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1. Introduction

For X a nondegenerate Peano continuum, let C(X) denote the hyperspace of subcontinua, topologized by the Hausdorff metric. $M \in C(X)$ is *stable* if, for every map $f: C(X) \to C(X)$ sufficiently close to the identity map, $M \in f(C(X))$. Thus, M is *unstable* if and only if {M} is a Z-set in C(X). If X does not contain a free arc then C(X) is homeomorphic to the Hilbert cube [2], hence each point is unstable. On the other hand, if X contains a free arc then C(X) has stable points (for instance, every subarc in the interior of a free arc). We obtain the following surprisingly simple set of conditions for determining which points of C(X) are stable.

1.1. *Theorem.* $M \in C(X)$ *is stable if and only if:*

a) M *is a finite union of free arcs in* X; *and*

b) *the quotient space* $M/\mathrm{bd}_X M$ *(or* M *if* M = X*) has no cut points.*

In particular, $X \in C(X)$ *is stable if and only if* X *is a finite graph with no cut points.*

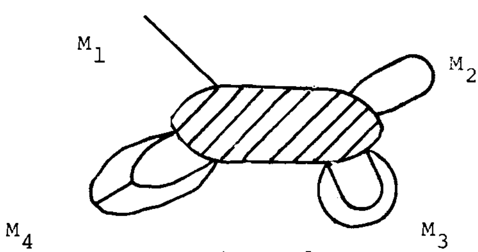

Figure 1

To illustrate the Theorem, consider X as in Figure 1. The elements M_1 and M_3 are unstable in C(X), whereas M_2 and M_4 are stable.

The only previous work on stable points in hyperspaces is due to Neil Gray [5], [6], who showed that every point in the hyperspace 2^X is unstable, and that for X a finite polyhedron, every point in C(X) is unstable if and only if X has no free arcs. (This was done before the Hilbert cube characterization of such hyperspaces.) For X a finite graph, Roman Duda [3], [4] described and analyzed polyhedral models for C(X). Such models were very useful for suggesting an initial version of the above condition b).

The consideration of stable points in C(X) is simplified by the fact that C(X) is an AR, and therefore locally equiconnected. This implies that stability is determined locally, i.e., if $\mathcal{U}$ is any neighborhood of M in C(X), then M is stable in C(X) if and only if M is stable in $\mathcal{U}$.

2. Stable Points in C(X)

In this section we demonstrate the necessity of the conditions a) and b) for stable $M \in C(X)$. For the argument that M is a finite union of free arcs, with bd M not separating M, we use just two basic types of maps in C(X), namely, "local channelization" maps and "global expansion" maps.

To describe the first type of map, we find it convenient to use partitions of X. A *partition* of a space is a finite collection $\mathcal{P}$ of mutually disjoint connected open subsets whose closures cover the space. It will be assumed that each partition element is the interior of its closure. Given a partition $\mathcal{P}$ of X and $x \in X$, we may obtain a partition which has an element containing x by replacing (if necessary) the subcollection $\mathcal{P}_x = \{P \in \mathcal{P}: x \in \overline{P}\}$ with a single larger element $P_x = \text{int}(\text{cl}(\cup\mathcal{P}_x))$. The basic result needed concerning partitions of a Peano continuum X is due to Bing [1] and Moise [8]: for every $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of X such that mesh $\mathcal{P} < \varepsilon$ and the closure of each partition element is a Peano continuum. For each element P of such a partition, it is easily seen that there exists a finite graph $T \subset \overline{P}$ such that the space $\cup\{\overline{Q}: Q \in \mathcal{P}\backslash\{P\}$ with $\overline{Q} \cap \overline{P} \neq \emptyset\} \cup T$ is connected (and therefore a Peano continuum). We call T a P-*connecting graph.*

2.1. *Lemma. For every* $x \in X$ *and* $\varepsilon > 0$*, there exists a neighborhood* P *of* x*, a Peano continuum* $K \subset X$*, and a finite graph* $T \subset K$ *such that:*

i) diam $(K \cup P) < \varepsilon$;

ii) bd $P \subset K$;

iii) $K \cap P \subset T$.

Proof. Choose a partition $\mathcal{P}$ of X such that $x \in P \in \mathcal{P}$, the closure of each partition element is a Peano continuum, and mesh $\mathcal{P} < \varepsilon/3$. Choose a P-connecting graph T, and take $K = \cup\{\overline{Q}: Q \in \mathcal{P}\backslash\{P\}$ with $\overline{Q} \cap \overline{P} \neq \emptyset\} \cup T$. Then P,K, and T have the required properties.

2.2. *Proposition. Every stable* $M \in C(X)$ *is a finite graph.*

Proof. Choose $\varepsilon > 0$ such that for every map $f: C(X) \to C(X)$ with $d(f,\text{id}) < \varepsilon$, $M \in f(C(X))$. (For simplicity of notation, we let d denote both the metric on X and the corresponding Hausdorff metric on C(X)). It suffices to show that at each point M is locally a finite graph. Given $x \in M$, choose a neighborhood P of x in X, a Peano continuum $K \subset X$, and a finite graph $T \subset K$ satisfying the conditions of (2.1). Since C(K) is an AR, there exists a map $\phi: \overline{P} \to C(K)$ with $\phi(y) = \{y\}$ for each $y \in \text{bd } P$. Extend to a map $\phi: X \to C(X)$ by setting $\phi(y) = \{y\}$ for each $y \in X\backslash\overline{P}$. Then ϕ induces a map $f: C(X) \to C(X)$, defined by $f(A) = \cup\{\phi(a): a \in A\}$. Since $d(f,\text{id}) < \varepsilon$, $M = f(A)$ for some $A \in C(X)$. Since $f(A) \cap P \subset K \cap P \subset T$, M is locally a finite graph.

2.3. *Proposition. For every stable* $M \in C(X)$, $\text{int}_X M$ *is a dense connected subset of* M.

Proof. Assume the metric d is convex. Given $\varepsilon > 0$, let $\phi: X \to C(X)$ be the map defined by $\phi(x) = \{y \in X: d(x,y) < \varepsilon\}$. Then ϕ induces an expansion map $f: C(X) \to C(X)$, with $d(f,\text{id}) < \varepsilon$. If $M = f(A)$ for some $A \in C(X)$, then $A \subset \text{int } M$ and $d(A,M) < \varepsilon$. Letting $\varepsilon \to 0$, we see that int M must be dense and connected.

Clearly, (2.2) and (2.3) together imply that for M stable, bd M is a finite set. It follows that the finite graph M is a finite union of free arcs in X.

2.4. *Proposition. For every stable* $M \in C(X)$, M/bd M *(or* M *if* $M = X$*) has no cut points.*

Proof. Suppose there exists a cut point p. By (2.3), $p \in \text{int } M$. Then there exists a nondegenerate finite graph

$G \subset \text{int } M$ and a nondegenerate subcontinuum $Y \subset X$ such that $X = G \cup Y$ and $G \cap Y = \{p\}$. (Some component F of $M \setminus \{p\}$ lies in int M; take $G = F \cup \{p\}$ and $Y = M \setminus F$). It follows that $C_p(X) = \{A \in C(X): p \in A\}$ is a neighborhood of M in C(X), hence M is stable in $C_p(X)$. Let $h: C_p(X) \to C_p(G) \times C_p(Y)$ be the natural homeomorphism defined by $h(A) = (A \cap G, A \cap Y)$. Then $h(M) = (G, M \cap Y)$ is stable in $C_p(G) \times C_p(Y)$, and G must be stable in $C_p(G)$. However, it has been shown by Mark Lynch [7] that G is always unstable in $C_p(G)$. (In fact, Lynch shows that X is unstable in $C_p(X)$, for every continuum X.) Thus there are no cut points in M/bd M.

Note. If X is a non-locally connected continuum, the situation regarding stable points in C(X) can be quite different from that above. For example, let X be the union of two copies of the $\sin(1/x)$-continuum, meeting only at a common endpoint p of their limit arcs. Then it can be shown that $\{p\}$ is stable in C(X); in fact, for every map $f: C(X) \to C(X)$ sufficiently close to the identity map, $f(\{p\}) = \{p\}$. Moreover, by adding a countable collection of stickers to X, we may obtain a continuum Y which has no free arcs, and yet $\{p\}$ is stable in C(Y).

3. Reduction to X = G

In this section and the next two, we give the proof of sufficiency of the conditions a) and b) for $M \in C(X)$ to be stable. The following result reduces the general problem to the special case $X = G$, a finite graph.

3.1. *Proposition. Let* $M \in C(X)$ *be a finite union of free arcs, with* $M \neq X$. *Let* $\{J_b\colon b \in \text{bd } M\}$ *be a collection of mutually disjoint arcs in* X *such that* b *is an endpoint of* J_b *and* $J_b \cap M = \{b\}$, *for each* b. *Consider the finite graph* $G = \cup\{J_b\colon b \in \text{bd } M\} \cup M$, *and suppose that* M *is stable in* C(G). *Then* M *is stable in* C(X).

Proof. Choose a collection $\{K_b\colon b \in \text{bd } M\}$ of mutually disjoint continua in X such that:

i) $J_b \subset K_b$;

ii) $K_b \cap M = \{b\}$; and

iii) $K_b \cup M$ is a neighborhood of b in X.

Set $Y = \cup\{K_b\colon b \in \text{bd } M\} \cup M$. We show first that M is stable in C(Y). For each b, choose a retraction $r_b\colon K_b \to J_b$ such that $r_b(x) = b$ only if $x = b$, and let $r\colon Y \to G$ be the retraction defined by the collection $\{r_b\}$. Let $R\colon C(Y) \to C(G)$ denote the induced hyperspace retraction. Note that $R(B) = M$ only if $B = M$. If $f\colon C(Y) \to C(Y)$ is a map sufficiently close to the identity, then the map $g\colon C(G) \to C(G)$ defined by $g(A) = R(f(A))$ will also be close to the identity, and by hypothesis we may assume $M = g(A)$ for some $A \in C(G)$. But this implies $M = f(A)$. Thus M is stable in C(Y). Since $M \subset \text{int } Y$, C(Y) is a neighborhood of M in C(X), and it follows that M is stable in C(X).

4. Cone Neighborhoods in C(G)

Since the hyperspace C(G) of a finite graph is homeomorphic to a polyhedron, every $M \in C(G)$ has a neighborhood homeomorphic to the cone over some polyhedron S, with M corresponding to the cone vertex. It is easily shown that

M is stable in such a cone neighborhood (and therefore in C(G)) if and only if the cone base S is non-contractible. Our objective in this section is the construction, for nondegenerate M, of a particular cone neighborhood for which the Euler characteristic $\chi(S)$ of the base is given in terms of a certain index $\lambda(M/\text{bd } M)$ of the quotient graph M/bd M. Analysis of this index in section 5 will show that if M/bd M has no cut points then $\chi(S) \neq 1$, hence S is non-contractible, and M is stable in C(G).

4.1. *Proposition. For* S *a compact metric* ANR, *consider* Cone S = S × [0,1]/S × {0}, *with cone vertex* Θ *corresponding to* S × {0}. *Then* Θ *is an unstable point in* Cone S *if and only if* S *is contractible.*

Proof. If S is contractible, there exists for every $t > 0$ a retraction of S × [0,t]/S × {0} onto S × {t}. Let r_t: Cone S $\to$ S × [t,1] be the extension of such a retraction by the identity. Then $d(r_t, \text{id}) \to 0$ as $t \to 0$, and Θ is unstable in Cone S.

Conversely, suppose there exist maps f: Cone S $\to$ S × (0,1] arbitrarily close to the identity. By local equiconnectedness of S, we may assume that $f|S \times \{1\} = \text{id}$. Then f followed by the projection onto S defines a contraction of S to a point.

Let G be a nondegenerate finite graph, considered with respect to a specific subdivision. We say that a subgraph L (possibly degenerate) is *large* if L meets each edge of G. Let $\mathcal{L}(G)$ denote the collection of large subgraphs of G, and for each vertex v of G, let $\mathcal{L}_v(G) = \{L \in \mathcal{L}(G):$

$v \in L\}$. For each $L \in \mathcal{L}(G)$ let def L be the number of edges in $G \setminus L$. We define the *stability index* $\lambda(G) = \sum\{(-1)^{\text{def } L}: L \in \mathcal{L}(G)\}$. Similarly, for each vertex v define $\lambda_v(G) = \sum\{(-1)^{\text{def } L}: L \in \mathcal{L}_v(G)\}$.

We will show in section 5 that $\lambda_v(G) = 0$ for all v; $\lambda(G)$ is a topological invariant of G (provided the subdivision of G has more than one edge); and $\lambda(G) = 0$ if and only if G has a cut point.

Referring to Figure 1, with $G_i = M_i/\text{bd } M_i$ for i = 1, 2, 3, 4, it's not difficult to see that $\lambda(G_1) = 0 = \lambda(G_3)$ and $\lambda(G_2) = 1 = \lambda(G_4)$.

For nondegenerate $M \in C(G)$, let $M^* = M/\text{bd } M$ if $M \neq G$, and $M^* = M$ if $M = G$. Thus M^* is topologically a finite graph.

4.2. *Theorem. For every nondegenerate* $M \in C(G)$, *there exists a finite polyhedron* S *and an imbedding* $h: \text{Cone } S \to C(G)$ *such that:*

i) $h(\Theta) = M$;

ii) $h(\text{Cone } S)$ *is a neighborhood of* M *in* $C(G)$; *and*

iii) $\chi(S) = 1 - \lambda(M^*)$.

Proof. We assume that M is a subgraph of G. Since st M is a neighborhood of M in G, we may in fact assume that M is a large subgraph of G. Let $\tilde{M}$ denote a barycentric subdivision of M, and consider the subdivision $\text{sd } G = \tilde{M} \cup (G \setminus M)$ of G. Let $E = \{e_1, \cdots, e_k\}$ be the collection of edges of sd G. For each $L \in \mathcal{L}(M)$, set $e(L) = \{e_i: e_i$ meets L in a single point$\} \subset E$. Note that $e(L) \neq \emptyset$, unless $L = M = G$.

Let $T = \{(t_i) \in \prod_1^k [0,1]_i : \sum t_i \leq 1$ and $\text{supp}(t_i) \subset e(L)$ for some $L \in \mathcal{L}(M)\}$, where $\text{supp}(t_i) = \{e_i : t_i > 0\}$. T can be considered as the cone over the polyhedron $S = \{(t_i) \in T: \sum t_i = 1\}$, with cone vertex $\Theta = (0,\cdots,0)$. We will construct an imbedding $h: T \to C(G)$ such that $h(\Theta) = M$ and $h(T)$ is a neighborhood of M in C(G).

Assume that $\{e_{2i-1}, e_{2i}\}$ is the subdivision of an edge α_i of M, for each $1 \leq i \leq j$, and that $\{e_{2j+1},\cdots,e_k\}$ is the collection of edges of G\M. Let d be the shortest path-length metric on G, with each edge of sd G having unit length. Thus for each $i \leq j$, the arc $\alpha_i = e_{2i-1} \cup e_{2i}$ has length 2. Let C_i denote the quotient space obtained from the hyperspace $J \in C(\alpha_i)$: diam $J \leq 1\}$ by collapsing the subspace of degenerate elements to a point. Then C_i is a 2-cell, and for $\Delta_i = \{(s,t) \in [0,1]_{2i-1} \times [0,1]_{2i}: s + t \leq 1\}$, there exists a homeomorphism $g_i: \Delta_i \to C_i$ such that:

1) $g_i(\{0\} \times [0,1]_{2i}) \subset C_{v_i}(\alpha_i)$, where v_i is the endpoint of α_i lying in e_{2i-1};

2) $g_i([0,1]_{2i-1} \times \{0\}) \subset C_{w_i}(\alpha_i)$, where w_i is the endpoint of α_i lying in e_{2i};

3) diam $g_i(s,t) = s + t$.

For each edge e_i of G\M, $2j < i \leq k$, let w_i be the endpoint of e_i not lying in M, and define a homeomorphism $g_i: [0,1]_i \to C_{w_i}(e_i)$ such that

4) diam $g_i(t) = 1 - t$.

The desired imbedding $h: T \to C(G)$ is defined by the formula

$$h((t_i)) = \cup_{i\leq j}\{cl(\alpha_i \backslash g_i(t_{2i-1},t_{2i}))\} \cup$$
$$\cup_{i>2j}\{cl(e_i \backslash g_i(t_i))\}.$$

Note that $h(\Theta) = M$. Since $supp(t_i) \subset e(L)$ for some $L \in \mathcal{L}(M)$, it is clear that $L \subset h((t_i))$ and that $h((t_i)) \in C(G)$. For any $A \subset G$, let $\delta[A] = \sum_1^k diam(A \cap e_i)$. Then 3) and 4) imply that for each $(t_i) \in T$,

$$\delta[(h((t_i))\backslash M) \cup (M\backslash h((t_i)))] = \sum t_i.$$

And in fact, $h(T) = \{H \in C(G): \delta[(H\backslash M) \cup (M\backslash H)] \leq 1\}$. (Since each edge of M has length 2, each H in the above set must contain a large subgraph of M). Thus h(T) is a neighborhood of M in C(G).

It remains to be shown that $\chi(S) = 1 - \lambda(M^*)$. The collection {F: F is a face of S} is in 1-1 correspondence with the collection $\{A: \emptyset \neq A \subset e(L)$ for some $L \in \mathcal{L}(M)\}$, the correspondence being given by $F_A = \{(t_i) \in S: supp(t_i) \subset A\}$. For $F = F_A$ we write supp F = A. Then dim F = # supp F - 1. For each face F, let L_F denote the *largest* $L \in \mathcal{L}(M)$ for which supp $F \subset e(L)$. For each $L \in \mathcal{L}(M)$ we will compute $\sigma(L) = \sum\{(-1)^{\dim F}$: F is a face with $L_F = L\}$. Then $\chi(S) = \sum\{\sigma(L): L \in \mathcal{L}(M)\}$.

For $L \in \mathcal{L}(M)$, the set e(L) may contain the following types of edges or pairs of edges of sd G:

a) pairs of edges $\{e_{2i-1},e_{2i}\}$ in $\tilde{M}$, $i \leq j$;

b) edges e_i in $\tilde{M}$, $i \leq 2j$, which do not occur as members of pairs in a);

c) edges e_i in $G\backslash M$, $i > 2j$.

Let m be the number of pairs of edges in e(L) of type a); n the number of edges of type b); and p the number of edges of type c). Note that def L = m + n.

For each face F of S with $L_F = L$, we have supp $F \subset e(L)$, with supp F containing all edges in e(L) of type b) and at least one member from each pair of type a). In other words, for some $0 \le \ell \le m$ and $0 \le q \le p$, supp F contains ℓ pairs of edges of type a), and $m - \ell$ single members of the remaining pairs; all n edges of type b); and q edges of type c). We must have $m + n + q > 0$, and dim F = #supp F - 1 = $\ell + m + n + q - 1$.

We now compute $\sigma(L) = \sum\{(-1)^{\dim F}: L_F = L\}$. There are four cases to consider.

Case i): $L \neq M$ and $L \cap \text{bd } M = \emptyset$. Then $m + n > 0$ and $p = 0$, and

$$\begin{aligned}\sigma(L) &= \textstyle\sum_{\ell=0}^{m} \binom{m}{\ell} 2^{m-\ell} (-1)^{\ell+m+n-1} \\ &= (-1)^{m+n-1} \textstyle\sum_{\ell=0}^{m} \binom{m}{\ell} 2^{m-\ell} (-1)^{\ell} \\ &= (-1)^{m+n-1}(2 - 1)^{m} \\ &= -(-1)^{\text{def } L}.\end{aligned}$$

Case ii): $L \neq M$ and $L \cap \text{bd } M \neq \emptyset$. Then $m + n > 0$ and $p > 0$, and

$$\begin{aligned}\sigma(L) &= \textstyle\sum_{\ell=0}^{m} \sum_{q=0}^{p} \binom{m}{\ell} 2^{m-\ell} \binom{p}{q} (-1)^{\ell+m+n+q-1} \\ &= (-1)^{m+n-1} \textstyle\sum_{\ell=0}^{m} \binom{m}{\ell} 2^{m-\ell} (-1)^{\ell} \sum_{q=0}^{p} \binom{p}{q} (-1)^{q} \\ &= (-1)^{m+n-1}(2 - 1)^{m}(1 - 1)^{p} \\ &= 0.\end{aligned}$$

Case iii): $L = M \neq G$. Then $m + n = 0$ and $p > 0$, and

$$\begin{aligned}\sigma(L) &= \textstyle\sum_{q=1}^{p} \binom{p}{q} (-1)^{q-1} \\ &= -(\textstyle\sum_{q=0}^{p} \binom{p}{q} (-1)^{q} - 1) \\ &= -((1 - 1)^{p} - 1) \\ &= 1.\end{aligned}$$

Case iv): $L = M = G$. Then $e(L) = \emptyset$, there are no faces F associated with L, and $\sigma(L) = 0$.

Finally, we compute $\chi(S) = \sum\{\sigma(L): L \in \mathcal{L}(M)\}$. Suppose first that $M \neq G$. Then $\chi(S) = 1 - \sum\{(-1)^{\text{def } L}: L \in \mathcal{L}(M)$ with $L \cap \text{bd } M = \emptyset\}$. If the quotient space $M^* = M/\text{bd } M$ is given the natural subdivision induced by M, then the collection $\{L \in \mathcal{L}(M): L \cap \text{bd } M = \emptyset\}$ may be identified with the collection $\mathcal{L}(M^*) \setminus \mathcal{L}_v(M^*)$, where v is the vertex of M^* corresponding to bd M. For each L in these collections, def L has the same value with respect to either M or M^*. Thus,

$$\begin{aligned}\chi(S) &= 1 - \sum\{(-1)^{\text{def } L}: L \in \mathcal{L}(M^*)\setminus\mathcal{L}_v(M^*)\}\\ &= 1 - (\lambda(M^*) - \lambda_v(M^*))\\ &= 1 - \lambda(M^*).\end{aligned}$$

Now suppose $M = G$. Then $\text{bd } M = \emptyset$, and

$$\begin{aligned}\chi(S) &= -\sum\{(-1)^{\text{def } L}: L \in \mathcal{L}(M) \text{ with } L \neq M\}\\ &= 1 - \lambda(M).\end{aligned}$$

This completes the proof of the theorem.

5. The Stability Index $\lambda(G)$

We show in this section that every graph without cut points has non-zero stability index. By (4.1) and (4.2), this implies that a nondegenerate $M \in C(G)$ is stable if M/bd M (or M if $M = G$) has no cut points. By the reduction (3.1), this completes the proof of the stable point characterization theorem (1.1).

In considering $\lambda(G)$, we will always assume that the graph G has at least two edges. With this proviso, we obtain the following invariance property.

5.1. *Proposition. The stability index* $\lambda(G)$ *is invariant under subdivision, and therefore a topological invariant of* G.

Proof. Given an edge $e = \langle v,w \rangle$ of G, and an interior point z of e, consider the subdivision $\tilde{G}$ whose edges are $\langle v,z \rangle$, $\langle w,z \rangle$, and the edges of $G \backslash e$. For each $L \in \mathcal{L}(G)$ there are four cases to consider.

Case i): $L \supset e$. Then $L \in \mathcal{L}(\tilde{G})$, and def L has the same value with respect to either G or $\tilde{G}$.

Case ii): $L \cap e = \{v\}$. Then $L \cup \langle v,z \rangle \in \mathcal{L}(\tilde{G})$, and def L = def($L \cup \langle v,z \rangle$).

Case iii): $L \cap e = \{w\}$. This is analogous to ii).

Case iv): $L \cap e = \{v,w\}$. Then L, $L \cup \langle v,z \rangle$, and $L \cup \langle w,z \rangle$ are all members of $\mathcal{L}(\tilde{G})$, and their net contributions to $\lambda(\tilde{G})$ is the same as the contribution $(-1)^{\text{def } L}$ of L to $\lambda(G)$.

Since each member of $\mathcal{L}(\tilde{G})$ is uniquely associated with some $L \in \mathcal{L}(G)$, under one of the above cases, we have $\lambda(G) = \lambda(\tilde{G})$. By iteration, $\lambda(G)$ is invariant under all subdivisions.

5.2. *Proposition. For each vertex* v *of* G, $\lambda_v(G) = 0$.

Proof. By induction on the number n of edges in G. The statement is easily seen to be true for n = 2. Assume it is true for n = k, and consider a graph G with k + 1 edges. Given a vertex v of G, let $e = \langle v,w \rangle$ be an edge containing v. We may assume that w is not an endpoint of G, since otherwise it is clear that $\lambda_v(G) = 0$. Let G/e be the quotient graph obtained by collapsing e to a point. For each $L \in \mathcal{L}_v(G)$, there are two possibilities:

1) if $e \subset L$, then $L/e \in \mathcal{L}_v(G/e)$, and every member of $\mathcal{L}_v(G/e)$ is so obtained;

2) if $e \not\subset L$, then $G\backslash \text{int } e$ is a graph, $L \in \mathcal{L}_v(G\backslash \text{int } e)$, and every member of $\mathcal{L}_v(G\backslash \text{int } e)$ is so obtained.

Since the inductive hypothesis applies to both G/e and $G\backslash \text{int } e$ (assuming the latter is a graph), this shows that the contribution to $\lambda_v(G)$ made by those members of $\mathcal{L}_v(G)$ containing e is 0, as is the contribution made by those members not containing e. Hence $\lambda_v(G) = 0$.

5.3. *Theorem. Let* G *be a finite graph with* m *vertices and* n *edges. Then* $\lambda(G) = (-1)^{m-n}|\lambda(G)|$, *and* $\lambda(G) = 0$ *if and only if* G *has a cut point.*

Proof. By induction on m. If $m = 3$, then either $n = 2$ or $n = 3$. For $n = 2$, $G \approx$ arc and $\lambda(G) = (-1)^0 + 2(-1)^1 + (-1)^2 = 0$ (there is a degenerate large subgraph). For $n = 3$, $G \approx$ circle and $\lambda(G) = (-1)^0 + 3(-1)^1 + 3(-1)^2 = 1$. Thus the theorem holds for $m = 3$.

Assume the theorem holds for all $m < k$; we verify that it holds for $m = k$. Consider a graph G with k vertices and n edges. If G has a cut point v, then $\mathcal{L}(G) = \mathcal{L}_v(G)$, and $\lambda(G) = \lambda_v(G) = 0$ by (5.2). Now suppose G does not have a cut point. It is easily seen that there exists a finite sequence of graphs $G_1,\cdots,G_\ell = G$ such that G_1 is a cycle with 3 vertices, and each G_{i+1} is obtained from G_i either by subdivision or by addition of an edge between a pair of nonadjacent vertices. For each i, let k_i and n_i denote the number of vertices and the number of edges, respectively, of G_i. We show inductively that the theorem holds for each G_i (note that each G_i is without cut points).

As shown in the first paragraph, the theorem holds for G_1. Suppose it holds for some G_i, $i < \ell$. Thus, $\lambda(G_i) = (-1)^{k_i - n_i}|\lambda(G_i)| \neq 0$. If G_{i+1} is obtained from G_i by subdivision, then $k_{i+1} - n_{i+1} = k_i - n_i$, $\lambda(G_i) = \lambda(G_{i+1})$ by (5.1), and the theorem holds for G_{i+1}. Thus, we have only to consider the case that $G_{i+1} = G_i \cup e$, where e is an edge of G_{i+1} with endpoints v and w in G_i.

Let $G_i^* = G_{i+1}/e = G_i/\{v,w\}$ denote the quotient graph, with vertex z corresponding to $\{v,w\}$. Consider the partition $\{L \in \mathcal{L}(G_i)\colon L \cap \{v,w\} = \emptyset\} \cup \{L \in \mathcal{L}(G_i)\colon L \cap \{v,w\} \neq \emptyset\}$ of the collection $\mathcal{L}(G_i)$. The subcollection $\{L \in \mathcal{L}(G_i)\colon L \cap \{v,w\} = \emptyset\}$ corresponds to $\mathcal{L}(G_i^*)\backslash\mathcal{L}_z(G_i^*)$, hence the contribution made by this subcollection to the index $\lambda(G_i)$ is $\lambda(G_i^*) - \lambda_z(G_i^*) = \lambda(G_i^*)$, by (5.2). We have $\{L \in \mathcal{L}(G_i)\colon L \cap \{v,w\} \neq \emptyset\} = \mathcal{L}(G_{i+1})\backslash\mathcal{L}_e(G_{i+1})$, where $\mathcal{L}_e(G_{i+1}) = \{L \in \mathcal{L}(G_{i+1})\colon e \subset L\}$. Since $\mathcal{L}_e(G_{i+1})$ corresponds to $\mathcal{L}_z(G_i^*)$, its contribution to the index $\lambda(G_{i+1})$ is 0. Thus, the contribution made to $\lambda(G_{i+1})$ by the subcollection $\{L \in \mathcal{L}(G_i)\colon L \cap \{v,w\} \neq \emptyset\}$ is $\lambda(G_{i+1})$; since $n_{i+1} = n_i + 1$, its contribution to $\lambda(G_i)$ is $-\lambda(G_{i+1})$. This shows that $\lambda(G_i) = \lambda(G_i^*) - \lambda(G_{i+1})$.

Since G_i^* has $k_i - 1$ vertices, and $k_i \leq k$, our induction hypothesis gives $\lambda(G_i^*) = (-1)^{k_i - 1 - n_i}|\lambda(G_i^*)|$. In other words, if $\lambda(G_i^*)$ is non-zero, it has the opposite sign from $\lambda(G_i)$. It follows that $\lambda(G_{i+1}) = \lambda(G_i^*) - \lambda(G_i)$ is non-zero, with opposite sign from $\lambda(G_i)$. Since $k_{i+1} - n_{i+1} = k_i - n_i - 1$, the theorem holds for G_{i+1}. This completes the inductive proof.

6. Stable Points in $C_V(X)$

For a nonempty finite subset V of the Peano continuum X, let $C_V(X) = \{M \in C(X): V \subset M\}$.

6.1. *Theorem.* $M \in C_V(X)$ *is stable if and only if:*

a) M *is a finite union of free arcs;*

b) bd M *does not separate* M*;*

c) $V \subset \text{int } M$*; and*

d) *for each* $p \in \text{int } M$*, each component of* $M\backslash\{p\}$ *meets* $V \cup \text{bd } M$.

Proof. We first show the conditions are necessary. Suppose M is stable in $C_V(X)$. Then M must be a finite graph, by the same arguments used for (2.1) and (2.2). (We may assume that the finite graph $T \subset K \cap \overline{P}$ contains the finite set $V \cap P$, and that $\phi: \overline{P} \to C(K)$ is a map with $\phi(y) = \{y\}$ for each $y \in T \cup \text{bd } P$. Then the induced map $f: C(X) \to C(X)$ sends $C_V(X)$ into itself.)

Likewise, the proof of (2.3) shows that int M is a dense connected subset of M. Thus bd M is a finite set which does not separate M, and M is a finite union of free arcs in X. And, since $V \subset \text{int } f(A)$ for each $A \in C_V(X)$, we must have $V \subset \text{int } M$.

Finally, suppose there exists $p \in \text{int } M$ such that some component N of $M\backslash\{p\}$ is disjoint from $V \cup \text{bd } M$. Then N is also a component of $X\backslash\{p\}$, and $X = G \cup Y$, where $G = \overline{N} \subset M$ is a nondegenerate graph, $Y = X\backslash N$ is a continuum containing V, and $G \cap Y = \{p\}$. If $p \in V$, $C_V(X)$ is homeomorphic to $C_p(G) \times C_V(Y)$, under a homeomorphism sending M to $(G, M \cap Y)$. Then the stability of M in $C_V(X)$ implies stability of G in $C_p(G)$, contradicting the result of Lynch referred to in the proof of (2.4). If $p \notin V$, set $W = V \cup \{p\}$. Then $C_W(X)$ is a neighborhood of M in $C_V(X)$, thus M is stable in $C_W(X)$, and this leads to a contradiction as above.

Now suppose $M \in C_V(X)$ satisfies the conditions a) through d). For $V = \{v_1,\cdots,v_n\}$, consider the Peano continuum $\hat{X} = X \cup I_1 \cup \cdots \cup I_n$, where each I_i is an arc with endpoints $\{v_i,w_i\}$, $I_i \cap X = \{v_i\}$, and $I_i \cap I_j = \emptyset$ for $i \neq j$. Set $\hat{M} = M \cup J_1 \cup \cdots \cup J_n$, where each J_i is a proper subarc of I_i containing the endpoint v_i. Since $C_V(\hat{X})$ is homeomorphic to $C_V(X) \times \prod_1^n C_{v_i}(I_i)$, with $\hat{M}$ corresponding to $(M,J_1,\cdots,J_n)$, it suffices to show that $\hat{M}$ is stable in $C_V(\hat{X})$. And since $C_V(\hat{X})$ is a neighborhood of $\hat{M}$ in $C(\hat{X})$, this is equivalent to showing that $\hat{M}$ is stable in $C(\hat{X})$.

We verify that $\hat{M}$ satisfies the conditions for stability given in (1.1). Clearly, $\hat{M}$ is a finite union of free arcs in $\hat{X}$. We have $\text{bd } \hat{M} = \text{bd}_X M \cup \{w_1,\cdots,w_n\}$. Since $\text{bd}_X M$ does not separate M and is disjoint from V, $\text{bd } \hat{M}$ does not separate $\hat{M}$. To complete the argument that the quotient space $\hat{M}/\text{bd } \hat{M}$ has no cut points, we show that for each $p \in \text{int } \hat{M}$, each component $\hat{N}$ of $\hat{M}\setminus\{p\}$ meets $\text{bd } \hat{M}$. If $p \in J_i\setminus\{v_i\}$ for some i, this is clear (note that in the case $M = X$, condition d) forces V to be nondegenerate). On the other hand, if $p \in \text{int}_X M$, then each component $\hat{N}$ of $\hat{M}\setminus\{p\}$ either contains a component N of $M\setminus\{p\}$ or is of the form $J_i\setminus\{v_i\}$ for some i. In the latter case, $w_i \in \hat{N}$, and we are done. In the former case, condition d) guarantees that the component N of $M\setminus\{p\}$ meets either $bd_X M$ or V. If N meets $bd_X M \subset bd\ \hat{M}$, we are done. And if $v_i \in N$ for some i, then $w_i \in \hat{N}$. Thus in any case $\hat{N}$ meets bd $\hat{M}$.

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