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## ON PRODUCTS OF COUNTABLY COMPACT SPACES

by

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## ON PRODUCTS OF COUNTABLY COMPACT SPACES

Shoulian Yang

Ginsburg and Saks have proved that if the power  $X^{2^C}$  is countably compact,  $X$  is a Hausdorff space, then every power of  $X$  is countably compact ([1]). Some natural questions were raised. Is the cardinal number  $2^C$  minimal in this context? Does there exist a family of countably compact spaces  $\{X_i: i \in I\}$  such that (1)  $|I| = 2^C$ , (2)  $\prod_{i \in I} X_i$  is not countably compact and (3) if  $J \subset I$  with  $0 < |J| < 2^C$  then  $\prod_{i \in J} X_i$  is countably compact? (These two questions were first raised by W. W. Comfort in Mathematical Review 52# 1633.) Saks discussed the second question in [2]. He raised a conjecture:  $\omega^*$  is not the union of  $< 2^C$  cluster sets (a set  $C$  is called a cluster set if there exist an  $x \in \omega^*$  and a sequence  $\{x_n: n < \omega\}$  in  $\beta\omega$  such that  $C = \{p \in \omega^*: x = p - \lim_n x_n \text{ and } \{n: x_n \neq x\} \in p\}$ ). He indicated that the conjecture implies the affirmative answer of the second question.

The conjecture is consistent with ZFC. Is it true in ZFC? Let us consider some propositions in set theory.

It is well known that there are  $2^C$  points in  $\omega^*$  such that they are pairwise incomparable in Rudin-Keisler order ([3]). But given a point  $p \in \omega^*$  does there exist a point  $q \in \omega^*$  such that  $p, q$  are incomparable? This is another open problem (see [4]) and the answer is unknown in ZFC. This also related to work of Neil Hindman (Number Theory) of the last 5 years. Taking  $< 2^C$  points instead of one point such as

*Proposition 1. For every  $I \subset \omega^*$  if  $|I| < 2^{\mathfrak{c}}$ , then there exists  $p \in \omega^*$  such that  $p$  is incomparable in Rudin-Keisler order with all  $q \in I$ .*

It appears to be difficult to prove the proposition is true in ZFC. However, in this paper our main result is following.

*Theorem 2. The following are equivalent.*

(1) *There exist a Hausdorff space  $X$  such that  $X^{2^{\mathfrak{c}}}$  is not countable compact and for every  $\alpha < 2^{\mathfrak{c}}$   $X^{\alpha}$  is countably compact.*

(2) *The proposition 1.*

(3) *Saks' conjecture:  $\omega^*$  is not the union of  $< 2^{\mathfrak{c}}$  cluster set.*

We are now going to construct an example to show (2)  $\Rightarrow$  (1).

Recall that a point  $z \in X$  is said to be a  $p$ -limit point of a sequence  $\{x_n: n < \omega\}$  in  $X$ , where  $p \in \omega^*$ , if  $\{n: x_n \in W\} \in p$  for every neighbourhood  $W$  of  $z$ . When  $p$ -limit exist, they are unique and are denoted by  $z = p\text{-}\lim_n x_n$ .

Kunen (see [5]) has given out a famous result: there exist  $2^{\mathfrak{c}}$  weak-P-points in  $\omega^*$  (a point  $p$  in  $\omega^*$  is called a weak-P-point provided that  $p \bar{\in} B'$  for any countable subset  $B$  of  $\omega^*$ ). Then we can get a set  $H = \{h_i: i < 2^{\mathfrak{c}}\}$  such that every  $h_i$  is a faithfully indexed sequence of weak-P-points in  $\omega^*$  and  $h_i \cap h_j = \emptyset$  if  $i \neq j$ . On the other hand enumerate  $\omega^*$  as  $\{p_i: i < 2^{\mathfrak{c}}\}$ . Because of the compactness

of  $\omega^*$  there is a point  $z_i$  for each  $i < 2^C$  such that

$$z_i = p_i - \lim_n h_i(n). \text{ Let}$$

$$Z = \{z_i : i < 2^C\}, \quad Y = \omega^* \setminus Z.$$

The Hausdorff space  $Y$  is just an example what we want.

The power  $Y^{2^C}$  is not countably compact. In fact, let the sequence  $s = \{s(n) : n < \omega\}$  in  $Y^{2^C}$  be defined by

$$s_i(n) = h_i(n), \quad n < \omega, \quad i < 2^C,$$

where  $s_i(n)$  denotes the  $i$ -th coordinate of the point  $s(n)$  in  $Y^{2^C}$ . It is clear that  $p_i - \lim_n s_i(n)$  does not exist. Thus  $p_i - \lim_n s(n)$  does not exist for every  $i < 2^C$ , i.e.  $\{s(n) : n < \omega\}$ , as an infinite subset of  $Y^{2^C}$ , has not any accumulation point.

*Lemma 3.* If  $B, E$  are such two countable subsets of  $\omega^*$  that  $\overline{B} \cap E = \emptyset$  and  $B \cap \overline{E} = \emptyset$ , then  $\overline{B} \cap \overline{E} = \emptyset$ .

It is easy to check.

*Lemma 4.* If  $B \subset \omega^*$  with  $|B| = \omega$  and  $E = B \setminus \cup\{\overline{h}_i : i < 2^C\}$ , then  $\overline{E} \cap Z = \emptyset$ .

*Proof.* Since  $h_i$  consist of weak-P-points, we have  $h_i \cap \overline{E} = \emptyset$ . On the other hand  $E \cap \overline{h}_i = \emptyset$ . By Lemma 3 we have  $\overline{E} \cap \overline{h}_i = \emptyset$  for every  $i < 2^C$ . Notice that  $Z \subset \cup\{\overline{h}_i : i < 2^C\}$ , we have  $\overline{E} \cap Z = \emptyset$ .

*Lemma 5.* If  $B$  is a countable subset of  $\omega^*$ , then  $|Z \cap \overline{B}| \leq \omega$ .

*Proof.* By Lemma 4 we have  $\overline{E} \cap Z = \emptyset$ . Then  $Z \cap \overline{B} = Z \cap \overline{B \setminus E}$ . Since  $h_i (i < 2^C)$  is a set of weak-P-points, we

have  $\bar{h}_i \cap \bar{h}_j = \emptyset$  ( $i \neq j$ ). It implies that every  $x_n \in B \setminus E$  only belongs to one  $\bar{h}_i$ . Then

$$|\{i < 2^C: \overline{B \setminus E} \cap \bar{h}_i \neq \emptyset\}| \leq \omega,$$

and easy to check  $|\overline{B \setminus E} \cap Z| \leq \omega$ .

*Definition 6.* Let  $\{Q_n: n < \omega\}$  be a sequence of subsets of  $\beta\omega$ . A point  $z \in \omega^*$  is called a  $p$ -limit point of the sequence for  $p \in \omega^*$ , provided that  $\{n: Q_n \cap W \neq \emptyset\} \in p$  for every neighbourhood  $W$  of  $z$ . It is denoted by  $z = p - \lim_n Q_n$ .

*Lemma 7.* Let  $\{Q_n: n < \omega\}$  be a sequence of non-empty closed-open subsets of  $\beta\omega$  which are pairwise disjoint. If  $z \in \overline{\bigcup_{n < \omega} Q_n} \setminus \bigcup_{n < \omega} Q_n$ , then there exist a unique  $p \in \omega^*$  such that  $z = p - \lim_n Q_n$ .

*Proof.* The proof of the uniqueness is routine. Let us show the existence. We claim that

$$p = \{\mathcal{G}(W) = \{n: Q_n \cap W \neq \emptyset\}: W \text{ is a closed-open neighbourhood of } z\}$$

is an ultrafilter, then it is clear that  $z = p - \lim_n Q_n$ . In fact, it is easy to check  $p$  is a base of filter. We are going to prove  $p$  is really an ultrafilter. Let  $A$  be such a subset of  $\omega$  that  $|A \cap H| = \omega$  for every  $H \in p$ . It is sufficient to show  $A \in p$ . Let

$$Q(A) = \bigcup \{Q_n: n \in A\}.$$

Then

$$Q(A) \cap W \neq \emptyset$$

for all closed-open neighbourhood  $W$  of  $z$ . Thus  $z \in \overline{Q(A)}$ .  $\overline{Q(A)}$  is also a closed-open neighbourhood of  $z$ . In fact,

$Q(A)$  is open and  $\beta\omega$  is extremely disconnected, so  $\overline{Q(A)}$  is also open. It implies that

$$\mathcal{G}(\overline{Q(A)}) \in p.$$

Notice that  $A = \{n: Q_n \cap Q(A) \neq \emptyset\}$ . We conclude that  $A = \mathcal{G}(\overline{Q(A)})$ , so  $A \in p$ . In fact, if there is a natural number  $m \in \mathcal{G}(\overline{Q(A)}) \setminus A$ , then  $Q_m \cap Q(A) = \emptyset$ . Since  $\overline{Q_m} = Q_m$ , we have  $Q_m \cap \overline{Q(A)} = \emptyset$ . It is contradictory to the fact  $m \in \mathcal{G}(\overline{Q(A)})$ .

*Lemma 8.* Let  $X$  be a countable subset of  $\beta\omega$  and  $z \in \omega^*$ . If  $z \in \overline{X} \setminus X$ , then there exist countable many closed-open sets  $\{Q_n: n < \omega\}$ , which are pairwise disjoint, such that  $X \subset \bigcup \{Q_n: n < \omega\}$  and  $z \in \overline{\bigcup_{n < \omega} Q_n} \setminus \bigcup_{n < \omega} Q_n$ .

*Proof.* Enumerate  $X$  by  $\{x_n: n < \omega\}$ . It is easy to construct for each  $n < \omega$  a closed-open set  $Q_n$  inductively such that (1)  $\{x_k: k \leq n\} \subset \bigcup_{k \leq n} Q_k$ , (2)  $Q_k \cap Q_h = \emptyset$  if  $k \neq h, k, h \leq n$ , (3)  $z \notin \bigcup_{k \leq n} Q_k$ .

*Lemma 9.* Let  $\{x_n: n < \omega\}$  be a sequence in  $\beta\omega$ ,  $z \in \overline{\{x_n: n < \omega\}} \setminus \{x_n: n < \omega\}$ . There exist a  $p \in \omega^*$  and  $f \in {}^\omega\omega$  such that if  $q \in \omega^*$  satisfies  $z = q - \lim_n x_n$ , then  $\overline{f}(q) = p$ .

*Proof.* According to the Lemma 8 there exists a sequence  $\{Q_n: n < \omega\}$  of closed-open sets such that they are pairwise disjoint,  $\{x_n: n < \omega\} \subset \bigcup_{n < \omega} Q_n$  and  $z \in \overline{\bigcup_{n < \omega} Q_n} \setminus \bigcup_{n < \omega} Q_n$ . Then there exists a unique  $p \in \omega^*$  such that  $z = p - \lim_n Q_n$  by the Lemma 7. Let  $f \in {}^\omega\omega$  be such a function that

$$f^{-1}(n) = \{i < \omega: x_i \in Q_n\}.$$

It is easy to check  $p$ ,  $f$  is just what we want.

*Theorem 10.* Let  $J \subset 2^C$  with  $|J| < 2^C$ . The proposition 1 implies that  $Y^J$  is countably compact. So (2)  $\Rightarrow$  (1).  
 ((1), (2) are the items in Theorem 2.)

*Proof.* Let  $s = \{s(n) : n < \omega\}$  be a sequence in  $Y^J$ , and for every  $i \in J$

$$R_i = \{z \in Z : z \text{ is an accumulation point of } s_i\}.$$

By the Lemma 5  $|R_i| \leq \omega$ . Let  $R = \cup\{R_i : i \in J\}$  we have  $|R| < 2^C$ . By the Lemma 9 there is a point  $p(z) \in \omega^*$  for each  $z \in R$  such that if  $z = q - \lim_n s_i(n)$  for some  $i \in J$  then  $q \geq p(z)$  ( $\geq$  denotes Rudin-Keisler order). Assume the Proposition 1 is true, then there is a point  $r \in \omega^*$  such that

$$r \notin \{t \in \omega^* : t \geq \text{some } p(z)\}$$

because  $|\{p(z) : z \in R\}| < 2^C$ . It implies that

$$z \neq r - \lim_n s_i(n)$$

for any  $z \in R$  and  $i \in J$ . So  $r - \lim_n s(n)$  exists in  $Y^J$ , i.e.  $Y^J$  is countably compact.

We have completed the proof of the implication (2)  $\Rightarrow$  (1). Now about the inverse implication.

*Lemma 11.* Let  $h, k \in {}^\omega X$ .  $X$  is a topological space and  $f \in {}^\omega \omega$  such that  $k(m) = h(n)$  whenever  $f(m) = n$ . Then

$$q - \lim_n k(n) = p - \lim_n h(n)$$

if  $f(q) = p$  and the limit exists.

*Proof.* Suppose  $z = p - \lim_n h(n)$  or  $z = q - \lim_n k(n)$ . Let  $W$  be any neighbourhood of  $z$ . Since  $\bar{f}(q) = p$ , if  $A, B \subset \omega$  and  $f[A] = B$ , then  $A \in q$  iff  $B \in p$ . Let

$$\mathcal{G}(W) = \{n : h(n) \in W\}, \mathcal{K}(W) = \{n : k(n) \in W\}.$$

It is clear that  $f[K(W)] = \mathcal{G}(W)$ . So  $\mathcal{G}(W) \in p$  iff  $K(W) \in q$ . It means  $q - \lim_n k(n) = p - \lim_n h(n)$  if the limit point exists.

*Lemma 12.* The Proposition 1 is equivalent to the following statement: for every subset  $J \subset \omega^*$  with  $|J| < 2^C$

$$\omega^* \neq \{p \in \omega^*: p \leq \text{some } q \in J\}.$$

It is easy to check, if we notice that  $|\{q \in \omega^*: q \leq p\}| \leq c$  for every  $p \in \omega^*$ .

*Theorem 13.* (1)  $\Rightarrow$  (2).

*Proof.* Assume the negation of (2), that the Proposition 1 is not true. By the Lemma 12 there is a subset  $J \subset \omega^*$  with  $|J| < 2^C$  such that for each  $q \in \omega^*$  there is such a point  $p \in J$  that  $p \leq q$ . In this case (1) is also false. If it is not, i.e. there exists a space  $X$  which is described in (1).  $X^{2^C}$  is not countably compact. Thus for each  $p \in J$  there is an ordinal  $i(p) < 2^C$  such that  $p - \lim_n h_{i(p)}(n)$  does not exist. Let

$$K = \{\langle f, p \rangle: p \in J, f \in {}^\omega \omega \text{ and } f[\omega] \in p\}.$$

It is clear that  $|K| < 2^C$ . The sequence  $\{s(n): n < \omega\}$  in  $X^K$  is defined by

$$s_{\langle f, p \rangle}(m) = h_{i(p)}(n), f(m) = n.$$

Because for each  $q \in \omega^*$  there is  $p \in J$  such that  $p \leq q$ , i.e. there is  $f \in {}^\omega \omega$  such that  $f[\omega] \in p$ . By the Lemma 11  $q - \lim s_{\langle f, p \rangle}(n)$  does not exist, since  $p - \lim_n h_{i(p)}(n)$  does not exist. Then  $X^K$  is not countably compact. It implies  $X^{|K|}$  is not countably compact. But  $|K| < 2^C$ . So (1) is false.



All that remains is to prove  $(2) \Leftrightarrow (3)$ .

*Theorem 14.*  $(2) \Rightarrow (3)$ .

*Proof.* Let  $\mathcal{Y}$  be a union of  $< 2^C$  cluster sets, which is denoted by

$$\mathcal{Y} = \cup \{C(z_e, X_e) : e \in E\}, \quad |E| < 2^C$$

where  $z_e \in \omega^*$ ,  $X_e$  is a sequence in  $\beta\omega$ ,  $X_e = \{x_e(n) : n < \omega\}$  such that  $z_e \in X'_e$  and  $C(z_e, X_e)$  denotes the cluster set about  $z_e$  and  $X_e$ .

Without loss of generality we may assume that  $z_e \in \overline{X_e}$  for each  $e \in E$ . By the Lemma 9 there exist  $p_e \in \omega^*$  and  $f_e \in {}^\omega\omega$  for each  $e \in E$  such that if  $q \in \omega^*$  satisfies  $z_e = q - \lim_n x_e(n)$ , then  $\overline{f_e}(q) = p_e$ . Since the cardinality of the set  $\mathcal{J} = \{p_e : e \in E\}$  is  $< 2^C$ , there is  $p \in \omega^*$  such that  $p$  is incomparable with every  $p_e \in \mathcal{J}$  by the hypothesis. It is clear  $p \notin \mathcal{Y}$ . It means (3) is true.

*Lemma 15.* Let  $f \in {}^\omega\omega$ .  $f[\omega] \in p$ ;  $\rho \in {}^\omega\beta\omega$  be an 1-1 function and  $h = \rho \circ f$ . Assume  $h[\omega]$  is discrete in  $\beta\omega$  and  $p - \lim_n \rho(n) = z$ , then an ultrafilter  $q$  satisfies

$$q - \lim_n h(n) = z$$

iff  $q \in (\overline{f})^{-1}(p)$ .

*Proof.* Assume  $q \in (\overline{f})^{-1}(p)$ . Let  $W$  be a neighbourhood of  $z$ . We have

$$\mathcal{G}(W) = \{n : \rho(n) \in W\} \in p.$$

It implies  $f^{-1}[\mathcal{G}(W)] \in q$  because  $q \in (\overline{f})^{-1}(p)$  iff  $f^{-1}[A] \in q$  for all  $A \in p$ . Then  $z = q - \lim_n h(n)$ .

Inversely, we assume  $q \in (\overline{f})^{-1}(p)$ . There is  $A \in p$  such that  $f^{-1}[A] \notin q$ . So there is  $B \in q$  such that  $B \cap f^{-1}[A] = \emptyset$ . It implies

$$h[B] \cap h[f^{-1}[A]] = \emptyset.$$

Since  $h[\omega]$  is discrete we have

$$\overline{h[B]} \cap \overline{h[f^{-1}[A]]} = \emptyset.$$

Then  $z \in \overline{h[B]}$  because  $z \in \overline{\rho[A]}$  and  $\rho[A] = h[f^{-1}[A]]$ .  $B \in q$  and  $z \in \overline{h[B]}$  implies  $z \neq q - \lim_n h(n)$ .

*Lemma 16.* Let  $\beta = \{\langle p, f \rangle : p \in \omega^*, f \in {}^\omega \omega \text{ and } f[\omega] \in p\}$ . Suppose  $J \subset \beta$  and  $|J| < 2^{\mathfrak{c}}$ . Then there exist a point  $z(p, f) \in \omega^*$  and a sequence  $h_{p, f} = \{h_{p, f}(n) : n < \omega\}$  of weak-P-points such that

$$(1) \quad z(p, f) = q - \lim_n h_{p, f}(n) \text{ iff } \bar{f}(q) = p;$$

$$(2) \quad z(p, f) \neq z(r, g) \text{ if } \langle p, f \rangle \neq \langle r, g \rangle.$$

Apply Kunen's result: there are  $2^{\mathfrak{c}}$  weak-p-points, and the Lemma 15. It is easy to check.

*Theorem 17.* (3)  $\Rightarrow$  (2).

*Proof.* If (2) is false, then there is a subset  $J \subset \omega^*$  with  $|J| < 2^{\mathfrak{c}}$  such that

$$\omega^* = \bigcup \{(\bar{f})^{-1}(p) : \langle p, f \rangle \in \mathcal{J}\} \quad (*)$$

by the Lemma 12, where

$$\mathcal{J} = \{\langle p, f \rangle : p \in J \text{ and } f[\omega] \in p\}.$$

By the Lemma 16 the collection  $(\bar{f})^{-1}(p)$  is a cluster set. But  $|\mathcal{J}| < 2^{\mathfrak{c}}$  then the equality (\*) contradicts the hypothesis.

The proof of the main result Theorem 2 is complete. Let us go back to the second question stated at the beginning of this paper. It is clear that (1) implies the affirmative answer to the second question. Saks gave out

an example in [2] to show (3) implies the affirmative answer to the second question: Let  $K_x = \beta\omega \setminus \{x\}$  for every  $x \in \omega^*$ . The products  $\prod\{K_x: x \in \omega^*\}$  is not countably compact and  $|\omega^*| = 2^{\mathfrak{c}}$ . He indicated the following

*Theorem 18. (3) implies  $\prod\{K_x: x \in J\}$  is countably compact for all  $J \subset \omega^*$  with  $0 < |J| < 2^{\mathfrak{c}}$ .*

We can prove that, as a corollary of the Theorem 17, the inverse proposition of the Theorem 18 is also true.

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