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1. Preliminaries

All spaces considered in this paper are assumed to be Hausdorff. If A is a subset of a space X, then $cl_X A$ (resp. $int_X A$, $bd_X A$) will denote the *closure* (resp. *interior boundary*) of A in X. For a space X, X_s will denote the *semi-regularization* of X (see [16], page 212), $\tau(X)$ will denote the topology on X and $|X|$ denotes the cardinal number of X. Also, RO(X) (resp. R(X)) denotes the complete Boolean algebra of *regular open* (resp. *regular closed*) subsets of X, and CO(X) will denote the algebra of clopen (= closed and open) subsets of X. An *open filter* on X is a filter in the lattice $\tau(X)$, and an *open ultrafilter* on X is a maximal (with respect to set inclusion) open filter. If $\mathcal{F}$ is a filter on X then $ad_X(\mathcal{F}) = \cap\{cl_X F\colon F \in \mathcal{F}\}$ denotes the *adherence* of $\mathcal{F}$ in X. A filter $\mathcal{F}$ on X is called *free* if $ad_X(\mathcal{F}) = \emptyset$; otherwise, $\mathcal{F}$ is called *fixed*. If $\mathcal{A}$ is any nonempty family of subsets of X with the finite intersection property, then $\langle \mathcal{A} \rangle$ will denote the filter on X generated by $\mathcal{A}$. For an open filter $\mathcal{F}$ on X, we shall denote by $\mathcal{F}_s$ the open filter on X generated by the filterbase $\{int_X cl_X A\colon A \in \mathcal{F}\}$. In what follows, for a space X, $F(X) = \{\mathcal{U}\colon \mathcal{U}$ is a

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free open ultrafilter on X}, $F_s(X) = \{\mathcal{U}_s: \mathcal{U} \in F(X)\}$. Also, $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ will denote the space of positive integers, the rationals and the reals (with usual topology) respectively.

A *map* f: X → Y is a (not necessarily continuous) function from X to Y. A map f: X → Y is called *compact* if for each y ∈ Y, $f^{\leftarrow}(y)$ (= {x ∈ X: f(x) = y}) is a compact subset of X; f is called *perfect* if it is both a compact and a closed map, and f is called *irreducible* if f is onto, closed, and, for each proper closed subset A of X, f(A) ≠ Y. A map f: X → Y is called *θ-continuous at a point* x ∈ X (see [6]) if for each open neighborhood G of f(x) in Y, there is an open neighborhood U of x in X such that $f(\mathrm{cl}_X U) \subseteq \mathrm{cl}_Y G$. If f is θ-continuous at each x ∈ X then f is called *θ-continuous*. A map f: X → Y is called a *θ-homeomorphism* provided that f is one-to-one, onto and both f and $f^{\leftarrow}$ are θ-continuous, and in this case the spaces X and Y are called *θ-homeomorphic*.

1.1 With each Hausdorff space X there is associated the space EX (called the *Iliadis absolute* of X [7]) consisting of all the convergent open ultrafilters on X with the topology τ(EX) generated by the open base $\{O_X U: U \in \tau(X)\}$, where

$$O_X U = \{\mathcal{U} \in EX: U \in \mathcal{U}\}.$$

The space EX is unique (up to homeomorphism) with respect to possessing these properties: EX is extremally disconnected and zero-dimensional (see [24] for definitions), and there exists a perfect, irreducible and θ-continuous surjection k_X: EX $\to$ X (given by $k_X(\mathcal{U}) = ad_X(\mathcal{U})$, $\mathcal{U} \in$ EX). The *Hausdorff absolute* (see [10], [19]) is the space PX whose underlying set is the set of EX with the topology $\tau(PX)$ generated by the open base $\{O_X U \cap k_X^{\leftarrow}(V) : U, V \in \tau(X)\}$. The space PX is unique (up to homeomorphism) with respect to possessing these properties: PX is extremally disconnected (but not necessarily zero-dimensional) and there exists a perfect, irreducible, continuous surjection π_X: PX $\to$ X, given by $\pi_X(\mathcal{U}) = ad_X(\mathcal{U})$, $\mathcal{U} \in$ PX. For a space X, EX = $(PX)_s$, $\tau(EX) \subseteq \tau(PX)$ and RO(EX) = RO(PX) = CO(EX) = CO(PX) = $\{O_X U: U \in \tau(X)\}$. For further details about EX and PX, the reader may refer to [7], [10], [12], [19], [20], [21] and [26].

1.2 An *extension* of a space X is a Hausdorff space Y such that X is a dense subspace of Y. If Y and Z are extensions of a space X, then Y is said to be *projectively larger* than Z, written hereafter Y $\geq_X$ Z, if there is a continuous mapping ϕ: Y $\to$ Z such that $\phi|_X = i_X$, the identity map on X. Two extensions Y and Z of a space X are called *equivalent* if Y $\geq_X$ Z and Z $\geq_X$ Y. We shall identify two equivalent extensions of X. If Y is an extension of X, then Y_s is an extension of X_s. Let Y be an extension of a space X. If $\mathcal{U}$ is an open (ultra) filter on X, then

$$\mathcal{U}^* = \{U \in \tau(Y): U \cap X \in \mathcal{U}\}$$

is an open (ultra) filter on Y which converges in Y if and only if $\mathcal{U}$ converges in Y; if $\mathcal{W}$ is an open (ultra) filter on Y, then

$$\mathcal{W}_* = \{W \cap X\colon W \in \mathcal{W}\}$$

is an open (ultra) filter on X which converges in Y if and only if $\mathcal{W}$ converges in Y. If more than one extension is involved, the meanings of $\mathcal{U}^*$ and $\mathcal{W}_*$ will be clear from the context.

Each extension Y of a space X induces the extensions $Y^\#$ and Y^+ of X. The extensions $Y^\#$ and Y^+ were introduced by Banaschewski [2] in 1964 (see also [16]). Let Y be an extension of X. For a point $y \in Y$, let

(a) $\mathcal{O}_Y^Y = (\mathcal{N}_y)_* = \{U \cap X\colon U \in \mathcal{N}_y\}$

where $\mathcal{N}_y$ is the open neighborhood filter of y in Y. For an open subset U of X, let

(b) $o_Y(U) = \{y \in Y\colon U \in \mathcal{O}_Y^Y\}$.

The family $\{o_Y(U)\colon U \in \tau(X)\}$ (respectively, $\{U \cup \{y\}\colon y \in Y\setminus X,\ U \in \mathcal{O}_Y^Y\} \cup \tau(X)$) forms an open base for a coarser (resp. finer) Hausdorff topology $\tau^\#$ (resp. τ^+) on Y. The space $(Y,\tau^\#)$ (resp. (Y,τ^+), denoted by $Y^\#$ (resp. Y^+) is an extension of X. An extension Y of a space X is called a *strict* (resp. *simple*) extension of X if $Y = Y^\#$ (resp. $Y = Y^+$). It can be shown very easily that Y is a simple extension of X if and only if X is open in Y and $Y\setminus X$ is a discrete subspace of Y. It is proved in [16] that for any extension Y of X,

(c) $\mathrm{int}_Y \mathrm{cl}_Y W = \mathrm{int}_Y \mathrm{cl}_Y (W \cap X) = o_Y(\mathrm{int}_X \mathrm{cl}_X (W \cap X))$

for each $W \in \tau(Y)$.

1.3 *Definition* [9]. Let Y be an extension of a space X. Then,

(a) X is said to be *paracombinatorially embedded* in Y if, for each pair G_1, G_2 of disjoint open subsets of X, $cl_Y(G_1) \cap cl_Y(G_2) \subseteq X$.

(b) X is said to be *hypercombinatorially embedded* in Y if for each pair F_1, F_2 of closed subsets of X such that $F_1 \cap F_2$ is nowhere dense in X, $cl_Y F_1 \cap cl_Y F_2 = F_1 \cap F_2$.

It follows from the definition that if Y is an extension of X, then X is paracombinatorially embedded in Y if and only if X is paracombinatorially embedded in $Y^{\#}$ (resp. Y^+). The following result will be used subsequently.

1.4 *Proposition* [18]. *Let* T *be an extension of a space* X, S *a space and* $f: S \twoheadrightarrow T$ *a perfect, irreducible and continuous surjection. If* X *is hypercombinatorially embedded in* T, *then* $f^{\leftarrow}(X)$ *is hypercombinatorially embedded in* S.

1.5 Recall that a space X is called H-*closed* (see [1] provided that X is closed in every Hausdorff space Y in which X is embedded. X is called *minimal Hausdorff* if $\tau(X)$ does not contain any coarser Hausdorff topology on X. A subset $A \subseteq X$ is called a H-*set* in X (see [23]) if whenever $\mathcal{C}$ is any cover of A by open sets in X, then there is a finite subfamily $\{C_i: i = 1,2,\cdots,n\} \subseteq \mathcal{C}$ such that $A \subseteq \cup\{cl_X C_i: i = 1,2,\cdots,n\}$; this is equivalent to saying that for every open filter $\mathcal{F}$ on X if $A \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, then $A \cap (ad_X(\mathcal{F})) \neq \emptyset$. The *Katetov extension* (see [9]) of a space X is the simple H-closed extension κX of X whose underlying set is the set $X \cup \mathsf{F}(X)$ with the topology $\tau(\kappa X)$ generated by the open base $\tau(X) \cup \{U \cup \{\mathcal{U}\}: U \in \mathcal{U} \in \mathsf{F}(X), U \in \tau(X)\}$. The *Fomin extension* (see [6]) of a space X is the strict H-closed extension σX of X whose underlying set is the set of κX and whose topology $\tau(\sigma X)$ is generated by the open base $\{o_{\sigma X}(U): U \in \tau(X)\}$, where for each $U \in \tau(X)$, $o_{\sigma X}(U) = U \cup \{\mathcal{U}: U \in \mathcal{U} \in \mathrm{F}(X)\}$. The space κX is the projective maximum in the set of all the H-closed extensions of X, $\sigma X = (\kappa X)^{\#}$, $\kappa X = (\sigma X)^{+}$ and $(\kappa X)_s = (\sigma X)_s$; moreover, the identity map i: $\sigma X \to \kappa X$ is perfect, irreducible and θ-continuous (see [1], [6], [8], [9], [15], [16], [17] and [20] for further details).

2. The Banaschewski—Fomin—Shanin (BFS)—Extension μX

The minimal Hausdorff extension $(\sigma X)_s$ (generally denoted by μX and called the BFS-extension in the existing literature) has been extensively studied by many authors for a semiregular space X (see for example [5], [13], [14], [15], [16] and [17]). It has been an open problem for a long time whether an extension of the type μX can be obtained for a general Hausdorff space X. Ovsepjan [11] gave a definition in this direction. In what follows, we shall explicitly describe an extension of the type μX for a general Hausdorff space X and study some of its properties.

2.1 Let X be a Hausdorff space and let $\tilde{X} = X \cup \mathrm{F}_s(X)$. For each $G \in \tau(X)$, let $o_{\tilde{X}}(G) = G \cup \{\mathcal{U}_s: \mathcal{U}_s \in \tilde{X}\backslash X, G \in \mathcal{U}_s\}$. Then,

$$o_{\tilde{X}}(G) \cap o_{\tilde{X}}(H) = o_{\tilde{X}}(G \cap H) \text{ if } G,H \in \tau(X).$$

Hence, the family $\{o_{\tilde{X}}(G): G \in \tau(X)\}$ forms an open base for a topology $\tau^{\#}$ on $\tilde{X}$. A routine verification shows that that $(\tilde{X},\tau^{\#})$, briefly denoted by $\tilde{X}_1$, is a strict H-closed extension of X. The map $j: \sigma X \to \tilde{X}_1$ defined by:

$$j(x) = x \text{ if } x \in X$$

$$j(\mathcal{U}) = \mathcal{U}_s \text{ if } \mathcal{U} \in F(X) \qquad (\text{note } \mathcal{U} \neq \mathcal{U}_s \text{ in general})$$

is a bijection.

We now define a topology τ^+ on $\tilde{X}$ by declaring that X is open in $\tilde{X}$, and, for $\mathcal{U}_s \in \tilde{X}\backslash X$, a τ^+-basic neighbourhood of $\mathcal{U}_s$ is $U \cup \{\mathcal{U}_s\}$ where U is open in X and $U \in \mathcal{U}_s$. Then $(\tilde{X},\tau^+)$ is a simple H-closed extension of X. A direct application of the definition leads to the following result.

2.2 *Proposition. For each open subset* U *of a space* X,

(a) $cl_{\tilde{X}_1}(U) = cl_{\tilde{X}_1}(o_{\tilde{X}_1}(U)) = (cl_X U) \cup o_{\tilde{X}_1}(int_X cl_X U) \subseteq cl_{\sigma X} j^{\leftarrow}(U)$, *and*

(b) $o_{\tilde{X}_1}(int_X cl_X(U))\backslash X = o_{\sigma X} j^{\leftarrow}(U)\backslash X$.

2.3 *Theorem. Let* X *be a space. Then:*

(a) *the mapping* $j^{\leftarrow}: \tilde{X}_1 \to \sigma X$ *is a* θ*-homeomorphism, and*

(b) $\tilde{X}_1\backslash X \simeq \sigma X\backslash X$.

Proof. The proof of (a) follows by 2.2(a) and [16; 1.2]. To prove (b), we note that from 2.2(b) it follows that the mapping $j^{\leftarrow}|\tilde{X}_1\backslash X: \tilde{X}_1\backslash X \to \sigma X\backslash X$ is continuous. Further, if $\mathcal{U}_s \in o_{\tilde{X}_1}(U)$, then there is a regular open set $V \in RO(X)$ such that $V \in \mathcal{U}_s$ and $V \subseteq U$. So, $\mathcal{U} \in o_{\sigma X}(V)$. By 2.2(b), $o_{\sigma X}(V)\backslash X = o_{\tilde{X}_1}(int_X cl_X(V))\backslash X = o_{\tilde{X}_1}(V)\backslash X \subseteq o_{\tilde{X}_1}(U)\backslash X$. Hence the map $(j^{\leftarrow}|_{\tilde{X}_1\backslash X})^{\leftarrow}: \sigma X\backslash X \to \tilde{X}_1\backslash X$ is continuous, and (b) follows.

2.4 *Proposition. The following statements are equivalent for a space* X.

(a) X *is semiregular.*

(b) $\tilde{X}_1$ *is semiregular.*

(c) $\tilde{X}_1 = (\sigma X)_s$.

Proof. Since the proof of (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is obvious, we show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Now, (b) $\Rightarrow$ (c) follows from 2.3(a) and [16]. To prove (a) $\Rightarrow$ (b), since $\mathrm{int}_{\tilde{X}_1}\mathrm{cl}_{\tilde{X}_1}(o_{\tilde{X}_1}(G)) = \mathrm{int}_{\tilde{X}_1}\mathrm{cl}_{\tilde{X}_1}G = o_{\tilde{X}_1}(\mathrm{int}_X\mathrm{cl}_X G)$ for all $G \in \tau(X)$, it suffices to show that the family $\{o_{\tilde{X}_1}(\mathrm{int}_X\mathrm{cl}_X(H): H \text{ open in } X\}$ is an open base for $\tilde{X}_1$. Let $\tilde{U}$ be a nonempty open subset of $\tilde{X}_1$. If $\mathcal{U}_s \in \tilde{U}\backslash X$, then there is a nonempty open subset G of X such that $\mathcal{U}_s \in o_{\tilde{X}_1}(G) \subseteq \tilde{U}$. Since $G \in \mathcal{U}_s$, $G \supseteq \mathrm{int}_X\mathrm{cl}_X H$ for some $H \in \mathcal{U}$. Now $\mathrm{int}_X\mathrm{cl}_X H \in \mathcal{U}_s$. So, $\mathcal{U}_s \in o_{\tilde{X}_1}(\mathrm{int}_X\mathrm{cl}_X H) \subseteq o_{\tilde{X}}(G) \subseteq \tilde{U}$. Now, let $x \in \tilde{U} \cap X$, and let G be open in X such that $x \in o_{\tilde{X}_1}(G) \subseteq U$. Then $x \in G$. Since X is semiregular, there is an open set $H \subseteq X$ such that $x \in \mathrm{int}_X\mathrm{cl}_X H \subseteq G$. Hence, $x \in o_{\tilde{X}_1}(\mathrm{int}_X\mathrm{cl}_X H) \subseteq o_{\tilde{X}_1}(G) \subseteq \tilde{U}$, and (b) follows.

2.5 *Remark.* In view of 2.4 we shall, henceforth, denote $\tilde{X}_1$ by μX, and call it the BFS-extension of X. For each Hausdorff space X, $\mu X = \sigma X$ (or, equivalently, $\kappa X = \mu^+ X$) if and only if $\mathcal{U} = \mathcal{U}_s$ for each $\mathcal{U} \in \mathcal{F}(X)$. One can show very easily that a space X is extremally disconnected if and only if μX is extremally disconnected, if and only if $\mu^+ X$ is extremally disconnected. It would be interesting to characterize those Hausdorff spaces X for which $\mu X = \sigma X$. In the next two propositions, we provide a partial answer to this problem.

2.6 *Proposition. If every closed and nowhere dense subset of a space* X *is contained in a* H-*set, then* $\sigma X = \mu X$.

Proof. Let $\mathcal{U} \in F(X)$. If $U \in \mathcal{U}$ and U is not regular open, then by hypothesis, there exists a H-set $H \subset X$ such that $\emptyset \neq cl_X(int_X cl_X U \backslash U) \subseteq H$. Since $\mathcal{U}$ is free, then, for each $p \in H$ there exist open subsets T_p and W_p of X such that $p \in T_p$, $W_p \in \mathcal{U}$, $W_p \subseteq U$ and $T_p \cap W_p = \emptyset$. Since H is a H-set in X, the open covering $\{T_p: p \in H\}$ of H contains a finite subfamily $\{T_{p_i}: i = 1,2,\cdots,n\}$ such that $H \subseteq \cup\{cl_X(T_{p_i}): i = 1,2,\cdots,n\} = cl_X(T)$, where $T = \cup\{T_{p_i}: i = 1,2,\cdots,n\}$. Let W_{p_i} be the corresponding members of $\mathcal{U}$ with $W_{p_i} \cap T_{p_i} = \emptyset$, and $W_{p_i} \subseteq U$ for all $i = 1,2,\cdots,n$, and let $W = \cap\{W_{p_i}: i = 1,2,\cdots,n\}$. Then $W \in \mathcal{U}$, $W \subseteq U$ and $(int_X cl_X W) \cap H = \emptyset$. Now $int_X cl_X W \subseteq int_X cl_X(U) = U \cup (int_X cl_X U \backslash U) \subseteq U \cup H$ and the above fact implies that $int_X cl_X W \subseteq U$. Hence, $U \in \mathcal{U}_s$. Thus $\mathcal{U} = \mathcal{U}_s$ and the result follows by 2.5.

2.7 *Proposition. Let* X *be semiregular and extremally disconnected. Then* $\sigma X = \mu X$ *if and only if every closed and nowhere dense subset of* X *is compact.*

2.8 *Definition.* (a) [13]. A Hausdorff space X is said to be *almost* H-*closed* if, for every pair of disjoint nonempty open subsets of X, the closure of at least one of them is H-closed.

(b) [9]. A subset A of a space X is called *regularly nowhere dense* if there are disjoint open sets U and V such that $cl_X A = cl_X U \cap cl_X V$.

2.9 *Theorem. Let* X *be a space. The following statements are equivalent.*

(a) $\kappa X = \sigma X$.

(b) $|\kappa X \backslash X| < \aleph_0$.

(c) X *has a finite cover of almost* H-*closed spaces.*

(d) $\mu^+ X = \mu X$.

Proof. See [16, Thm. 4.2] and [5, Thm. 12].

If the space μX is compact then X must be semiregular. It is proved in [15] that for a space X, μX is compact if and only if $\mu X = \beta X$, if and only if X is semiregular and every closed regularly nowhere dense subset of X is compact. We prove the analogous result for $\mu^+ X$.

2.10 *Theorem. For a space* X, *the following statements are equivalent.*

(a) $\mu^+ X$ *is compact.*

(b) (i) X *has a finite cover of almost* H-*closed spaces, and*

(ii) X *is semiregular and every closed regularly nowhere dense subset of* X *is compact.*

Proof. The proof is a direct consequence of 2.9, [15, Thm. 6.2] and the fact that $\mu^+ X \backslash X$ is discrete.

3. Characterization of the Spaces μX and $\mu^+ X$

3.1 *Definition.* (a) A point p of a space X is called a *semiregular point* (respectively, a *regular point*) if whenever G is any open neighborhood of p in X, then there exists an open subset $U \subseteq X$ such that $p \in \mathrm{int}_X \mathrm{cl}_X U \subseteq G$ (respectively, $p \in \mathrm{cl}_X U \subseteq G$).

(b) A filter $\mathcal{F}$ on the Boolean algebra R(X) of a space X will be called a rc-*filter* on X. An open filter $\mathcal{F}$ on a space X is called a *regular filter* if, for each $U \in \mathcal{F}$, there is a $V \in \mathcal{F}$ such that $cl_X V \subseteq U$.

The next two propositions characterize the spaces μX and $\mu^+ X$. We omit their straightforward proofs.

3.2 *Proposition. The space* μX *is uniquely determined by the following properties:*

(a) μX *is a strict* H-*closed extension of* X,

(b) X *is paracombinatorially embedded in* μX, *and*

(c) *each point* $p \in \mu X \setminus X$ *is a semiregular point in* μX.

3.3 *Proposition. The space* $\mu^+ X$ *is uniquely determined by the following properties:*

(a) $\mu^+ X$ *is a simple* H-*closed extension of* X,

(b) X *is hypercombinatorially embedded in* $\mu^+ X$, *and*

(c) *each point* $p \in \mu^+ X \setminus X$ *is a semiregular point in* $(\mu^+ X)^\#$.

3.4 *Lemma. Let* $\mathcal{W}$ *be a free* rc-*ultrafilter on* X, *and let* $\mathcal{W}^0 = \{int_X cl_X W: W \in \mathcal{W}\}$. *Then* $\mathcal{W}^0 = \mathcal{U}_s$ *for some* $\mathcal{U} \in F(X)$.

Proof. Clearly, $\mathcal{W}^0$ is a free open filter base and is contained in some free open ultrafilter $\mathcal{U}$. Moreover, $\mathcal{W}^0 \subseteq \mathcal{U}_s$. Now, if V is a regular open set in $\mathcal{U}_s$, then $V \cap int_X W \neq \emptyset$ for all $W \in \mathcal{W}$. Thus, $(cl_X V) \cap W \neq \emptyset$ for all $W \in \mathcal{W}$. Since $X = (cl_X V) \cup (X \setminus V) \in \mathcal{W}$ and $\mathcal{W}$ is a rc-ultrafilter, either $cl_X(V) \in \mathcal{W}$ or, $X \setminus V \in \mathcal{W}$. However, $(cl_X V) \wedge (X \setminus V) = \emptyset$. So, $X \setminus V \notin \mathcal{W}$. Thus $cl_X(V) \in \mathcal{W}$, whence, $V = int_X cl_X V \in \mathcal{W}^0$. Thus, $\mathcal{U}_s = \mathcal{W}^0$.

Recall that an open cover $\mathcal{C}$ of a space X is called a p-*cover* of X if there exist finitely many members $C_1, C_2, \cdots, C_n$ in $\mathcal{C}$ such that $X = \cup_{i=1}^{n} cl_X C_i$. If X is a space and $\mathcal{F}$ is a filter on X, then, a subset $A \subseteq X$ is said to *miss* $\mathcal{F}$ if $A \cap F = \emptyset$ for some $F \in \mathcal{F}$; otherwise, we say that $\mathcal{F}$ *meets* A.

3.5 *Theorem. For a space* X, *the following statements are equivalent.*

(a) *If* A *is any closed regularly nowhere dense subset of* X, *then* A *misses every free* rc-*filter on* X.

(b) $\mathcal{U}_s$ *is a regular filter for each* $\mathcal{U} \in \mathsf{F}(X)$.

(c) *If* $\mathcal{C}$ *is any regular open cover of* X *such that* $\mathcal{C}$ *is not a* p-*cover, then for each closed regularly nowhere dense subset* A *of* X *there exist finitely many* $C_1, C_2, C_3, \cdots, C_n$ *in* $\mathcal{C}$ *such that* $A \subseteq int_X cl_X[\cup_{i=1}^{n} C_i]$.

Proof. (a) $\Rightarrow$ (c). Let $A \subseteq bd_X U$, $U \in RO(X)$ be any closed regularly nowhere dense subset of X, and let $\mathcal{C}$ be an open cover of X consisting of regular open subsets of X, which is not a p-cover. Then, $\mathcal{F} = \{cl_X int_X(X \setminus \cup_{i=1}^{n} C_i)$: $C_1, C_2, \cdots, C_n \in \mathcal{C}$, $n \in \mathbb{N}\}$ is a free rc-filter base. Hence, by (a) there is a finite family $C_1, C_2, \cdots, C_n$ in $\mathcal{C}$ such that $A \cap cl_X int_X[X \setminus \cup_{i=1}^{n} C_i] = \emptyset$. Consequently, $A \subseteq int_X cl_X[\cup_{i=1}^{n} C_i]$ and (c) follows.

(c) $\Rightarrow$ (b). Let $\mathcal{U} \in \mathsf{F}(X)$ and let $U \in RO(X) \cap \mathcal{U}_s$. The family $\{X \setminus cl_X(W)\colon W \in \mathcal{U}_s\}$ is a regular open cover of X which is not a p-cover. Hence, by (c), there are finitely many $W_1, W_2, \cdots, W_n$ in $\mathcal{U}_s$ such that $bd_X U \subseteq int_X cl_X[\cup_{i=1}^{n} (X \setminus cl_X(W_i))] = X \setminus cl_X int_X[\cap_{i=1}^{n} (cl_X(W_i))] \subseteq X \setminus cl_X[\cap_{i=1} W_i]$. Let

$V = U \cap \cap_{i=1}^{n} W_i$. Then, $V \in \mathcal{U}_s$ and $(\mathrm{cl}_X V) \cap [X \backslash \mathrm{cl}_X(\cap_{i=1}^{n} W_i)] = \emptyset$. Hence $\mathrm{cl}_X V \subseteq U$ and (b) follows.

(b) $\Rightarrow$ (a). Let A be a closed regularly nowhere dense subset of X, say $A \subseteq \mathrm{bd}_X U$ for some $U \in RO(X)$. Let $\mathcal{J}$ be any free rc-filter on X. Assume that $\mathcal{J}$ meets A. Then $\mathcal{J}$ meets $\mathrm{cl}_X U$. Hence, the family $\mathcal{J} \cup \{\mathrm{cl}_X V: V \in \tau(X), V \supseteq \mathrm{bd}_X U\}$ has the finite intersection property, and there is a free rc-ultrafilter $\mathcal{W}$ containing this family. By 3.4, $\mathcal{W}^0 = \{\mathrm{int}_X \mathrm{cl}_X W: W \in \mathcal{W}\} = \mathcal{U}_s$ for some $\mathcal{U} \in \mathrm{F}(X)$. Now since $U \in RO(X)$, either $U \in \mathcal{U}_s$ or $X \backslash \mathrm{cl}_X U \in \mathcal{U}_s$. Suppose that $U \in \mathcal{U}_s$. Since $\mathcal{U}_s$ is a regular filter by hypothesis, there is a set $V \in \mathcal{U}_s$ such that $\mathrm{cl}_X V \subseteq U$. So, $X \backslash \mathrm{cl}_X V \supseteq \mathrm{bd}_X U$, and, hence, $\mathrm{cl}_X(X \backslash \mathrm{cl}_X V) \in \mathcal{W}$. But then $X \backslash \mathrm{cl}_X V = \mathrm{int}_X \mathrm{cl}_X(X \backslash \mathrm{cl}_X V) \in \mathcal{U}_s$, which is impossible, since $V \in \mathcal{U}_s$. Now if $X \backslash \mathrm{cl}_X U \in \mathcal{U}_s$, then there is a set $V' \in \mathcal{U}_s$ such that $\mathrm{cl}_X V' \subseteq X \backslash \mathrm{cl}_X U$, and since $\mathrm{bd}_X U = \mathrm{bd}_X(X \backslash \mathrm{cl}_X U)$, by the same reasoning as above, $X \backslash \mathrm{cl}_X V' \in \mathcal{U}_s$, which is impossible. Thus $\mathcal{J}$ misses A, and the theorem follows.

3.6 *Proposition. For a space* X, *each point* $p \in \mu X \backslash X$ *is regular in* μX *if and only if* $\mathcal{U}_s$ *is a regular filter on* X *for each* $\mathcal{U} \in \mathrm{F}(X)$.

Proof. Suppose that $\mathcal{U}_s$ is a regular filter for each $\mathcal{U} \in \mathrm{F}(X)$. Let $o_{\mu X}(G)$ be a basic open neighborhood of $\mathcal{U}_s$ in μX, where $G \in \mathcal{U}_s$. There is a regular open set $H \in \mathcal{U}_s$ such that $\mathrm{cl}_X H \subseteq G$. Then, $\mathcal{U}_s \in o_{\mu X}(H) \subseteq \mathrm{cl}_{\mu X}(o_{\mu X}(H)) = \mathrm{cl}_X(H) \cup o_{\mu X}(\mathrm{int}_X \mathrm{cl}_X H) \subseteq o_{\mu X}(G)$, whence, $\mathcal{U}_s$ is a regular point in μX. Conversely suppose that each point $\mathcal{U}_s \in \mu X \backslash X$ is a regular point in μX. Let $\mathcal{U}_s \in \mu X \backslash X$, and let $G \in \mathcal{U}_s$. Then, there exists a basic open neighborhood $o_{\mu X}(H)$ of $\mathcal{U}_s$ such that $\mathcal{U}_s \in o_{\mu X}(H) \subseteq cl_{\mu X}H \subseteq o_{\mu X}(G)$. Hence, $H \in \mathcal{U}_s$ and $cl_X H \subseteq X \cap o_{\mu X}(G) = G$, whence $\mathcal{U}_s$ is a regular filter, and the proof of the proposition is complete.

3.7 *Proposition. For a space* X, *each point of* $\sigma X \backslash X$ *is regular in* σX *if and only if* $\mathcal{U}$ *is a regular filter for each* $\mathcal{U} \in F(X)$.

Proof. Similar to the proof of 3.6.

3.8 *Example.* Let $X = \beta\mathbb{N}\backslash\{P\}$ where $p \in \beta\mathbb{N}\backslash\mathbb{N}$. By [15], $\mu X = \beta X$ $(=\beta\mathbb{N})$. Moreover, p is a regular point in X. However, $\sigma X \neq \beta X$, and p is not a regular point on σX. In particular, $\sigma X \neq \mu X$ and $\kappa X \neq \mu^+ X$.

It is easy to see that if X is any regular space, then each point of X is a regular point in σX (resp. μX).

4. Commutativity of the Absolutes E and P with the Extensions μ and $\mu+$

Let hX be a H-closed extension of a space X. We identify EX with $\overleftarrow{k}_{hX}(X)$ and PX with $\overleftarrow{\pi}_{hX}(X)$. Let h'EX (respectively, h'PX) be a H-closed extension of EX (resp. PX). We say that h'EX = EhX (resp. h'PX = PhX) provided that there exists a homeomorphism ϕ: h'EX $\rightarrow$ EhX (resp. ϕ: h'PX $\rightarrow$ PhX) that fixes EX (resp. PX) pointwise. Various such commutativity relations h'EX = EhX have already been investigated in the literature. In [7] it is shown that EhX = βEX for every space X and every H-closed extension hX of X. In [9] and [17] it is shown that $E\sigma X = \sigma EX$ if and only if the set of nonisolated points of EX is compact, if and only if every closed and nowhere dense subset of EX is compact. In [10] and [18] it is shown that $P\kappa X = \kappa PX$ for every space X. Recently it was shown in [18] that $P\sigma X = \sigma PX$ for every space X, $E\mu X = \mu EX$ for every semiregular space X, and, for a regular space X, $P\mu X = \mu PX$ if and only if every closed regularly nowhere dense subset of X is compact. In what follows, we develop various commutativity relations between the two absolutes E and P and the extensions μX and μ^+X. We begin with the next result.

4.1 *Theorem. For every Hausdorff space* X, $E\mu X = \mu EX$.

Proof. Now $\mu EX = \beta EX = E\mu X$ by [7] and [15].

4.2 *Theorem. For a space* X, $\mu^+EX = E\mu^+X$ *if and only if* X *is a finite union of almost* H-*closed spaces.*

Proof. Since $|\sigma EX \backslash EX| = |\mu EX \backslash EX| = |\beta EX \backslash EX| = |\sigma X \backslash X|$, it follows by 2.9 that X is a finite union of almost H-closed spaces if and only if EX is a finite union of almost H-closed spaces. Since $E\mu^+X = E\mu X = \mu EX$, the theorem follows from 2.9.

4.3 *Remark.* Let $X = \beta\mathbb{N}\backslash\{p\}$ be the space of 3.8. Then X is extremally disconnected, and by [15], $\mu^+X = \mu X = \beta X$. Also $EX = PX = X$, $\mu^+PX = \mu^+X = P\mu^+X$, $\mu PX = \mu X = P\mu X$. Moreover, $\kappa X = \sigma X \neq \beta X$. Since $P\kappa X = \kappa PX = \kappa X$, it follows that $P\kappa X \neq P\mu^+X$ and $P\sigma X \neq P\mu X$. (Incidently it follows that there are θ-homeomorphic spaces $Y = \sigma X$, $Z = \mu X$, such that $EY = EZ$, but $PY \neq PZ$.) However, the commutativity of P and μ is, in general, more delicate.

4.4 *Example.* Let $m\mathbb{N}$ be the following space defined by Urysohn [22]:

$m\mathbb{N} = \{(0,1),(0,-1)\} \cup \{(1/n,0): n \in \mathbb{N}\} \cup \{(1/n,1/m): n \in \mathbb{N}, |m| \in \mathbb{N}\}$. Define $\tau(m\mathbb{N})$ as follows: a subset $U \in m\mathbb{N}$ is open if $U\setminus\{(0,1),(0,-1)\}$ is open in the topology that $m\mathbb{N}\setminus\{(0,1),(0,-1)\}$ inherits from the usual topology of $\mathcal{R}^2$, and $(0,1) \in U$ (respectively, $(0,-1) \in U$) implies that there is some $k \in \mathbb{N}$ such that $\{(1/n,1/m): n \geq k, m \in \mathbb{N}$ (resp., $-m \in \mathbb{N}$)$\} \subseteq U$. Then

(a) $m\mathbb{N}$ is minimal Hausdorff, but not Urysohn (and hence is not regular),

(b) $m\mathbb{N}$ contains a countable dense discrete subspace, and, hence, $m\mathbb{N}$ is a strict minimal Hausdorff extension of $\mathbb{N}$.

Now, the space $Pm\mathbb{N}$ is a H-closed extension of $\overleftarrow{\pi}_{m\mathbb{N}}(\mathbb{N})$ such that $\kappa\mathbb{N} \geq Pm\mathbb{N} \geq \sigma\mathbb{N}$. However, $Pm\mathbb{N} \neq \sigma\mathbb{N}$ since $m\mathbb{N}$ is not compact. Also, $(Pm\mathbb{N})^{\#} = \sigma\mathbb{N}$. Thus, even though $m\mathbb{N}$ is a strict H-closed extension of $\mathbb{N}$, $Pm\mathbb{N}$ is not a strict extension of $\overleftarrow{\pi}_{m\mathbb{N}}(\mathbb{N})$.

The proof of the next lemma is straightforward and is omitted.

4.5 *Lemma. Let* X *be a Hausdorff space.*

(a) *The map* $\pi_{\mu X}|_{P\mu X\setminus\overleftarrow{\pi}_{\mu X}(X)} : P\mu X\setminus\overleftarrow{\pi}_{\mu X}(X) \to \mu X\setminus X$ *is a continuous bijection.*

(b) *Each point of* $P\mu X\setminus\overleftarrow{\pi}_{\mu X}(X)$ *is semiregular in* $P\mu X$ *if and only if* $\mathcal{U}_s$ *is a regular filter on* X *for each* $\mathcal{U} \in \mathcal{F}(X)$.

4.6 *Theorem. For a space* X, $P\mu X = \mu PX$ *if and only if* $P\mu X$ *is a strict extension of* $\overleftarrow{\pi}_{\mu X}(X)$ *and* $\mathcal{U}_s$ *is a regular* *filter on* X *for each* $\mathcal{U} \in \mathcal{F}(X)$.

Proof. Since $P\mu X$ is extremally disconnected, $\overleftarrow{\pi}_{\mu X}(X)$ is paracombinatorially embedded in $P\mu X$. The theorem now follows directly from 3.2 and 4.5.

4.7 *Proposition. Let* X *be a regular space. Then,* $P\mu X = \mu PX$ *if and only if* $\mathcal{U}_s$ *is a regular filter on* X *for each* $\mathcal{U} \in \mathcal{F}(X)$.

Proof. We first show that if X is a regular space and $\mathcal{U}_s$ is a regular filter on X for each $\mathcal{U} \in \mathcal{F}(X)$, then $P\mu X$ is a strict extension of $\overleftarrow{\pi}_{\mu X}(X)$. Let $W = \overleftarrow{\pi}_{\mu X}(U) \cap 0_{\mu X}(V)$ (where U and V are open subsets of μX) be a basic open subset of $P\mu X$, and let $\alpha \in W$. We show that there is an open subset $B \subseteq \mu X$ such that $\alpha \in o_{P\mu X}[0_{\mu X}B \cap \overleftarrow{\pi}_{\mu X}(X)] \subseteq W$. If $\alpha \in W\setminus \overleftarrow{\pi}_{\mu X}(X)$, then $\lambda = (\alpha_*)_s = \pi_{\mu X}(\alpha) \in U\setminus X$. So, there is a set $G \in \lambda$ such that $\lambda \in o_{\mu X}(G) \subseteq U$. Since λ is a regular filter, there is a regular open set $H \in \lambda$ such that $cl_X H \subseteq G$. Then $\lambda \in o_{\mu X}(H) \subseteq cl_{\mu X}(o_{\mu X}(H)) = cl_X(H) \cup o_{\mu X}(H) \subseteq o_{\mu X}(G)$. Hence, $\alpha \in \overleftarrow{\pi}_{\mu X}(o_{\mu X}(H)) \subseteq cl_{P\mu X}[\overleftarrow{\pi}_{\mu X}(o_{\mu X}(H))] = int_{P\mu X}cl_{P\mu X}[\overleftarrow{\pi}_{\mu X}(o_{\mu X}(H))] = int_{P\mu X}[\overleftarrow{\pi}_{\mu X}(cl_{\mu X}(o_{\mu X}(H)))] \subseteq int_{P\mu X}\overleftarrow{\pi}_{\mu X}(o_{\mu X}(G)) \subseteq \overleftarrow{\pi}_{\mu X}(U)$. Since $P\mu X$ is extremally disconnected, $cl_{P\mu X}\overleftarrow{\pi}_{\mu X}(o_{\mu X}(H))] = 0_{\mu X}A$ for some open subset $A \subseteq \mu X$. Take $B = A \cap V$. Then, $\alpha \in 0_{\mu X}B = o_{P\mu X}[0_{\mu X}B \cap \overleftarrow{\pi}_{\mu X}(X)] \subseteq W$. The case when $\alpha \in W\setminus\overleftarrow{\pi}_{\mu X}(X)$ is dealt in an analogous manner using the fact that X is regular. Thus, $P\mu X$ is a strict extension of $\overleftarrow{\pi}_{\mu X}(X)$. Now, $\overleftarrow{\pi}_{\mu X}(X)$ is paracombinatorially embedded in $P\mu X$. Hence, by 3.2 and 4.5 it follows that $P\mu X = \mu PX$. The converse follows from 4.6.

4.8 *Corollary.* *If* X *is a regular space, then* $\mathcal{U}_s$ *is a regular filter on* X *for each* $\mathcal{U} \in \mathcal{F}(X)$ *if and only if every closed and regularly nowhere dense subset of* X *is compact.*

Proof. The proof follows from 4.7 and [18, Thm. 7.1].

We conclude this section with the following remarks.

4.9 *Remarks.* (1) Let $Y = E\mathbb{R}$. Then $\mu Y = \mu E\mathbb{R} = E\mu\mathbb{R} = \beta E\mathbb{R} = \beta Y$, and $\sigma Y \underset{\neq}{\supset} \mu Y$.

(2) Now, let $X = \mathbb{Q} \cup \mathbb{Q}(\sqrt{2})$ with the topology $\tau(X)$ induced by the usual topology on $\mathbb{R}$. Let Y be the space with the underlying set of X and the topology $\tau(Y)$ generated by the family $\{\tau(X) \cup \{\mathbb{Q}\}\}$ (i.e. $\mathbb{Q}$ is open in Y). Since $(-\sqrt{2},\sqrt{2}) \cap X$ is an open neighborhood of 0 in X, $(-\sqrt{2},\sqrt{2}) \cap \mathbb{Q}$ is an open neighborhood of 0 in Y. If $\mathcal{U}_s \in o_{\mu Y}((-\sqrt{2},\sqrt{2}) \cap \mathbb{Q})\backslash Y$, then there exists an open set $U \in \mathcal{U}$ $(\in \mathcal{F}(Y))$ such that $\mathrm{int}_Y \mathrm{cl}_Y(U) \subseteq (-\sqrt{2},\sqrt{2}) \cap \mathbb{Q}$, which is impossible. Thus $o_{\mu Y}((-\sqrt{2},\sqrt{2}) \cap \mathbb{Q})\backslash Y = \emptyset$. On the other hand, for each non-empty open subset $V \subset Y$, $o_{\sigma Y}(V \cap \mathbb{Q})\backslash Y \neq \emptyset$. This shows that 0 is not an interior point of $o_{\mu Y}((-\sqrt{2},\sqrt{2}) \cap \mathbb{Q})$ in σY and hence $o_{\mu Y}((-\sqrt{2},\sqrt{2}) \cap \mathbb{Q})$ is not open in σY. The above examples show that the topologies $\tau(\sigma Z)$ and $\tau(\mu Z)$ are not (in general) comparable, and that the following diagram cannot be completed:

$$\begin{tikzcd}
\kappa Z \arrow[r, "\text{continuous}"'] \arrow[d, dash, "\text{continuous}"'] \arrow[dr, "\text{continuous}"] & \sigma Z \arrow[d, phantom, "\text{continuous}"] \\
\mu^{+} Z \arrow[r] & \mu Z
\end{tikzcd}$$

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