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by

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1. Introduction

We introduce and study the concepts of subregular refinability[1] and sub-κ-regular refinability. These generalize regular refinability [A] and κ-regular refinability [W]. The first of these is involved in the classical Alexandroff-Urysohn metrization theorem [AU] of which the following theorem is a slight variation.

1.1 *Theorem. A space is metrizable if and only if it is a Moore space, i.e., a* T_3 *developable* [B] *space, which is regularly refinable.*

Worrell [W] proved the following theorem.

1.2 *Theorem. A space is regular and paracompact if and only if it is essentially* T_1[2] *and* ω*-regularly refinable.*

In this paper we replace the open refinements entering into the definitions of regular and κ-regular refinability by sequences of open refinements in a natural way illustrated by the relation between full normality and one of the equivalent forms of subparacompactness ([B], Theorem 3.1(v)) or

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[1]Definitions are given in Section 2.

[2]A space X is essentially T_1 [WW] if and only if for all $x,y \in X$, if $x \in \{\overline{y}\}$ then $y \in \{\overline{x}\}$. Regular spaces have the property and normal spaces with the property are regular.

that between metacompactness and θ-refinability [WW] (= submetacompactness). These concepts permit us to obtain Theorem 1.3, a natural analogue of Theorem 1.2, and Theorem 1.4 which is related to Theorem 1.1.

1.3 *Theorem. A regular space is subparacompact if and only if it is sub-ω-regularly refinable.*

1.4 *Theorem. A regular space is a Moore space if and only if it is sub-ω-regularly refinable and has a base of countable order.*[3]

The next two theorems are variations of Theorems 1.4 and 1.2, respectively.

1.5 *Theorem. A regular space is a Moore space if and only if it is subregularly refinable, submetaLindelöf, and has a base of countable order.*

1.6 *Theorem. A space is regular paracompact if and only if it is essentially* T_1, *collectionwise normal and sub-ω-regularly refinable.*

As a corollary to this we have a metrization theorem.

1.7 *Theorem. A space is metrizable if and only if it is* T_1 *collectionwise normal, subregularly refinable, submetaLindelöf and has a base of countable order.*

These theorems provide some evidence that the "sub" concepts introduced here have a natural place in the theory

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[3]For a definition see [WW].

of previously defined concepts such as subparacompactness, submetacompactness (θ-refinability) and submetaLindelöfness ($\delta\theta$-refinability).

The characterization of subparacompactness given in Theorem 1.3 is deduced from Theorem 3.1 below which states that in a sub-κ-regularly refinable space every open cover has a κ-discrete refinement (where $\kappa \geq \omega$).

Section 2 of the paper contains definitions and a lemma; section 3 presents the proof of Theorem 3.1. Section 4 contains a general theorem on subparacompactness which includes the Theorem 1.3. We also give the proofs of the other new results of the introduction in section 4. In section 5 we list some relevant examples. Definitions of other covering properties not in section 2 can be found in [Bu]. We use ZFC set theory in the notation of [Mo]. Topological terminology used is standard; we are not assuming that the concepts of regularity or collectionwise normality imply T_1, however.

2. Subregular Refinability and Related Concepts

In this section we give the definitions of the two basic concepts involved in the results and prove a lemma. If $\mathcal{U}$ and $\mathcal{V}$ are covers of a set X, the collection $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subseteq U$. If $\mathcal{V}$ is not required to cover X, then we say that $\mathcal{V}$ *partially refines* $\mathcal{U}$. We use κ to denote a cardinal number such that $\kappa \geq 1$. The reference [Bu] can be used for definitions not found here.

2.1 *Definitions* [W]. Let $\mathcal{H}$ and $\mathcal{K}$ be collections of sets. The collection $\mathcal{K}$ is said to be $(\kappa,1)$-*regularly* ($(\kappa,2)$-*regularly*) *inscribed in* $\mathcal{H}$ *at* p if and only if $p \in \cup\mathcal{K}$ and there exists $\mathcal{W} \subseteq \mathcal{H}$ such that $|\mathcal{W}| \leq \kappa$ and if $U,V \in \mathcal{K}$ and $p \in U \cap V$ ($U,V \in \mathcal{K}$, $U \cap V \neq \emptyset$, and $p \in U$), then there is $W \in \mathcal{W}$ such that $U \cup V \subseteq W$.

The collection $\mathcal{K}$ is said to be *regularly inscribed in* $\mathcal{H}$ *at* p if and only if $\mathcal{K}$ is $(|\mathcal{H}|,1)$-regularly inscribed in $\mathcal{H}$ at p. The collection $\mathcal{K}$ is said to be *regularly inscribed in* $\mathcal{H}$ [P] if and only if $\mathcal{K}$ is regularly inscribed in $\mathcal{H}$ at all $p \in \cup\mathcal{K}$. If, in addition, $\cup\mathcal{H} = \cup\mathcal{K}$, then $\mathcal{K}$ is called a *regular refinement* [A] *of* $\mathcal{H}$.

2.2 *Definitions*. A space X is called *regularly refinable* [A] if and only if every open cover of X has an open regular refinement. A space X is called κ-*regularly refinable* [W] if and only if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ that is $(\kappa,1)$-regularly inscribed in $\mathcal{U}$ at all $p \in X$.

The concept of subregular refinability (sub-κ-regular refinability) is related to regular refinability (κ-regular refinability) in a fashion analogous to the way in which subparacompactness is related to paracompactness or θ-refinability (= submetacompactness) is related to metacompactness.

2.3 *Definitions*. A space X is called *subregularly refinable (sub-κ-regularly refinable)* if and only if for every open cover $\mathcal{U}$ of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of open refinements of $\mathcal{U}$ such that for all $p \in X$ there is $n \in \omega$ such that $\mathcal{V}_n$ is regularly $((\kappa,1)$-regularly) inscribed in $\mathcal{U}$ at p. We call such a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ a *subregularly (sub-κ-regularly) refining sequence of* $\mathcal{U}$.

2.4 *Lemma. Let* X *be a sub-κ-regularly refinable space. Let* $\mathcal{U}$ *be an open cover of* X. *Then there exists a sequence* $\langle \mathcal{U}_n : n \in \omega \rangle$ *of open covers of* X *such that for all* $n \in \omega$:

(1) $\mathcal{U}_0 = \mathcal{U}$

(2) $\mathcal{U}_{n+1}$ *refines* $\mathcal{U}_n$

(3) *For each* $p \in X$, *there is* $j \in \omega$ *such that* $j \geq n$ *and* $\mathcal{U}_j$ *is* $(\kappa,1)$*-regularly inscribed in* $\mathcal{U}_n$ *at* p.

Proof. For collections $\mathcal{H}$ and $\mathcal{K}$ let $\mathcal{H} \wedge \mathcal{K} = \{H \cap K: H \in \mathcal{H}$ and $K \in \mathcal{K}\}$. Define $\mathcal{W}(0,0) = \mathcal{U}$. Assume $\mathcal{W}(k,m)$ is defined for $m \leq k$ and $0 \leq k < n + 1$ and that $\langle \mathcal{V}(j,k) : j \geq k \rangle$ is a sub-κ-regularly refining sequence of $\mathcal{W}(k,k)$ for $0 \leq k < n + 1$. Then we define

$\mathcal{W}(n+1,0) = \mathcal{W}(n,n) \wedge \mathcal{V}(n,0)$ and, if $\mathcal{W}(n+1,j)$ is defined for $0 \leq j < m \leq n$ let

$\mathcal{W}(n+1,m) = \mathcal{W}(n+1,m-1) \wedge \mathcal{V}(n,m)$ and, finally, let

$\mathcal{W}(n+1,n+1) = \mathcal{W}(n+1,n)$.

Let $\langle \mathcal{V}(j,n+1) : j \geq n + 1 \rangle$ be a sub-κ-regularly refining sequence of $\mathcal{W}(n+1,n+1)$. Thus $\mathcal{W}(n,m)$ is defined for all $\langle n,m \rangle \in \omega \times \omega$ such that $m \leq n$. Let $\mathcal{U}_n = \mathcal{W}(n,n)$ for all $n \in \omega$. Then $\mathcal{U}_0 = \mathcal{U}$ and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$. Suppose $p \in X$ and $n \in \omega$. Then $p \in \cup\mathcal{U}_n$ and there is $j \geq n$ such that $\mathcal{V}(j,n)$ is $(\kappa,1)$-regularly inscribed in $\mathcal{U}_n$ at p. Thus $\mathcal{W}(j+1,n)$ is also $(\kappa,1)$-regularly inscribed in $\mathcal{U}_n$ at p and so is $\mathcal{U}_{j+1} = \mathcal{W}(j+1,j+1)$.

3. Sub-κ-Regular Refinability and κ-Discrete Refinements

In this section we prove a principal theorem on the existence of κ-discrete refinements which, together with regularity, implies the characterization of subparacompactness of Theorem 1.3. We also prove Theorem 3.2 which gives a simple necessary and sufficient condition that a subregularly refinable space be sub-κ-regularly refinable.

3.1 *Theorem. Suppose* $\kappa \geq \omega$ *and* X *is a sub-*κ*-regularly refinable space. Then every open cover of* X *has a* κ*-discrete refinement.*

Proof. Suppose X is such a space and $\mathcal{U}$ is an open cover of X. There is a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X, with $\mathcal{U}_0 = \mathcal{U}$, which satisfies the conclusion of Lemma 2.4. We use this sequence to construct a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$, with a special relationship to $\langle \mathcal{U}_n : n \in \omega \rangle$, to which Theorem 1 of [W] may be applied. The construction proceeds by making several auxiliary definitions and proving four statements concerning these concepts. We use the notation ${}^n\omega$ for the set of all functions from the integer $n \in \omega$ into ω. We also use $R(\mathcal{V},\mathcal{U},\kappa,p)$ to stand for the statement that $\mathcal{V}$ is $(\kappa,1)$-regularly inscribed in $\mathcal{U}$ at p. We employ the following notation: If $\mathcal{W}$ is a collection of sets and E is a set, then $\mathcal{W}|E = \{W \cap E : W \in \mathcal{W}\}$ and $(\mathcal{W})_p = \{W \in \mathcal{W} : p \in W\}$. Also $\phi|n$ denotes the restriction of ϕ to the set n.

We define the concept of n-*admissible* for $n \in \omega\setminus 1$ by induction.

ϕ is 1-*admissible* if and only if $\phi \in {}^1\omega$ and there is $p \in X$ such that $\phi(0) = \min\{j \in \omega : R(\mathcal{U}_j, \mathcal{U}_0, \kappa, p)\}$. Assume m-admissibility is defined for all $m \in n$ and suppose $n > 1$: Then ϕ is n-*admissible* if and only if $\phi \in {}^n\omega$ and for all $m \in n\setminus 1$, $\phi|m$ is m-admissible and there is $p \in X$ such that

$$\phi(n - 1) = \min\{j \in \omega: j \geq \phi(n - 2) \text{ and } R(\mathcal{U}_j, \mathcal{U}_{\phi(n-2)}, \kappa, p)\}.$$

For each 1-admissible ϕ define

$$E_\phi = \{p \in X: \phi(0) = \min\{j \in \omega: R(\mathcal{U}_j, \mathcal{U}_0, \kappa, p)\}\}.$$

If E_ϕ is defined for all m-admissible ϕ such that $m < n$, then for $n > 1$ and n-admissible ϕ, define

$$E_\phi = \{p \in E_{\phi|(n-1)}: \phi(n - 1) = \min\{j \in \omega: j \geq \phi(n - 2) \text{ and } R(\mathcal{U}_j, \mathcal{U}_{\phi(n-2)}, \kappa, p)\}\}.$$

Then the following four statements hold:

(1) For all $n \in \omega\setminus 1$, $\{E_\phi: \phi \text{ is n-admissible}\}$ is a partition of X.

For if $n = 1$ and $p \in X$, then by (3) of Lemma 2.4 there is a 1-admissible ϕ such that $R(\mathcal{U}_{\phi(0)}, \mathcal{U}_0, \kappa, p)$ and $p \in E_\phi$. If $\phi' \neq \phi$ and ϕ' is 1-admissible, then $\phi'(0) \neq \phi(0)$, thus $p \notin E_{\phi'}$. Suppose $n \in \omega\setminus 2$ and that (1) is true for all $m \in n\setminus 1$. Let $p \in X$. Then $p \in E_\psi$ for some unique (n-1)-admissible ψ. Define $\phi \in {}^n\omega$ by letting $\phi|(n - 1) = \psi$ and defining $\phi(n - 1)$ as follows: By (3) of Lemma 2.4,

$$k = \min\{j \in \omega: j \geq \psi(n - 2) \text{ and } R(\mathcal{U}_j, \mathcal{U}_{\psi(n-2)}, \kappa, p)\} \text{ exists.}$$

Let $\phi(n - 1) = k$. Then ϕ is n-admissible and $p \in E_\phi$. If $\phi' \in {}^n\omega$ and $p \in E_{\phi'}$, then $\phi'|(n - 1) = \phi|(n - 1)$ so that $\phi'(n - 1) = \phi(n - 1)$ by the definition of E_ϕ and $E_{\phi'}$. Hence $\phi = \phi'$.

(2) If $n > 0$, ϕ is $(n+1)$-admissible, and $p \in E_\phi$ then $\mathcal{U}_{\phi(n)}|E_\phi$ is $(\kappa,1)$-regularly inscribed (and, hence, regularly inscribed) in $\mathcal{U}_{\phi(n-1)}|E_{\phi|n}$ at p.

By the definition of E_ϕ, $p \in E_{\phi|n}$ and $\mathcal{U}_{\phi(n)}$ is $(\kappa,1)$-regularly inscribed in $\mathcal{U}_{\phi(n-1)}$ at p. Hence there is $\mathcal{W} \subseteq \mathcal{U}_{\phi(n-1)}$ with $|\mathcal{W}| \leq \kappa$ such that $U,V \in \mathcal{U}_{\phi(n)}$ and $p \in U \cap V$ implies that there is $W \in \mathcal{W}$ such that $U \cup V \subseteq W$. Since $E_\phi \subseteq E_{\phi|n}$ it follows that $U \cap V \cap E_\phi \subseteq W \cap E_{\phi|n}$. Since $|\{W \cap E_{\phi|n}: W \in \mathcal{W}\}| \leq \kappa$ the result follows.

(3) For any 2-admissible ϕ and $p \in E_\phi$, $\mathcal{U}_{\phi(1)}|E_\phi$ is $(\kappa,2)$-regularly inscribed in $\mathcal{U}_0$ at p.

Suppose $U,V \in \mathcal{U}_{\phi(1)}$, $q \in U \cap V \cap E_\phi$ and $p \in U$. By (2) there is $W \in \mathcal{U}_{\phi(0)}|E_{\phi|1}$ such that $U \cup V \cap E_\phi \subseteq W$. Since $\phi|1$ is 1-admissible, there is $\mathcal{M} \subseteq \mathcal{U}_0$ with $|\mathcal{M}| \leq \kappa$ such that $(\mathcal{U}_{\phi(0)}|E_{\phi|1})_p$[4] partially refines $\mathcal{M}$. Hence $U \cup V \cap E_\phi$ is a subset of some member of $\mathcal{M}$.

Now define $\mathcal{V}_0 = \mathcal{U}_0$ and, for $n \in \omega$,

$$\mathcal{V}_{n+1} = \cup\{\mathcal{U}_{\phi(n)}|E_\phi: \phi \text{ is } (n+1)\text{-admissible}\}.$$

(4) For all $n \in \omega$, $\mathcal{V}_n$ covers X, $\mathcal{V}_{n+1}$ regularly refines $\mathcal{V}_n$, and $\mathcal{V}_1$ is $(\kappa,2)$-regularly inscribed in $\mathcal{V}_0$ at all $p \in X$.

$\mathcal{V}_0$ is a cover of X. If $n \in \omega$ and $p \in X$, there is an $(n+1)$-admissible ϕ such that $p \in E_\phi$. Since $\mathcal{U}_{\phi(n)}$ covers X, there is an element of $\mathcal{U}_{\phi(n)}|E_\phi$ that contains p. By (2), $\mathcal{U}_{\phi(n)}|E_\phi$ is $(\kappa,1)$-regularly inscribed in $\mathcal{U}_{\phi(n-1)}|E_{\phi|n}$ at p. Since the only elements of $\mathcal{V}_{n+1}$ that contain p belong to $\mathcal{U}_{\phi(n)}|E_\phi$ and $\mathcal{U}_{\phi(n-1)}|E_{\phi|n} \subseteq \mathcal{V}_n$, it follows that $\mathcal{V}_{n+1}$ is $(\kappa,1)$-regularly inscribed in $\mathcal{V}_n$ at p. The last statement

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[4] $(\mathcal{W})_p = \{W \in \mathcal{W}: p \in W\}$.

of (4) follows from (3).

Theorem 1 of [W] implies that for such a sequence $\langle \mathcal{V}_n: n \in \omega \rangle$ there exist $\mu \subseteq \omega \setminus 1$ and a function $f: \mu \to \mathcal{P}(\mathcal{P}(X))$ such that: (a) $\cup f(\mu)$ covers X, (b) if $n \in \mu$ and $\mathcal{C}$ is a coherent subcollection of $\mathcal{V}_n$ (i.e., $\mathcal{C}$ is not the union of two nonempty collections $\mathcal{A}$ and $\mathcal{B}$ such that $\cup\mathcal{A} \cap \cup\mathcal{B} = \emptyset$) such that $|\mathcal{C}| \leq 3$, then $\cup\mathcal{C}$ does not intersect two elements of $f(n)$, and (c) if $n \in \mu$ and $A \in f(n)$, then $\{V \in \mathcal{V}_n: V \cap A \neq \emptyset\}$ is regularly inscribed in some subcollection of $\mathcal{V}_0$ of cardinal number $\leq\kappa$.

Let $n + 1 \in \mu$ and let ϕ be $(n+1)$-admissible. Let $k = \phi(n)$. Then no element of $\mathcal{U}_k$ meets two elements of $\{A \cap E_\phi: A \in f(n + 1)\}$. For if $U \in \mathcal{U}_k$ then $U \cap E_\phi \in \mathcal{U}_k | E_\phi \subseteq \mathcal{V}_{n+1}$. The set $\{U \cap E_\phi\}$ is a coherent subcollection of $\mathcal{V}_{n+1}$ and thus does not intersect two elements of $f(n + 1)$. Hence for all $n \in \omega$, if $n + 1 \in \mu$ and ϕ is $(n+1)$-admissible, $\{A \cap E_\phi: A \in f(n + 1)\}$ is a discrete collection since $\mathcal{U}_{\phi(n)}$ is an open cover of X. If $n + 1 \in \mu$, the collection of all elements of $\mathcal{V}_{n+1}$ intersecting $A \in f(n + 1)$ is regularly inscribed in a subcollection $\mathcal{V}_0'(A)$ of $\mathcal{V}_0$ of cardinality $\leq\kappa$, by (c) above. Let $\langle V_{\alpha,A}: \alpha < \kappa \rangle$ be an enumeration of $\mathcal{V}_0'(A)$ (with repetitions possible). Then each set $\mathcal{D}(\alpha,n,\phi) = \{A \cap E_\phi \cap V_{\alpha,A}: A \in f(n + 1)\}$, where ϕ is $(n+1)$-admissible, is discrete and partially refines $\mathcal{V}_0 = \mathcal{U}_0$. Since there are $\leq\kappa$ such collections for each $n \in \omega$ and $(n+1)$-admissible ϕ, there are at most κ such sets. If $p \in X$, then by (a) there exist $n \in \omega$ with $n + 1 \in \mu$, $A \in f(n + 1)$, and an $(n+1)$-admissible ϕ such that $p \in A \cap E_\phi$. There is also some

$W \in \mathcal{V}_{n+1}$ with $p \in W$, so there exists $\alpha < \kappa$ such that $W \subseteq V_{\alpha,A}$. Hence $p \in \cup\mathcal{D}(\alpha,n,\phi)$. Thus the collection of all $\mathcal{D}(\alpha,n,\phi)$ is a κ-discrete refinement of $\mathcal{U}$.

3.2 *Theorem. Let* X *be a subregularly refinable space. Then* X *is sub-*κ*-regularly refinable if and only if for every open cover* $\mathcal{U}$ *of* X *there is a sequence* $\langle \mathcal{V}_n : n \in \omega \rangle$ *of open refinements of* $\mathcal{U}$ *such that for all* $p \in X$ *there exist* $n \in \omega$ *and* $\mathcal{M}_p \subseteq \mathcal{U}$ *such that* $|\mathcal{M}_p| \leq \kappa$ *and* $(\mathcal{V}_n)_p$[5] *partially refines* $\mathcal{M}_p$.

Proof. The necessity is clear. Let $\mathcal{U}$ be an open cover of X and let $\langle \mathcal{V}_n : n \in \omega \rangle$ be a sequence of refinements of $\mathcal{U}$ with the property indicated above. For each $n \in \omega$, there is a subregular refining sequence $\langle \mathcal{W}_{nm} : m \in \omega \rangle$ of $\mathcal{V}_n$. Suppose $p \in X$. Then for some $n \in \omega$ and $\mathcal{M}_p \subseteq \mathcal{U}$, $|\mathcal{M}_p| \leq \kappa$ and $(\mathcal{V}_n)_p$ partially refines $\mathcal{M}_p$. For some $m \in \omega$, $\mathcal{W}_{nm}$ is regularly inscribed in $\mathcal{V}_n$ at p. Suppose $W_1, W_2 \in \mathcal{W}_{nm}$ and $p \in W_1 \cap W_2$. There is $V \in (\mathcal{V}_n)_p$ such that $W_1 \cup W_2 \subseteq V$. There is also $G \in \mathcal{M}_p$ such that $V \subseteq G$. Thus $\langle \mathcal{W}_{nm} : \langle n,m \rangle \in \omega\times\omega \rangle$ is a sequence of open refinements of $\mathcal{U}$ such that for all $p \in X$ there is some $\mathcal{W}_{nm}$ which is $(\kappa,1)$-regularly inscribed in $\mathcal{U}$ at p.

4. Subparacompactness and Related Topics

4.1 *Theorem. The following are equivalent for a regular space* X:

(a) X *is subparacompact.*

(b) *For every open cover* $\mathcal{U}$ *of* X *there exists a sequence*

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[5] $(\mathcal{W})_p = \{W \in \mathcal{W} : p \in W\}$.

$\langle \mathcal{V}_n\colon n \in \omega \rangle$ of open refinements of $\mathcal{U}$ such that for all $p \in X$ there exist $m,n \in \omega$ such that $\mathcal{V}_n$ is $(m,1)$-regularly inscribed in $\mathcal{U}$ at p.

(c) *X is sub-ω-regularly refinable.*

(d) *X is subregularly refinable and for every open cover $\mathcal{U}$ of X there is a sequence $\langle \mathcal{V}_n\colon n \in \omega \rangle$ of open refinements such that for all $p \in X$ there is $n \in \omega$ and $\mathcal{M} \subseteq \mathcal{U}$ such that $|\mathcal{M}| \leq \omega$ and $(\mathcal{V}_n)_p$ partially refines $\mathcal{M}$.*

(e) *X is subregularly refinable and submetaLindelöf.*

Proof. (a) ⇒ (b). Let $\mathcal{U}$ be an open cover of X. By Theorem 3.1(v) of [Bu] there is a sequence $\langle \mathcal{U}_n\colon n \in \omega \rangle$ of open refinements of $\mathcal{U}$ such that for every $p \in X$ there is $n \in \omega$ such that $st(p,\mathcal{V}_n) \subseteq U$ for some $U \in \mathcal{U}$. Hence (b) holds with $m \equiv 1$.

(b) ⇒ (c) and (c) ⇒ (d) are both clear from Definition 2.3.

(c) ⇒ (a). Let $\mathcal{U}$ be an open cover of X. By regularity there is an open refinement $\mathcal{W}$ of $\mathcal{U}$ such that $\{\overline{W}\colon W \in \mathcal{W}\}$ refines $\mathcal{U}$. By Theorem 3.1, $\mathcal{W}$ has a σ-discrete refinement $\cup_{n\in\omega}\mathcal{D}_n$, where each $\mathcal{D}_n$ is discrete. Then $\cup\{\overline{D}\colon D \in \mathcal{D}_n \wedge n \in \omega\}$ is a σ-discrete closed refinement of $\mathcal{U}$.

(d) ⇒ (c) by Theorem 3.2 for the case $\kappa = \omega$, and

(e) ⇒ (d) by definition.

(a) ⇒ submetacompactness by Theorem 3.1 (vi) of [Bu]. Also (a) ⇒ subregularly refinable because (a) ⇔ (c). Hence (a) ⇒ (e).

It is clear from (a) ⇒ (c) that Theorem 1.3 holds.

Proof of Theorems 1.4 *and* 1.5. Theorem 1.4 follows from 4.1 part (e) and the fact that a regular subparacompact space having a base of countable order is a Moore space. This is a direct consequence of Theorem 3 of [WW] and the fact that subparacompactness implies θ-refinability. Theorem 1.5 follows from 1.4 by applying Theorem 3.2 and the definition of submetaLindelöf.

Proof of Theorem 1.6. A regular paracompact space is essentially T_1 and collectionwise normal. Since such a space is ω-regularly refinable [W], it is sub-ω-regularly refinable. On the other hand if a space satisfies the conditions then it is regular and by Theorem 4.1 (c), it is subparacompact. But a collectionwise normal subparacompact space is paracompact [M].

Proof of Theorem 1.7. This follows from 1.4 and the fact that a collectionwise normal Moore space is metrizable [B].

5. Examples

We cite several examples showing the independence of some of the concepts involved above.

5.1 *Example*. A T_2 collectionwise normal regularly refinable but not sub-ω-regularly refinable space.

The space ω_1 with the order topology is T_2 collectionwise normal. It is also regularly refinable [W]. The space cannot be subparacompact since it is countably compact but not compact, hence it cannot be sub-ω-regularly refinable by Theorem 1.3.

5.2 *Example.* A sub-ω-regularly refinable space which is not regularly refinable.

Any non metrizable Moore space has this property since such a space is subparacompact but cannot be regularly refinable by Theorem 1.1.

5.3 *Example.* A T_2 normal metacompact space which is not subregularly refinable.

Example 4.9 (ii) of [Bu] is T_2 normal and metacompact. Since it is not subparacompact it cannot be sub-ω-regularly refinable. If it were subregularly refinable, it would be sub-ω-regularly refinable by Theorem 3.2.

References

[A] P. S. Alexandroff, *On some results concerning topological spaces and their continuous mappings.* In: General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), Academic Press, New York, 1962, 41-54.

[AU] ________ and P. Urysohn, *Une condition nécessaire et suffisante pour qu'une classe (L) soit une classe (D)*, C. R. Acad. Sci. Paris 177 (1923), 1274-1276.

[B] R. H. Bing, *Metrization of topological spaces*, Can. J. Math. 3 (1951), 175-186.

[Bu] D. K. Burke, *Covering properties.* In: Handbook of Set-Theoretic Topology, North Holland, Amsterdam, 1984, 347-422.

[M[L. F. McAuley, *A note on complete collectionwise normality and paracompactness*, Proc. Amer. Math. Soc. 9 (1958), 796-799.

[Mo] J. D. Monk, *Introduction to set theory*, McGraw-Hill, New York, 1969.

[P] V. Ponomarev, *Axioms of countability and continuous mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math., Astron. et Phys. 8 (1960), 127-133.

[W] J. M. Worrell, Jr., *On collections of domains inscribed in a covering of a space in the sense of Alexandroff and Urysohn*, Portug. Math. 26 (1967), 405-420.

[WW] ________ and H. H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. 17 (1965), 820-830.

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