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1. Preliminaries

All spaces considered in this paper are assumed to be Hausdorff. If A is a subset of a space X , then $\text{cl}_X A$ (resp. $\text{int}_X A$, $\text{bd}_X A$) will denote the *closure* (resp. *interior boundary*) of A in X . For a space X , X_s will denote the *semi-regularization* of X (see [16], page 212), $\tau(X)$ will denote the topology on X and $|X|$ denotes the cardinal number of X . Also, $\text{RO}(X)$ (resp. $\text{R}(X)$) denotes the complete Boolean algebra of *regular open* (resp. *regular closed*) subsets of X , and $\text{CO}(X)$ will denote the algebra of clopen (= closed and open) subsets of X . An *open filter* on X is a filter in the lattice $\tau(X)$, and an *open ultrafilter* on X is a maximal (with respect to set inclusion) open filter. If \mathcal{J} is a filter on X then $\text{ad}_X(\mathcal{J}) = \cap \{\text{cl}_X F : F \in \mathcal{J}\}$ denotes the *adherence* of \mathcal{J} in X . A filter \mathcal{J} on X is called *free* if $\text{ad}_X(\mathcal{J}) = \emptyset$; otherwise, \mathcal{J} is called *fixed*. If \mathcal{A} is any nonempty family of subsets of X with the finite intersection property, then $\langle \mathcal{A} \rangle$ will denote the filter on X generated by \mathcal{A} . For an open filter \mathcal{J} on X , we shall denote by \mathcal{J}_s the open filter on X generated by the filterbase $\{\text{int}_X \text{cl}_X A : A \in \mathcal{J}\}$. In what follows, for a space X , $\mathcal{F}(X) = \{\mathcal{U} : \mathcal{U} \text{ is a}$

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free open ultrafilter on X }, $F_s(X) = \{\mathcal{U}_s: \mathcal{U} \in F(X)\}$. Also, N , Q , and R will denote the space of positive integers, the rationals and the reals (with usual topology) respectively.

A map $f: X \rightarrow Y$ is a (not necessarily continuous) function from X to Y . A map $f: X \rightarrow Y$ is called *compact* if for each $y \in Y$, $f^+(y)$ ($= \{x \in X: f(x) = y\}$) is a compact subset of X ; f is called *perfect* if it is both a compact and a closed map, and f is called *irreducible* if f is onto, closed, and, for each proper closed subset A of X , $f(A) \neq Y$. A map $f: X \rightarrow Y$ is called θ -continuous at a point $x \in X$ (see [6]) if for each open neighborhood G of $f(x)$ in Y , there is an open neighborhood U of x in X such that $f(\text{cl}_X U) \subseteq \text{cl}_Y G$. If f is θ -continuous at each $x \in X$ then f is called θ -continuous. A map $f: X \rightarrow Y$ is called a θ -homeomorphism provided that f is one-to-one, onto and both f and f^+ are θ -continuous, and in this case the spaces X and Y are called θ -homeomorphic.

1.1 With each Hausdorff space X there is associated the space EX (called the *Iliadis absolute* of X [7]) consisting of all the convergent open ultrafilters on X with the topology $\tau(EX)$ generated by the open base $\{O_X U: U \in \tau(X)\}$, where

$$O_X U = \{\mathcal{U} \in EX: U \in \mathcal{U}\}.$$

The space EX is unique (up to homeomorphism) with respect to possessing these properties: EX is extremally disconnected and zero-dimensional (see [24] for definitions), and there exists a perfect, irreducible and θ -continuous

surjection $k_X: EX \rightarrow X$ (given by $k_X(\mathcal{U}) = \text{ad}_X(\mathcal{U})$, $\mathcal{U} \in EX$). The *Hausdorff absolute* (see [10], [19]) is the space PX whose underlying set is the set of EX with the topology $\tau(PX)$ generated by the open base $\{O_X U \cap k_X^+(V): U, V \in \tau(X)\}$. The space PX is unique (up to homeomorphism) with respect to possessing these properties: PX is extremally disconnected (but not necessarily zero-dimensional) and there exists a perfect, irreducible, continuous surjection $\pi_X: PX \rightarrow X$, given by $\pi_X(\mathcal{U}) = \text{ad}_X(\mathcal{U})$, $\mathcal{U} \in PX$. For a space X , $EX = (PX)_S$, $\tau(EX) \subseteq \tau(PX)$ and $RO(EX) = RO(PX) = CO(EX) = CO(PX) = \{O_X U: U \in \tau(X)\}$. For further details about EX and PX , the reader may refer to [7], [10], [12], [19], [20], [21] and [26].

1.2 An *extension* of a space X is a Hausdorff space Y such that X is a dense subspace of Y . If Y and Z are extensions of a space X , then Y is said to be *projectively larger* than Z , written hereafter $Y \succeq_X Z$, if there is a continuous mapping $\phi: Y \rightarrow Z$ such that $\phi|_X = i_X$, the identity map on X . Two extensions Y and Z of a space X are called *equivalent* if $Y \succeq_X Z$ and $Z \succeq_X Y$. We shall identify two equivalent extensions of X . If Y is an extension of X , then Y_S is an extension of X_S . Let Y be an extension of a space X . If \mathcal{U} is an open (ultra) filter on X , then

$$\mathcal{U}^* = \{U \in \tau(Y): U \cap X \in \mathcal{U}\}$$

is an open (ultra) filter on Y which converges in Y if and only if \mathcal{U} converges in X ; if \mathcal{W} is an open (ultra) filter on Y , then

$$\mathcal{W}_* = \{W \cap X: W \in \mathcal{W}\}$$

is an open (ultra) filter on X which converges in Y if and only if \mathcal{W} converges in Y . If more than one extension is involved, the meanings of \mathcal{W}^* and \mathcal{W}_* will be clear from the context.

Each extension Y of a space X induces the extensions $Y^\#$ and Y^+ of X . The extensions $Y^\#$ and Y^+ were introduced by Banaschewski [2] in 1964 (see also [16]). Let Y be an extension of X . For a point $y \in Y$, let

$$(a) \quad \mathcal{O}_Y^Y = (\mathcal{N}_Y)_* = \{U \cap X: U \in \mathcal{N}_Y\}$$

where \mathcal{N}_Y is the open neighborhood filter of y in Y . For an open subset U of X , let

$$(b) \quad \mathcal{o}_Y(U) = \{y \in Y: U \in \mathcal{O}_Y^Y\}.$$

The family $\{\mathcal{o}_Y(U): U \in \tau(X)\}$ (respectively, $\{U \cup \{y\}: y \in Y \setminus X, U \in \mathcal{O}_Y^Y \cup \tau(X)\}$) forms an open base for a coarser (resp. finer) Hausdorff topology $\tau^\#$ (resp. τ^+) on Y . The space $(Y, \tau^\#)$ (resp. (Y, τ^+)), denoted by $Y^\#$ (resp. Y^+) is an extension of X . An extension Y of a space X is called a *strict* (resp. *simple*) extension of X if $Y = Y^\#$ (resp. $Y = Y^+$). It can be shown very easily that Y is a simple extension of X if and only if X is open in Y and $Y \setminus X$ is a discrete subspace of Y . It is proved in [16] that for any extension Y of X ,

$$(c) \quad \text{int}_Y \text{cl}_Y W = \text{int}_Y \text{cl}_Y (W \cap X) = \mathcal{o}_Y(\text{int}_X \text{cl}_X (W \cap X))$$

for each $W \in \tau(Y)$.

1.3 Definition [9]. Let Y be an extension of a space X . Then,

(a) X is said to be *paracombinatorially embedded* in Y if, for each pair G_1, G_2 of disjoint open subsets of X , $\text{cl}_Y(G_1) \cap \text{cl}_Y(G_2) \subseteq X$.

(b) X is said to be *hypercombinatorially embedded* in Y if for each pair F_1, F_2 of closed subsets of X such that $F_1 \cap F_2$ is nowhere dense in X , $\text{cl}_Y F_1 \cap \text{cl}_Y F_2 = F_1 \cap F_2$.

It follows from the definition that if Y is an extension of X , then X is paracombinatorially embedded in Y if and only if X is paracombinatorially embedded in $Y^\#$ (resp. Y^+). The following result will be used subsequently.

1.4 Proposition [18]. *Let T be an extension of a space X , S a space and $f: S \rightarrow T$ a perfect, irreducible and continuous surjection. If X is hypercombinatorially embedded in T , then $f^+(X)$ is hypercombinatorially embedded in S .*

1.5 Recall that a space X is called *H-closed* (see [1] provided that X is closed in every Hausdorff space Y in which X is embedded. X is called *minimal Hausdorff* if $\tau(X)$ does not contain any coarser Hausdorff topology on X . A subset $A \subseteq X$ is called a *H-set* in X (see [23]) if whenever \mathcal{C} is any cover of A by open sets in X , then there is a finite subfamily $\{C_i: i = 1, 2, \dots, n\} \subseteq \mathcal{C}$ such that $A \subseteq \cup\{\text{cl}_X C_i: i = 1, 2, \dots, n\}$; this is equivalent to saying that for every open filter \mathcal{F} on X if $A \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, then $A \cap (\text{ad}_X(\mathcal{F})) \neq \emptyset$. The *Katetov extension* (see [9]) of a space X is the simple H-closed extension κX of X whose underlying set is the set $X \cup F(X)$ with the topology $\tau(\kappa X)$ generated by the open base $\tau(X) \cup \{U \cup \{U\}: U \in \mathcal{U} \in F(X), U \in \tau(X)\}$. The *Fomin extension* (see [6]) of

a space X is the strict H -closed extension σX of X whose underlying set is the set of κX and whose topology $\tau(\sigma X)$ is generated by the open base $\{o_{\sigma X}(U): U \in \tau(X)\}$, where for each $U \in \tau(X)$, $o_{\sigma X}(U) = U \cup \{\mathcal{U}: U \in \mathcal{U} \in \mathcal{F}(X)\}$. The space κX is the projective maximum in the set of all the H -closed extensions of X , $\sigma X = (\kappa X)^\#$, $\kappa X = (\sigma X)^+$ and $(\kappa X)_S = (\sigma X)_S$; moreover, the identity map $i: \sigma X \rightarrow \kappa X$ is perfect, irreducible and θ -continuous (see [1], [6], [8], [9], [15], [16], [17] and [20] for further details).

2. The Banaschewski—Fomin—Shanin (BFS)—Extension μX

The minimal Hausdorff extension $(\sigma X)_S$ (generally denoted by μX and called the BFS-extension in the existing literature) has been extensively studied by many authors for a semiregular space X (see for example [5], [13], [14], [15], [16] and [17]). It has been an open problem for a long time whether an extension of the type μX can be obtained for a general Hausdorff space X . Ovsepjan [11] gave a definition in this direction. In what follows, we shall explicitly describe an extension of the type μX for a general Hausdorff space X and study some of its properties.

2.1 Let X be a Hausdorff space and let $\tilde{X} = X \cup \mathcal{F}_S(X)$. For each $G \in \tau(X)$, let $o_{\tilde{X}}(G) = G \cup \{\mathcal{U}_S: \mathcal{U}_S \in \tilde{X} \setminus X, G \in \mathcal{U}_S\}$. Then,

$$o_{\tilde{X}}(G) \cap o_{\tilde{X}}(H) = o_{\tilde{X}}(G \cap H) \text{ if } G, H \in \tau(X).$$

Hence, the family $\{o_{\tilde{X}}(G): G \in \tau(X)\}$ forms an open base for a topology $\tau^\#$ on \tilde{X} . A routine verification shows that that $(\tilde{X}, \tau^\#)$, briefly denoted by \tilde{X}_1 , is a strict H -closed

extension of X . The map $j: \sigma X \rightarrow \tilde{X}_1$ defined by:

$$j(x) = x \text{ if } x \in X$$

$$j(\mathcal{U}) = \mathcal{U}_s \text{ if } \mathcal{U} \in F(X) \quad (\text{note } \mathcal{U} \neq \mathcal{U}_s \text{ in general})$$

is a bijection.

We now define a topology τ^+ on \tilde{X} by declaring that X is open in \tilde{X} , and, for $\mathcal{U}_s \in \tilde{X} \setminus X$, a τ^+ -basic neighbourhood of \mathcal{U}_s is $U \cup \{\mathcal{U}_s\}$ where U is open in X and $U \in \mathcal{U}_s$. Then (\tilde{X}, τ^+) is a simple H -closed extension of X . A direct application of the definition leads to the following result.

2.2 Proposition. For each open subset U of a space X ,

$$(a) \text{cl}_{\tilde{X}_1}^{\sim}(U) = \text{cl}_{\tilde{X}_1}^{\sim}(\sigma_{\tilde{X}_1}^{\sim}(U)) = (\text{cl}_X U) \cup \sigma_{\tilde{X}_1}^{\sim}(\text{int}_X \text{cl}_X U) \subseteq \text{cl}_{\sigma X} j^+(U), \text{ and}$$

$$(b) \sigma_{\tilde{X}_1}^{\sim}(\text{int}_X \text{cl}_X(U)) \setminus X = \sigma_{\sigma X} j^+(U) \setminus X.$$

2.3 Theorem. Let X be a space. Then:

(a) the mapping $j^*: \tilde{X}_1 \rightarrow \sigma X$ is a θ -homeomorphism, and

(b) $\tilde{X}_1 \setminus X \approx \sigma X \setminus X$.

Proof. The proof of (a) follows by 2.2(a) and [16;

1.2]. To prove (b), we note that from 2.2(b) it follows that the mapping $j^*|_{\tilde{X}_1 \setminus X}: \tilde{X}_1 \setminus X \rightarrow \sigma X \setminus X$ is continuous.

Further, if $\mathcal{U}_s \in \sigma_{\tilde{X}_1}^{\sim}(U)$, then there is a regular open set

$V \in \text{RO}(X)$ such that $V \in \mathcal{U}_s$ and $V \subseteq U$. So, $\mathcal{U} \in \sigma_{\sigma X}(V)$.

By 2.2(b), $\sigma_{\sigma X}(V) \setminus X = \sigma_{\tilde{X}_1}^{\sim}(\text{int}_X \text{cl}_X(V)) \setminus X = \sigma_{\tilde{X}_1}^{\sim}(V) \setminus X \subseteq \sigma_{\tilde{X}_1}^{\sim}(U) \setminus X$.

Hence the map $(j^*|_{\tilde{X}_1 \setminus X})^*: \sigma X \setminus X \rightarrow \tilde{X}_1 \setminus X$ is continuous, and

(b) follows.

2.4 Proposition. The following statements are equivalent for a space X .

- (a) X is semiregular.
- (b) \tilde{X}_1 is semiregular.
- (c) $\tilde{X}_1 = (\sigma X)_s$.

Proof. Since the proof of (c) \Rightarrow (b) \Rightarrow (a) is obvious, we show that (a) \Rightarrow (b) \Rightarrow (c). Now, (b) \Rightarrow (c) follows from 2.3(a) and [16]. To prove (a) \Rightarrow (b), since $\text{int}_{\tilde{X}_1} \text{cl}_{\tilde{X}_1} (o_{\tilde{X}_1} (G)) = \text{int}_{\tilde{X}_1} \text{cl}_{\tilde{X}_1} G = o_{\tilde{X}_1} (\text{int}_X \text{cl}_X G)$ for all $G \in \tau(X)$, it suffices to show that the family $\{o_{\tilde{X}_1} (\text{int}_X \text{cl}_X (H)) : H \text{ open in } X\}$ is an open base for \tilde{X}_1 . Let \tilde{U} be a nonempty open subset of \tilde{X}_1 . If $\mathcal{U}_s \in \tilde{U} \setminus X$, then there is a nonempty open subset G of X such that $\mathcal{U}_s \in o_{\tilde{X}_1} (G) \subseteq \tilde{U}$. Since $G \in \mathcal{U}_s$, $G \supseteq \text{int}_X \text{cl}_X H$ for some $H \in \mathcal{U}$. Now $\text{int}_X \text{cl}_X H \in \mathcal{U}_s$. So, $\mathcal{U}_s \in o_{\tilde{X}_1} (\text{int}_X \text{cl}_X H) \subseteq o_{\tilde{X}_1} (G) \subseteq \tilde{U}$. Now, let $x \in \tilde{U} \cap X$, and let G be open in X such that $x \in o_{\tilde{X}_1} (G) \subseteq \tilde{U}$. Then $x \in G$. Since X is semiregular, there is an open set $H \subseteq X$ such that $x \in \text{int}_X \text{cl}_X H \subseteq G$. Hence, $x \in o_{\tilde{X}_1} (\text{int}_X \text{cl}_X H) \subseteq o_{\tilde{X}_1} (G) \subseteq \tilde{U}$, and (b) follows.

2.5 Remark. In view of 2.4 we shall, henceforth, denote \tilde{X}_1 by μX , and call it the BFS-extension of X . For each Hausdorff space X , $\mu X = \sigma X$ (or, equivalently, $\kappa X = \mu^+ X$) if and only if $\mathcal{U} = \mathcal{U}_s$ for each $\mathcal{U} \in F(X)$. One can show very easily that a space X is extremally disconnected if and only if μX is extremally disconnected, if and only if $\mu^+ X$ is extremally disconnected. It would be interesting to characterize those Hausdorff spaces X for which $\mu X = \sigma X$. In the next two propositions, we provide a partial answer to this problem.

2.6 Proposition. If every closed and nowhere dense subset of a space X is contained in a H -set, then $\sigma X = \mu X$.

Proof. Let $\mathcal{U} \in F(X)$. If $U \in \mathcal{U}$ and U is not regular open, then by hypothesis, there exists a H -set $H \subset X$ such that $\emptyset \neq \text{cl}_X(\text{int}_X \text{cl}_X U \setminus U) \subseteq H$. Since \mathcal{U} is free, then, for each $p \in H$ there exist open subsets T_p and W_p of X such that $p \in T_p$, $W_p \in \mathcal{U}$, $W_p \subseteq U$ and $T_p \cap W_p = \emptyset$. Since H is a H -set in X , the open covering $\{T_p : p \in H\}$ of H contains a finite subfamily $\{T_{p_i} : i = 1, 2, \dots, n\}$ such that $H \subseteq \bigcup \{\text{cl}_X(T_{p_i}) : i = 1, 2, \dots, n\} = \text{cl}_X(T)$, where $T = \bigcup \{T_{p_i} : i = 1, 2, \dots, n\}$. Let W_{p_i} be the corresponding members of \mathcal{U} with $W_{p_i} \cap T_{p_i} = \emptyset$, and $W_{p_i} \subseteq U$ for all $i = 1, 2, \dots, n$, and let $W = \bigcap \{W_{p_i} : i = 1, 2, \dots, n\}$. Then $W \in \mathcal{U}$, $W \subseteq U$ and $(\text{int}_X \text{cl}_X W) \cap H = \emptyset$. Now $\text{int}_X \text{cl}_X W \subseteq \text{int}_X \text{cl}_X (U) = U \cup (\text{int}_X \text{cl}_X U \setminus U) \subseteq U \cup H$ and the above fact implies that $\text{int}_X \text{cl}_X W \subseteq U$. Hence, $U \in \mathcal{U}_s$. Thus $\mathcal{U} = \mathcal{U}_s$ and the result follows by 2.5.

2.7 Proposition. Let X be semiregular and extremally disconnected. Then $\sigma X = \mu X$ if and only if every closed and nowhere dense subset of X is compact.

2.8 Definition. (a) [13]. A Hausdorff space X is said to be almost H -closed if, for every pair of disjoint nonempty open subsets of X , the closure of at least one of them is H -closed.

(b) [9]. A subset A of a space X is called regularly nowhere dense if there are disjoint open sets U and V such that $\text{cl}_X A = \text{cl}_X U \cap \text{cl}_X V$.

2.9 Theorem. *Let X be a space. The following statements are equivalent.*

- (a) $\kappa X = \sigma X$.
- (b) $|\kappa X \setminus X| < \aleph_0$.
- (c) X has a finite cover of almost H -closed spaces.
- (d) $\mu^+ X = \mu X$.

Proof. See [16, Thm. 4.2] and [5, Thm. 12].

If the space μX is compact then X must be semiregular. It is proved in [15] that for a space X , μX is compact if and only if $\mu X = \beta X$, if and only if X is semiregular and every closed regularly nowhere dense subset of X is compact. We prove the analogous result for $\mu^+ X$.

2.10 Theorem. *For a space X , the following statements are equivalent.*

- (a) $\mu^+ X$ is compact.
- (b) (i) X has a finite cover of almost H -closed spaces, and
(ii) X is semiregular and every closed regularly nowhere dense subset of X is compact.

Proof. The proof is a direct consequence of 2.9, [15, Thm. 6.2] and the fact that $\mu^+ X \setminus X$ is discrete.

3. Characterization of the Spaces μX and $\mu^+ X$

3.1 Definition. (a) A point p of a space X is called a *semiregular point* (respectively, a *regular point*) if whenever G is any open neighborhood of p in X , then there exists an open subset $U \subseteq X$ such that $p \in \text{int}_X \text{cl}_X U \subseteq G$ (respectively, $p \in \text{cl}_X U \subseteq G$).

(b) A filter \mathcal{J} on the Boolean algebra $R(X)$ of a space X will be called a *rc-filter* on X . An open filter \mathcal{J} on a space X is called a *regular filter* if, for each $U \in \mathcal{J}$, there is a $V \in \mathcal{J}$ such that $\text{cl}_X V \subseteq U$.

The next two propositions characterize the spaces μX and $\mu^+ X$. We omit their straightforward proofs.

3.2 Proposition. *The space μX is uniquely determined by the following properties:*

- (a) μX is a strict H -closed extension of X ,
- (b) X is paracombinatorially embedded in μX , and
- (c) each point $p \in \mu X \setminus X$ is a semiregular point in μX .

3.3 Proposition. *The space $\mu^+ X$ is uniquely determined by the following properties:*

- (a) $\mu^+ X$ is a simple H -closed extension of X ,
- (b) X is hypercombinatorially embedded in $\mu^+ X$, and
- (c) each point $p \in \mu^+ X \setminus X$ is a semiregular point in $(\mu^+ X)^\#$.

3.4 Lemma. *Let \mathcal{W} be a free rc-ultrafilter on X , and let $\mathcal{W}^0 = \{\text{int}_X \text{cl}_X W : W \in \mathcal{W}\}$. Then $\mathcal{W}^0 = \mathcal{U}_S$ for some $\mathcal{U} \in F(X)$.*

Proof. Clearly, \mathcal{W}^0 is a free open filter base and is contained in some free open ultrafilter \mathcal{U} . Moreover, $\mathcal{W}^0 \subseteq \mathcal{U}_S$. Now, if V is a regular open set in \mathcal{U}_S , then $V \cap \text{int}_X W \neq \emptyset$ for all $W \in \mathcal{W}$. Thus, $(\text{cl}_X V) \cap W \neq \emptyset$ for all $W \in \mathcal{W}$. Since $X = (\text{cl}_X V) \cup (X \setminus V) \in \mathcal{W}$ and \mathcal{W} is a rc-ultrafilter, either $\text{cl}_X(V) \in \mathcal{W}$ or $X \setminus V \in \mathcal{W}$. However, $(\text{cl}_X V) \wedge (X \setminus V) = \emptyset$. So, $X \setminus V \notin \mathcal{W}$. Thus $\text{cl}_X(V) \in \mathcal{W}$, whence, $V = \text{int}_X \text{cl}_X V \in \mathcal{W}^0$. Thus, $\mathcal{U}_S = \mathcal{W}^0$.

Recall that an open cover \mathcal{C} of a space X is called a p -cover of X if there exist finitely many members C_1, C_2, \dots, C_n in \mathcal{C} such that $X = \bigcup_{i=1}^n \text{cl}_X C_i$. If X is a space and \mathcal{F} is a filter on X , then, a subset $A \subseteq X$ is said to miss \mathcal{F} if $A \cap F = \emptyset$ for some $F \in \mathcal{F}$; otherwise, we say that \mathcal{F} meets A .

3.5 Theorem. For a space X , the following statements are equivalent.

(a) If A is any closed regularly nowhere dense subset of X , then A misses every free rc-filter on X .

(b) \mathcal{U}_s is a regular filter for each $\mathcal{U} \in \mathcal{F}(X)$.

(c) If \mathcal{C} is any regular open cover of X such that \mathcal{C} is not a p -cover, then for each closed regularly nowhere dense subset A of X there exist finitely many $C_1, C_2, C_3, \dots, C_n$ in \mathcal{C} such that $A \subseteq \text{int}_X \text{cl}_X [\bigcup_{i=1}^n C_i]$.

Proof. (a) \Rightarrow (c). Let $A \subseteq \text{bd}_X U$, $U \in \text{RO}(X)$ be any closed regularly nowhere dense subset of X , and let \mathcal{C} be an open cover of X consisting of regular open subsets of X , which is not a p -cover. Then, $\mathcal{F} = \{\text{cl}_X \text{int}_X (X \setminus \bigcup_{i=1}^n C_i) : C_1, C_2, \dots, C_n \in \mathcal{C}, n \in \mathbb{N}\}$ is a free rc-filter base. Hence, by (a) there is a finite family C_1, C_2, \dots, C_n in \mathcal{C} such that $A \cap \text{cl}_X \text{int}_X [X \setminus \bigcup_{i=1}^n C_i] = \emptyset$. Consequently, $A \subseteq \text{int}_X \text{cl}_X [\bigcup_{i=1}^n C_i]$ and (c) follows.

(c) \Rightarrow (b). Let $\mathcal{U} \in \mathcal{F}(X)$ and let $U \in \text{RO}(X) \cap \mathcal{U}_s$. The family $\{X \setminus \text{cl}_X(W) : W \in \mathcal{U}_s\}$ is a regular open cover of X which is not a p -cover. Hence, by (c), there are finitely many W_1, W_2, \dots, W_n in \mathcal{U}_s such that $\text{bd}_X U \subseteq \text{int}_X \text{cl}_X [\bigcup_{i=1}^n (X \setminus \text{cl}_X(W_i))] = X \setminus \text{cl}_X \text{int}_X [\bigcap_{i=1}^n (\text{cl}_X(W_i))] \subseteq X \setminus \text{cl}_X [\bigcap_{i=1}^n W_i]$. Let

$V = U \cap \bigcap_{i=1}^n W_i$. Then, $V \in \mathcal{U}_S$ and $(\text{cl}_X V) \cap [X \setminus \text{cl}_X (\bigcap_{i=1}^n W_i)] = \emptyset$. Hence $\text{cl}_X V \subseteq U$ and (b) follows.

(b) \Rightarrow (a). Let A be a closed regularly nowhere dense subset of X , say $A \subseteq \text{bd}_X U$ for some $U \in \text{RO}(X)$. Let \mathcal{F} be any free rc-filter on X . Assume that \mathcal{F} meets A . Then \mathcal{F} meets $\text{cl}_X U$. Hence, the family $\mathcal{F} \cup \{\text{cl}_X V : V \in \tau(X), V \supseteq \text{bd}_X U\}$ has the finite intersection property, and there is a free rc-ultrafilter \mathcal{W} containing this family. By 3.4, $\mathcal{W}^0 = \{\text{int}_X \text{cl}_X W : W \in \mathcal{W}\} = \mathcal{U}_S$ for some $\mathcal{U} \in F(X)$. Now since $U \in \text{RO}(X)$, either $U \in \mathcal{U}_S$ or $X \setminus \text{cl}_X U \in \mathcal{U}_S$. Suppose that $U \in \mathcal{U}_S$. Since \mathcal{U}_S is a regular filter by hypothesis, there is a set $V \in \mathcal{U}_S$ such that $\text{cl}_X V \subseteq U$. So, $X \setminus \text{cl}_X V \supseteq \text{bd}_X U$, and, hence, $\text{cl}_X (X \setminus \text{cl}_X V) \in \mathcal{W}$. But then $X \setminus \text{cl}_X V = \text{int}_X \text{cl}_X (X \setminus \text{cl}_X V) \in \mathcal{U}_S$, which is impossible, since $V \in \mathcal{U}_S$. Now if $X \setminus \text{cl}_X U \in \mathcal{U}_S$, then there is a set $V' \in \mathcal{U}_S$ such that $\text{cl}_X V' \subseteq X \setminus \text{cl}_X U$, and since $\text{bd}_X U = \text{bd}_X (X \setminus \text{cl}_X U)$, by the same reasoning as above, $X \setminus \text{cl}_X V' \in \mathcal{U}_S$, which is impossible. Thus \mathcal{F} misses A , and the theorem follows.

3.6 Proposition. For a space X , each point $p \in \mu X \setminus X$ is regular in μX if and only if \mathcal{U}_S is a regular filter on X for each $\mathcal{U} \in F(X)$.

Proof. Suppose that \mathcal{U}_S is a regular filter for each $\mathcal{U} \in F(X)$. Let $o_{\mu X}(G)$ be a basic open neighborhood of \mathcal{U}_S in μX , where $G \in \mathcal{U}_S$. There is a regular open set $H \in \mathcal{U}_S$ such that $\text{cl}_X H \subseteq G$. Then, $\mathcal{U}_S \in o_{\mu X}(H) \subseteq \text{cl}_{\mu X}(o_{\mu X}(H)) = \text{cl}_X(H) \cup o_{\mu X}(\text{int}_X \text{cl}_X H) \subseteq o_{\mu X}(G)$, whence, \mathcal{U}_S is a regular point in μX . Conversely suppose that each point $\mathcal{U}_S \in \mu X \setminus X$ is a regular point in μX . Let $\mathcal{U}_S \in \mu X \setminus X$, and let $G \in \mathcal{U}_S$. Then,

there exists a basic open neighborhood $o_{\mu X}(H)$ of \mathcal{U}_S such that $\mathcal{U}_S \in o_{\mu X}(H) \subseteq cl_{\mu X} H \subseteq o_{\mu X}(G)$. Hence, $H \in \mathcal{U}_S$ and $cl_X H \subseteq X \cap o_{\mu X}(G) = G$, whence \mathcal{U}_S is a regular filter, and the proof of the proposition is complete.

3.7 Proposition. *For a space X , each point of $\sigma X \setminus X$ is regular in σX if and only if \mathcal{U} is a regular filter for each $\mathcal{U} \in F(X)$.*

Proof. Similar to the proof of 3.6.

3.8 Example. Let $X = \beta\mathbb{N} \setminus \{P\}$ where $p \in \beta\mathbb{N} \setminus \mathbb{N}$. By [15], $\mu X = \beta X (= \beta\mathbb{N})$. Moreover, p is a regular point in X . However, $\sigma X \neq \beta X$, and p is not a regular point on σX . In particular, $\sigma X \neq \mu X$ and $\kappa X \neq \mu^+ X$.

It is easy to see that if X is any regular space, then each point of X is a regular point in σX (resp. μX).

4. Commutativity of the Absolutes E and P with the Extensions μ and μ^+

Let hX be a H -closed extension of a space X . We identify EX with $k_{hX}^+(X)$ and PX with $\pi_{hX}^+(X)$. Let $h'EX$ (respectively, $h'PX$) be a H -closed extension of EX (resp. PX). We say that $h'EX = EhX$ (resp. $h'PX = PhX$) provided that there exists a homeomorphism $\phi: h'EX \rightarrow EhX$ (resp. $\phi: h'PX \rightarrow PhX$) that fixes EX (resp. PX) pointwise. Various such commutativity relations $h'EX = EhX$ have already been investigated in the literature. In [7] it is shown that $EhX = \beta EX$ for every space X and every H -closed extension hX of X . In [9] and [17] it is shown that $E\sigma X = \sigma EX$ if and only if the set of nonisolated points of EX is compact, if and only if

every closed and nowhere dense subset of EX is compact.

In [10] and [18] it is shown that $P_{\kappa}X = \kappa PX$ for every space X . Recently it was shown in [18] that $P_{\sigma}X = \sigma PX$ for every space X , $E_{\mu}X = \mu EX$ for every semiregular space X , and, for a regular space X , $P_{\mu}X = \mu PX$ if and only if every closed regularly nowhere dense subset of X is compact. In what follows, we develop various commutativity relations between the two absolutes E and P and the extensions μX and μ^+X . We begin with the next result.

4.1 Theorem. *For every Hausdorff space X , $E_{\mu}X = \mu EX$.*

Proof. Now $\mu EX = \beta EX = E_{\mu}X$ by [7] and [15].

4.2 Theorem. *For a space X , $\mu^+EX = E_{\mu^+}X$ if and only if X is a finite union of almost H -closed spaces.*

Proof. Since $|\sigma EX \setminus EX| = |\mu EX \setminus EX| = |\beta EX \setminus EX| = |\sigma X \setminus X|$, it follows by 2.9 that X is a finite union of almost H -closed spaces if and only if EX is a finite union of almost H -closed spaces. Since $E_{\mu^+}X = E_{\mu}X = \mu EX$, the theorem follows from 2.9.

4.3 Remark. Let $X = \beta\mathbb{N} \setminus \{p\}$ be the space of 3.8. Then X is extremally disconnected, and by [15], $\mu^+X = \mu X = \beta X$. Also $EX = PX = X$, $\mu^+PX = \mu^+X = P_{\mu^+}X$, $\mu PX = \mu X = P_{\mu}X$. Moreover, $\kappa X = \sigma X \neq \beta X$. Since $P_{\kappa}X = \kappa PX = \kappa X$, it follows that $P_{\kappa}X \neq P_{\mu^+}X$ and $P_{\sigma}X \neq P_{\mu}X$. (Incidentally it follows that there are θ -homeomorphic spaces $Y = \sigma X$, $Z = \mu X$, such that $EY = EZ$, but $PY \neq PZ$.) However, the commutativity of P and μ is, in general, more delicate.

4.4 *Example.* Let $m\mathbb{N}$ be the following space defined by Urysohn [22]:

$m\mathbb{N} = \{(0,1), (0,-1)\} \cup \{(1/n,0) : n \in \mathbb{N}\} \cup \{(1/n,1/m) : n \in \mathbb{N}, |m| \in \mathbb{N}\}$. Define $\tau(m\mathbb{N})$ as follows: a subset $U \in m\mathbb{N}$ is open if $U \setminus \{(0,1), (0,-1)\}$ is open in the topology that $m\mathbb{N} \setminus \{(0,1), (0,-1)\}$ inherits from the usual topology of \mathbb{R}^2 , and $(0,1) \in U$ (respectively, $(0,-1) \in U$) implies that there is some $k \in \mathbb{N}$ such that $\{(1/n,1/m) : n \geq k, m \in \mathbb{N} \text{ (resp., } -m \in \mathbb{N})\} \subseteq U$. Then

(a) $m\mathbb{N}$ is minimal Hausdorff, but not Urysohn (and hence is not regular),

(b) $m\mathbb{N}$ contains a countable dense discrete subspace, and, hence, $m\mathbb{N}$ is a strict minimal Hausdorff extension of \mathbb{N} .

Now, the space $Pm\mathbb{N}$ is a H-closed extension of $\pi_{m\mathbb{N}}^+(\mathbb{N})$ such that $\kappa\mathbb{N} \geq Pm\mathbb{N} \geq \sigma\mathbb{N}$. However, $Pm\mathbb{N} \neq \sigma\mathbb{N}$ since $m\mathbb{N}$ is not compact. Also, $(Pm\mathbb{N})^\# = \sigma\mathbb{N}$. Thus, even though $m\mathbb{N}$ is a strict H-closed extension of \mathbb{N} , $Pm\mathbb{N}$ is not a strict extension of $\pi_{m\mathbb{N}}^+(\mathbb{N})$.

The proof of the next lemma is straightforward and is omitted.

4.5 *Lemma.* Let X be a Hausdorff space.

- (a) The map $\pi_{\mu X} \big|_{P\mu X \setminus \pi_{\mu X}^+(X)} : P\mu X \setminus \pi_{\mu X}^+(X) \rightarrow \mu X \setminus X$ is a continuous bijection.
- (b) Each point of $P\mu X \setminus \pi_{\mu X}^+(X)$ is semiregular in $P\mu X$ if and only if \mathcal{U}_S is a regular filter on X for each $\mathcal{U} \in F(X)$.

4.6 *Theorem.* For a space X , $P\mu X = \mu PX$ if and only if $P\mu X$ is a strict extension of $\pi_{\mu X}^+(X)$ and \mathcal{U}_S is a regular

filter on X for each $\mathcal{U} \in \mathbf{F}(X)$.

Proof. Since $P_\mu X$ is extremally disconnected, $\pi_{\mu X}^+(X)$ is paracombinatorially embedded in $P_\mu X$. The theorem now follows directly from 3.2 and 4.5.

4.7 Proposition. *Let X be a regular space. Then, $P_\mu X = {}_\mu P X$ if and only if \mathcal{U}_S is a regular filter on X for each $\mathcal{U} \in \mathbf{F}(X)$.*

Proof. We first show that if X is a regular space and \mathcal{U}_S is a regular filter on X for each $\mathcal{U} \in \mathbf{F}(X)$, then $P_\mu X$ is a strict extension of $\pi_{\mu X}^+(X)$. Let $W = \pi_{\mu X}^+(U) \cap {}_0_{\mu X}(V)$ (where U and V are open subsets of ${}_\mu X$) be a basic open subset of $P_\mu X$, and let $\alpha \in W$. We show that there is an open subset $B \subseteq {}_\mu X$ such that $\alpha \in {}_0_{P_\mu X}[{}_0_{\mu X} B \cap \pi_{\mu X}^+(X)] \subseteq W$. If $\alpha \in W \setminus \pi_{\mu X}^+(X)$, then $\lambda = (\alpha_*)_S = \pi_{\mu X}(\alpha) \in U \setminus X$. So, there is a set $G \in \lambda$ such that $\lambda \in {}_0_{\mu X}(G) \subseteq U$. Since λ is a regular filter, there is a regular open set $H \in \lambda$ such that $\text{cl}_X H \subseteq G$. Then $\lambda \in {}_0_{\mu X}(H) \subseteq \text{cl}_{\mu X}({}_0_{\mu X}(H)) = \text{cl}_X(H) \cup {}_0_{\mu X}(H) \subseteq {}_0_{\mu X}(G)$. Hence, $\alpha \in \pi_{\mu X}^+({}_0_{\mu X}(H)) \subseteq \text{cl}_{P_\mu X}[\pi_{\mu X}^+({}_0_{\mu X}(H))] = \text{int}_{F_\mu X} \text{cl}_{P_\mu X}[\pi_{\mu X}^+({}_0_{\mu X}(H))] = \text{int}_{P_\mu X}[\pi_{\mu X}^+(\text{cl}_{\mu X}({}_0_{\mu X}(H)))] \subseteq \text{int}_{P_\mu X} \pi_{\mu X}^+({}_0_{\mu X}(G)) \subseteq \pi_{\mu X}^+(U)$. Since $P_\mu X$ is extremally disconnected, $\text{cl}_{P_\mu X} \pi_{\mu X}^+({}_0_{\mu X}(H)) = {}_0_{\mu X} A$ for some open subset $A \subseteq {}_\mu X$. Take $B = A \cap V$. Then, $\alpha \in {}_0_{\mu X} B = {}_0_{P_\mu X}[{}_0_{\mu X} B \cap \pi_{\mu X}^+(X)] \subseteq W$. The case when $\alpha \in W \setminus \pi_{\mu X}^+(X)$ is dealt in an analogous manner using the fact that X is regular. Thus, $P_\mu X$ is a strict extension of $\pi_{\mu X}^+(X)$. Now, $\pi_{\mu X}^+(X)$ is paracombinatorially embedded in $P_\mu X$. Hence, by 3.2 and 4.5 it follows that $P_\mu X = {}_\mu P X$. The converse follows from 4.6.

4.8 Corollary. *If X is a regular space, then \mathcal{U}_s is a regular filter on X for each $\mathcal{U} \in \mathcal{F}(X)$ if and only if every closed and regularly nowhere dense subset of X is compact.*

Proof. The proof follows from 4.7 and [18, Thm. 7.1].

We conclude this section with the following remarks.

4.9 Remarks. (1) Let $Y = ER$. Then $\mu Y = \mu ER = E\mu R = \beta ER = \beta Y$, and $\sigma Y \not\supseteq \mu Y$.

(2) Now, let $X = \mathbb{Q} \cup \mathbb{Q}(\sqrt{2})$ with the topology $\tau(X)$ induced by the usual topology on \mathbb{R} . Let Y be the space with the underlying set of X and the topology $\tau(Y)$ generated by the family $\{\tau(X) \cup \{Q\}\}$ (i.e. Q is open in Y). Since $(-\sqrt{2}, \sqrt{2}) \cap X$ is an open neighborhood of 0 in X , $(-\sqrt{2}, \sqrt{2}) \cap Q$ is an open neighborhood of 0 in Y . If $\mathcal{U}_s \in \sigma_{\mu Y}((-\sqrt{2}, \sqrt{2}) \cap Q) \setminus Y$, then there exists an open set $U \in \mathcal{U}$ ($\in \mathcal{F}(Y)$) such that $\text{int}_Y \text{cl}_Y(U) \subseteq (-\sqrt{2}, \sqrt{2}) \cap Q$, which is impossible. Thus $\sigma_{\mu Y}((-\sqrt{2}, \sqrt{2}) \cap Q) \setminus Y = \emptyset$. On the other hand, for each non-empty open subset $V \subset Y$, $\sigma_{\sigma Y}(V \cap Q) \setminus Y \neq \emptyset$. This shows that 0 is not an interior point of $\sigma_{\mu Y}((-\sqrt{2}, \sqrt{2}) \cap Q)$ in σY and hence $\sigma_{\mu Y}((-\sqrt{2}, \sqrt{2}) \cap Q)$ is not open in σY . The above examples show that the topologies $\tau(\sigma Z)$ and $\tau(\mu Z)$ are not (in general) comparable, and that the following diagram cannot be completed:

$$\begin{array}{ccc}
 & \kappa Z & \xrightarrow{\hspace{2cm}} \sigma Z \\
 \text{continuous} \quad & \downarrow & \searrow \text{continuous} \\
 & \mu^+ Z & \xrightarrow{\hspace{2cm}} \mu Z
 \end{array}
 \qquad \text{continuous}$$

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