected fiber function  $\alpha_i: \bar{M}_i \rightarrow C(\bar{M}_i)$ ,  $i = 1, 2$ . Then  $C(X)$  is contractible.

In [4], we introduced a notion of a space  $X$  being contractible im kleinen at a closed set  $K$  and proved that if  $C(X)$  is contractible and  $X$  is contractible im kleinen at  $K$  then the hyperspace  $C(X/K)$  is contractible. In this line, we use the  $T$ -admissibility condition on admissible fiber of  $X$  to investigate certain confluent maps associated with the  $\mathbb{M}$ -set of  $X$  and the contractibility of the hyperspace of the quotient space  $X/\bar{M}$ .

We recall the definition of a confluent map. Let  $X$  and  $\hat{X}$  be continua. A map  $f: X \rightarrow \hat{X}$  is called confluent if  $f$  is a continuous surjection such that for each component  $B$  of  $f^{-1}(\hat{B})$  of each subcontinuum  $\hat{B}$  of  $\hat{X}$  it is true that  $f(B) = \hat{B}$ . Clearly, continuous monotone surjections are confluent.

*Lemma 3.8.* Let  $f: X \rightarrow \hat{X}$  be a confluent map and  $M$  be the  $\mathcal{M}$ -set of  $X$ . Suppose  $M \cap f^{-1}(\hat{x}) = \emptyset$ . Then  $\hat{x}$  is a  $K$ -point of  $\hat{X}$ .

*Proof.* Let  $\hat{H}$  denote the Hausdorff metric on  $C(\hat{X})$  and  $f^*: C(X) \rightarrow C(\hat{X})$  be the map induced by  $f$ . Then  $f^*$  is uniformly continuous. Let  $\epsilon > 0$ . There is  $\delta_1 > 0$  such that  $H(A, B) < \delta_1$ ,  $A, B \in C(X)$  imply  $\hat{H}(f(A), f(B)) < \epsilon$ . Let  $x \in f^{-1}(\hat{x})$ . Suppose  $\hat{A} \in C(\hat{X})$  such that  $\hat{x} \in \hat{A}$  and denote by  $A$  the component of  $f^{-1}(\hat{A})$  containing  $x$ . Since  $x$  is a  $K$ -point of  $X$  there is  $\eta_x > 0$  such that for each  $y$  in the  $\eta_x$ -neighborhood of  $x$  there is  $B \in F(y)$  such that  $H(A, B) < \delta_1$ . By the confluency of  $f$  we have  $\hat{H}(\hat{A}, f(B)) = \hat{H}(f(A), f(B)) < \epsilon$ . The compactness of  $f^{-1}(\hat{x})$  implies there is  $\eta > 0$  such that for each  $y$  in the  $\eta$ -neighborhood  $V$  of  $f^{-1}(\hat{x})$  there is  $B \in F(y)$  such that  $\hat{H}(\hat{A}, f(B)) < \epsilon$ . There is  $\delta > 0$  such that the  $\delta$ -neighborhood  $W$  of  $\hat{x}$  in  $\hat{X}$  has  $f^{-1}(W) \subset V$ . For each  $\hat{y}$  in the  $\delta$ -neighborhood of  $\hat{x}$  we have  $f^{-1}(\hat{y}) \subset V$ . Let  $y \in f^{-1}(\hat{y})$ . Then there is  $B \in F(y)$  such that  $\hat{H}(\hat{A}, f(B)) < \epsilon$ . Since  $\hat{y} = f(y) \in f(B)$ , we have  $\hat{x}$  is a  $K$ -point of  $\hat{X}$ .

The above lemma includes a result of [9] where the  $\mathcal{M}$ -set is assumed to be empty.

*Lemma 3.9.* Let  $f: X \rightarrow \hat{X}$  be a confluent map and  $M$  be the  $\mathcal{M}$ -set of  $X$ . Suppose  $X$  is  $T$ -admissible and  $\hat{x} \in \hat{X}$  is such that, for each component  $M_\alpha$  of  $M$ , either  $M_\alpha \cap f^{-1}(\hat{x}) = \emptyset$  or  $f^{-1}(\hat{x}) \supset M_\alpha$ . Then  $\hat{x}$  is a  $K$ -point of  $\hat{X}$ .

*Proof.* The proof is similar to that of the previous lemma. Let  $x \in M \cap f^{-1}(\hat{x})$  and denote by  $M_\alpha$  the component

of  $M$  containing  $x$ . Then  $M_\alpha \subset f^{-1}(\hat{x})$ . As before, let  $\hat{A} \in C(\hat{X})$  such that  $\hat{x} \in \hat{A}$  and denote by  $A$  the component of  $f^{-1}(\hat{A})$  containing  $x$ . Then  $M_\alpha \subset A$  and hence  $A \in \sigma(x)$  by Proposition 3.3. Consequently, there is  $n_x > 0$  such that each  $y$  in the  $n_x$ -neighborhood of  $x$  has an element  $B \in F(y)$  such that  $H(A, B) < \delta_1$ . The proof is completed just as in Lemma 3.8.

Immediate consequences are following.

*Theorem 3.10.* Let  $X$  be  $T$ -admissible and  $f: X \rightarrow \hat{X}$  be confluent. If, for each component  $M_\alpha$  of the  $M$ -set of  $X$  and each  $\hat{x} \in \hat{X}$ , either  $M_\alpha \cap f^{-1}(\hat{x}) = \emptyset$  or  $M_\alpha \subset f^{-1}(\hat{x})$ , then  $\hat{X}$  has property  $K$  and hence  $C(\hat{X})$  is contractible.

*Corollary 3.11.* Let  $X$  be  $T$ -admissible. If the  $M$ -set  $M$  of  $X$  is connected then the quotient space  $X/\bar{M}$  has property  $K$  and hence  $C(X/\bar{M})$  is contractible.

#### 4. Obstructing Sets

In [5], we introduced the notion of  $S$ -point and proved that any space having an  $S$ -point does not have contractible hyperspaces. In this section, we generalize this notion. Let  $X$  be a nonvoid metric continuum. By Theorem 1.0, each admissible fiber  $\sigma(x)$  is nonempty. However, if  $X$  is not an admissible space, there is an element  $(x, t) \in X \times I$  such that  $\sigma_t(x) = \sigma(x) \cap \mu^{-1}(t) = \emptyset$ . This occurs at some point  $x$  of the  $M$ -set of  $X$ .

*Proposition 4.1.* Suppose  $\sigma_t(x) = \emptyset$  for some  $(x, t) \in X \times I$ . Let  $S = \{t \in I \mid \sigma_t(x) = \emptyset\}$ . Then  $S$  is nonempty open

subset of the reals  $R$  contained in  $I$ . Moreover, if  $t_0 \in I \setminus S$  such that  $t_0 = \text{glb } S'$ , for some nonempty subset  $S' \subset S$ , then  $a_{t_0}(x) \subset \mathcal{M}_x = \{A \in a_{t_0}(x) \mid A \subset M\}$ . In particular, if  $s_0 = \text{glb } S$ , then  $a_s(x) \subset \mathcal{M}_x$  for all  $0 \leq s \leq s_0$ .

*Proof.* Let  $s$  be a limit point of  $S$ . For each positive integer  $n$  there is  $s_n \in I \setminus S$  such that  $|s_n - s| < \frac{1}{n}$ . Let  $A_n \in a(x) \cap \mu^{-1}(s_n)$ . Since  $a(x)$  is compact in  $C(X)$ , we may assume that the sequence  $A_n$  converges to  $A_0 \in a(x)$ . Because  $\mu$  is continuous we have  $A_0 \in a(x) \cap \mu^{-1}(s)$ . Hence  $s \in I \setminus S$  and  $S$  is open in  $I$ . Since  $0 \notin S$ ,  $1 \notin S$  we have  $S$  is open in  $R$ .

The proof of the second assertion is similar to that of the last assertion. Let  $0 \leq s \leq s_0$ ,  $s_0 = \text{glb } S$ . We suppose there is  $B \in a_s(x)$  such that  $B \setminus M \neq \emptyset$ . Let  $\xi \in B \setminus M$ . Then  $F(\xi) = a(\xi)$ . There is a monotone segment  $\mathcal{J}$  from  $B$  to  $X$  in  $C(X)$  by [3]. Since  $\mu(\mathcal{J}) = [s, 1]$ , there is an element  $A \in \mathcal{J}$  such that  $\mu(A) \in S$  and  $A \supset B$ . This means that  $A$  is not admissible at  $x$ . On the other hand, we have  $A \in a(\xi)$ ,  $B \in a(x)$  and  $\xi \in A \cap B$ . So by Theorem 1.0,  $A = A \cup B \in a(x)$ , a contradiction. Hence  $a_s(x) \subset \mathcal{M}_x$  for  $0 \leq s \leq s_0$ . The proposition is now proved.

If  $s_0 = 0$  then  $a_{s_0}(x) = \{x\}$ . In this case  $x$  is an  $S$ -point as defined in [5]. It is clear that the concept of  $S$ -point is independent of the choice of the Whitney function  $\mu$ . By Theorem 1.1, an increasing continuous map  $h: X \times I \rightarrow C(X)$  with  $h(x, 0) = \{x\}$  must have  $h(x, t) \in a(x)$  for all  $(x, t) \in X \times I$ . We have by Proposition 4.1 that such an  $h$  must stabilize in a subcontinuum (element) of  $a_{t_0}(x)$ .

Let us call element of  $a_{t_0}(x)$  S-set.

*Proposition 4.2.* If a metric continuum  $X$  contains an S-set then  $C(X)$  is not contractible.

## 5. Examples

We will give three examples to illustrate Theorem 3.5 and Corollary 3.6 and an example of a space which contains an S-set.

*Example 5.1.* In the plane, let  $P_n$  and  $q_n$ , be points defined by  $P_n = (0, \frac{1}{n})$ ,  $q_n = (1, \frac{-1}{n})$ , for  $n = 1, 2, 3, \dots$  and  $P_0 = (0, 0)$ ,  $q_0 = (1, 0)$ . Let  $\overline{P_n q_0}$  and  $\overline{q_n P_0}$  be segments joining  $P_n$  to  $q_0$  and  $q_n$  to  $P_0$  respectively for  $n = 1, 2, \dots$  and  $\overline{P_0 q_0} = M$ . Let  $X = \bigcup_{n=1}^{\infty} (\overline{P_n q_0} \cup \overline{q_n P_0}) \cup M$ . Then it is easy to check that  $X$  is T-admissible and  $M$  is the  $\mathcal{H}$ -set of  $X$ . For each  $x \in M$ , every element  $A \in F(x) \cap C(M)$  is admissible at  $x$  in  $X$ . Therefore by Corollary 3.6,  $C(X)$  is contractible.

*Example 5.2.* Let  $X_1$  be the closure of the graph of  $\sin \frac{1}{x}$ ,  $0 < x \leq 1$ , and  $X_2$  the graph of  $\frac{1}{2} \sin \frac{1}{x}$ ,  $-1 \leq x < 0$ , and  $X = X_1 \cup X_2$ . Let  $P_i = (0, i)$ ,  $q_i = (0, \frac{-i}{2})$ ,  $i = \pm 1$ . Let  $M$  be the line segment joining  $P_1$  to  $P_{-1}$ , and  $N$  the line segment joining  $q_1$  to  $q_{-1}$ . Since the first coordinates of points of  $M$  are all 0, we will use the following notations. Denote the point  $(0, z)$  by  $z$  in  $M$ , and  $[z, w]$  denotes the closed segment in  $M$  joining the point  $(0, z)$  and  $(0, w)$ ,  $z < w$ .

Since  $X$  is locally connected at each point  $z \in X \setminus M$ , each element of  $F(z)$  is admissible at  $z$ . If  $z \in M$ , there are elements in  $F(z)$  which are not admissible at  $z$ . Thus  $M$

is the  $\mathcal{M}$ -set of  $X$ . Let  $\alpha: M \rightarrow C(M)$  be a set-valued function defined as follows:

$$\alpha'(z) = \begin{cases} (F(z) \cap C(M)) \cup \{[-\varepsilon, \varepsilon] \mid \frac{1}{2} \leq \varepsilon \leq 1\}, & z \in N \\ \{[z, \varepsilon] \mid z \leq \varepsilon \leq -z\} \cup \{[z-\varepsilon, z+\varepsilon] \mid 0 \leq \varepsilon \leq |1+z|\}, & \\ z \in M \setminus N, -1 \leq z \leq \frac{1}{2}, & \\ \{[\varepsilon, z] \mid -z \leq \varepsilon \leq z\} \cup \{[-z-\varepsilon, z+\varepsilon] \mid 0 \leq \varepsilon \leq |1-z|\}, & \\ \frac{1}{2} \leq z \leq 1 & \end{cases}$$

One can easily check that  $X$  is  $T$ -admissible and  $\alpha'$  monotone-connected connected fiber function. Thus by Corollary 3.6,  $X$  admits a  $c$ -function.

*Example 5.3.* Let  $X_n$  be the closure in the plane of the set  $\{(x, y+4n) \mid y = \sin \frac{1}{x}, 0 < x \leq 1\}$ ,  $n = 0, 1, 2, \dots$  and let  $X$  be the one-point compactification of  $\bigcup_{n=0}^{\infty} X_n$ . Let  $P \in X$  be the point at  $\infty$ ,  $q = (0, -1)$  and let  $M_n$  be the line segment joining the points  $(0, 4n-1)$  and  $(0, 4n+1)$ , and  $Z$  the segment joining  $P$  and  $q$ , and let  $M'_0 = M_0 \setminus \{q\}$ . Then  $M'_0, M_n, n = 1, 2, \dots$ , are the components of the  $\mathcal{M}$ -set  $M = (\bigcup_{n=1}^{\infty} M_n) \cup M'_0$  of  $X$ , and  $\bar{M} = \bigcup_{n=0}^{\infty} M_n$ . We note that  $P$  and  $q$  are  $K$ -points. To check the  $T$ -admissibility of  $X$ , it suffices to check  $\sigma(x)$ , when  $x \in M_n$ . Since each element of  $F(x) \cap C(M_n)$  is admissible at  $x$ , and every subcontinuum of  $X$  containing  $M_n$  is admissible at  $X$ , it is clear that  $\sigma(x)$  satisfies the  $T$ -admissibility condition. We define  $\alpha: \bar{M} \rightarrow C(Z)$  by, for  $x \in M_n$

$$\alpha(x) = (F(x) \cap C(M_n)) \cup \{A \in F(x) \cap C(Z) \mid M_n \subset A\}.$$

Then  $\alpha$  is a monotone-connected lower semicontinuous fiber function. Hence by Theorem 3.5,  $C(X)$  is contractible.

*Example 5.4.* Let  $X = X_1 \cup X_2$ , where  $X_1$  is the closure of the graph of  $\sin \frac{1}{x}$ ,  $0 < x \leq 1$ , and  $X_2$  is the closure of the graph of  $\frac{1}{2} + \sin \frac{1}{x}$ ,  $-1 \leq x < 0$ . Then the line segment joining the points  $(0,1)$  and  $(0,-\frac{1}{2})$  is the S-set of  $X$ .

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