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1. Introduction and Preliminaries

The results presented here are mainly of interest when applied to continua (connected, compact, metric spaces). However, connectedness plays no role in the proofs of the main theorems, whereas compactness is essential. Hence, with one exception, everything is presented in the general context of compacta (compact, metric spaces).

Notation. X and Y are compacta. "f: X $\to$ Y" means f is a function from X into Y. "$\twoheadrightarrow$" indicates an *on*to function. If A is a point set and ε is a positive number, then $N_\varepsilon(A)$ is the ε-neighborhood of A, and diam A is the diameter of A.

Definitions. A *map* is a continuous function (but the verb does not connote continuity). f: X $\to$ Y is *ε-continuous*, if, for each x in X, there is a neighborhood D of x such that $f[D] \subseteq N_\varepsilon(f(x))$. h: X $\to$ Y is an *ε-homeomorphism*, if h and h^{-1} are both ε-continuous. f: X $\twoheadrightarrow$ Y is an *ε-map*, if f is continuous and diam $(f^{-1}(y)) < \varepsilon$, for each y in f[X]. f: X $\twoheadrightarrow$ Y is a *strong ε-function*, if, for each y in Y, there is a neighborhood D of y such that diam $(f^{-1}[D]) < \varepsilon$.

[1] This paper is an extension of some of the work presented to the 1977 Spring Topology Conference at L.S.U. March 10, 1977, under the title "Refinable functions on finite graphs and on spaces having the proximate f.p.p.".

Note. f: X $\twoheadrightarrow$ Y is a strong ε-function, if it is an ε-map, since X is compact.

Definitions [3, 5]. f: X $\twoheadrightarrow$ Y is *(proximately) refinable*, if, for every positive number δ, there is a (δ-)continuous (strong) δ-function g: X $\twoheadrightarrow$ Y δ-near f, i.e., such that d(f(x),g(x)) < δ, for all x in X. f is *weakly refinable*, if f is continuous and, for every positive number δ, there is a δ-continuous function g: Y → X such that g^{-1} is δ-near f|g[Y], i.e., such that d(y,f(g(y)) < δ, for all y in Y. f is a *proximate near homeomorphism*, if, for every positive number δ, there is a δ-homeomorphism h: X $\twoheadrightarrow$ Y δ-near f.

Proximately refinable functions and proximate near homeomorphisms are easily seen to be continuous [5]. Refinable maps on compacta are proximately refinable, and they in turn are weakly refinable [5]. Proximate near homeomorphisms on compacta are clearly proximately refinable. Proximately refinable maps, on compacta with no isolated points, are proximate near homeomorphisms [5, Th. 1].

Definition [6, 7]. A metric space Z has the *proximate fixed point property (p.f.p.p.)*, if, for each positive number ε, there is a positive number δ such that every δ-continuous function f: Z → Z moves some point z less than ε (i.e., d(z,f(z)) < ε).

It is easily seen [6, p. 45] that a compactum with the p.f.p.p. also has the fixed-point property (f.p.p.).

Also, certain dog-chases-the-rabbit (or dead-end) proofs for the f.p.p. convert easily to the p.f.p.p., especially for locally connected spaces. Hence, there is a ready supply of continua with, and continua without, the p.f.p.p.

Metric spaces that are not connected clearly do not have the p.f.p.p., or even the f.p.p. However, some do have the p.f.p.p. for certain classes of functions, for example, the disjoint union of an arc and a singleton set has the p.f.p.p. for one-to-one functions.

The Sin $\frac{1}{x}$ Circle or Warsaw Circle (obtained by identifying the locally connected end point of the Sin $\frac{1}{x}$ Continuum with another end point) has the f.p.p. [2, Th. 13] (the limit bar is a "dead end"). The projection map onto the decomposition space (a simple, closed curve), resulting from "shrinking" the limiting bar to a point, is a refinable map. Hence refinable maps do not preserve the f.p.p. As we shall see, they do preserve the p.f.p.p., however.

Trees (acyclic graphs) not only have the f.p.p., but also the p.f.p.p. There is, however, a tree-like continuum [1] that does not have the f.p.p., and, hence, does not have the p.f.p.p. This shows that being $\mathcal{P}$-like, where $\mathcal{P}$ is a class of continua with the p.f.p.p., does not assure having the p.f.p.p. A compactum does, however, have the p.f.p.p., if, for every positive number ε, there is a *weakly refinable* ε-map onto some compactum with the p.f.p.p.

Lemma 1. *Suppose* Z *is a metric space and* $f: Y \to Z$ *is* $\frac{\varepsilon}{2}$*-continuous. Then there is a positive number* δ *such that* $f(g)$ *is* ε*-continuous, if* $g: X \to Y$ *is* δ*-continuous.*

This follows from the definition and the uniform ε-continuity of $\frac{\varepsilon}{2}$-continuous functions defined on compacta.

2. The Proximate Fixed-Point Property

Theorem 1. *Suppose* f: X $\twoheadrightarrow$ Y *is weakly refinable and* X *has the* p.f.p.p. *Then* Y *has the* p.f.p.p.

Proof. Let ε be a positive number. Since f is continuous, f is uniformly $\frac{\varepsilon}{4}$-continuous. Hence there is a positive number ε' such that diam $(f[D]) < \frac{\varepsilon}{2}$, if diam $D < \varepsilon'$. Since X has the p.f.p.p., there is a positive number $\delta' < \varepsilon$ such that any δ'-continuous function from X into X moves some point less than ε'. Let g: Y $\to$ X be a $\frac{\delta'}{2}$-continuous function such that g^{-1} is $\frac{\delta'}{2}$-near $f|g[Y]$. By Lemma 1, there is a positive number δ such that g(G) (and, hence, g(G(f))) is δ'-continuous, if G: Y $\to$ Y is δ-continuous. Let G: Y $\to$ Y be any δ-continuous function and let F = g(G(f)). Then F: X $\to$ X is δ'-continuous, and so, there is a x in X such that $d(x,F(x)) < \varepsilon'$. Therefore, $d(f(x),f(g(G(f(x))))) = d(f(x),f(F(x))) < \frac{\varepsilon}{2}$. Consequently $d(f(x),G(f(x))) \leq d(f(x),f(g(G(f(x))))) + d(f(g(G(f(x)))),G(f(x))) < \frac{\varepsilon}{2} + \frac{\delta'}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence G moves f(x) less than ε. It follows that Y has the p.f.p.p.

The main result of the paper follows immediately from Theorem 1.

Theorem 2. *The image of a compactum with the* p.f.p.p.. *under a refinable map, has the* p.f.p.p.

Theorem 3. *X has the* p.f.p.p., *if and only if, for every positive number* ε, *there is a weakly refinable* ε-*map from* X *onto some compactum with the* p.f.p.p.

Proof. X has the p.f.p.p. only if such map(s) exist, since the identity map on X is a weakly refinable ε-map, for each positive number ε.

Let ε be a positive number, and let f be a weakly refinable ε-map from X onto some compactum Y with the p.f.p.p. Since f is an ε-map, it is a uniform, strong ε-map, so there is a positive number ε' such that diam $(f^{-1}[D]) < \varepsilon$, if diam $D < \varepsilon'$. Since Y has the p.f.p.p., there is a positive number δ' such that any δ'-continuous function from Y into Y moves some point less than $\frac{\varepsilon'}{2}$. Since f is uniformly continuous, there is a positive number δ such that f(F) is $\frac{\delta'}{2}$-continuous, if $F: X \to X$ is δ-continuous. Let F be any δ-continuous function from X into X. By Lemma 1, there is a positive number $\gamma < \frac{\varepsilon'}{2}$ such that f(F(g)) is δ'-continuous if g is γ-continuous. Let $g: Y \to X$ be a γ-continuous function such that g^{-1} is γ-near $f|g[Y]$, and let $G: Y \to Y$ be f(F(g)). Then there is a y in Y such that $d(y,G(y)) < \frac{\varepsilon'}{2}$. But $d(y,f(g(y))) < \gamma < \frac{\varepsilon'}{2}$, since g^{-1} is γ-near $f|g[Y]$. Hence $d(f(F(g(y))),f(g(y))) = d(G(y),f(g(y))) \leq d(G(y),y) + d(y,f(g(y))) < \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'$. It follows that $d(F(g(y)), g(y)) < \varepsilon$, so X has the p.f.p.p.

An example at the end of the paper shows that the ε-map condition in Theorem 3 cannot be omitted, even if there is a refinable map from X onto a compactum with the p.f.p.p.

Definition. Let $\mathcal{P}$ be a class of compacta and X be a compactum. Then X is *(weakly, proximately) refinably $\mathcal{P}$-like*, if, for every positive number ε, there is a (weakly, proximately, respectively) refinable ε-map from X onto a member of $\mathcal{P}$.

Corollary 1. *Let $\mathcal{P}$ be the class of all compacta with the* p.f.p.p., *and let* X *be a compactum. Then* $X \in \mathcal{P}$ *if and only if* X *is weakly refinably $\mathcal{P}$-like.*

Definition [4, p. 140]. $f: X \twoheadrightarrow Y$ is 2-*refinable*, if, for every positive number ε, there is a refinable ε-map from X onto Y that is ε-near f.

Corollary 2. *If* $f: X \twoheadrightarrow Y$ *is* 2-*refinable then* X *has the* p.f.p.p. *if and only if* Y *has the* p.f.p.p.

3. The Proximate Fixed-Point Property for One-to One or Onto Functions

Definition. X has the *p.f.p.p. for* a class $\mathcal{J}$ of functions, if, for every positive number ε, there is a positive number δ such that each δ-continuous function $f: X \to X$ in $\mathcal{J}$ moves some point less than ε.

Theorem 4. *Suppose* $f: X \twoheadrightarrow Y$ *is a proximate near homeomorphism. Then* Y *has the* p.f.p.p. *for any one of the following classes of functions, if* X *has it for the same class of functions.*

(1) *Functions.*

(2) *One-to-one functions.*

(3) *Onto functions.*

(4) *One-to-one, onto functions.*

Proof. We prove (4). The other proofs are almost identical (and all are similar to the proof of Theorem 1).

Assume X has the p.f.p.p. for one-to-one, onto functions, and let ε be a positive number. Then there is a positive number ε' such that diam $(f[D]) < \frac{\varepsilon}{3}$, if diam $D < \varepsilon'$. Since X has the p.f.p.p. for one-to-one functions onto X, there is a positive number $\delta' < \varepsilon$ such that every δ'-continuous, one-to-one function from X onto X moves some point less than ε'. Let h be a $\frac{\delta'}{4}$-homeomorphism $\frac{\delta'}{4}$-near f. By Lemma 1, there is a positive number δ such that $h^{-1}(G)$ is $\frac{\delta'}{2}$-continuous, if $G: Y \to Y$ is δ-continuous. Let G be any δ-continuous, one-to-one function from Y onto Y. By Lemma 1, there is a positive number $\delta'' < \frac{\varepsilon}{3}$ such that $h^{-1}(G(g))$ is δ'-continuous if g is δ''-continuous. Let g be a δ''-homeomorphism δ''-near f. Let $F = h^{-1}(G(g))$. Then F is a δ'-continuous, one-to-one function from X onto X. Hence there is an x in X such that $d(x,F(x)) < \varepsilon'$. Therefore $d(f(x),f(F(x))) < \frac{\varepsilon}{3}$. Consequently, $d(g(x),G(g(x))) = d(g(x),h(F(x)) \leq d(g(x),f(x)) + d(f(x),f(F(x))) + d(f(F(x)), h(F(x)) < \delta'' + \frac{\varepsilon}{3} + \frac{\delta'}{4} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, i.e., G moves g(x) less than ε. It follows that Y has the p.f.p.p. for one-to-one, onto functions.

The theorem that relates to Theorem 3, as Theorem 4 does to Theorem 1, can be proved using Lemma 2 and ideas from the proofs of Theorems 3 and 4.

Lemma 2. *Let* $f: X \twoheadrightarrow Y$ *be an* ε*-map. Then there is a positive number* δ *such that, if* g *and* h *are* δ*-homeomorphisms* δ*-near* f*, then* $d(g^{-1}(y),h^{-1}(z)) < \varepsilon$*, if* $d(y,z) < \delta$.

Proof. Assume the lemma is false. For each positive integer i let g_i and h_i be $\frac{1}{i}$-homeomorphisms $\frac{1}{i}$-near f, and let y_i and z_i be points in Y such that $d(y_i, z_i) < \frac{1}{i}$, but $d(g_i^{-1}(y), h_i^{-1}(z)) \geq \varepsilon$. Assume without loss of generality that $y_1, y_2, \cdots$ and $z_1, z_2, \cdots$ converge to y_0, that $g_1^{-1}(y_1), g_2^{-1}(y_2), \cdots$ converges to x_0, and that $h^{-1}(z_1)$, $h^{-1}(z_2), \cdots$ converges to t_0. Then $d(x_0, t_0) \geq \varepsilon$. Let $x_i = g_i^{-1}(y_i)$, for $i = 1,2,\cdots$. Then $d(y_i, f(x_0)) = d(g_i(x_i), f(x_0)) \leq d(g_i(x_i), f(x_i)) + d(f(x_i), f(x_0)) \leq \frac{1}{i} + d(f(x_i), f(x_0))$. Since $f(x_1), f(x_2), \cdots$ converges to $f(x_0)$, $y_1, y_2, \cdots$ also converges to $f(x_0)$. Similarly $z_1, z_2, \cdots$ converges to $f(t_0)$. Hence $f(x_0) = y_0 = f(t_0)$, and f is not an ε-map, since $d(x_0, t_0) \geq \varepsilon$.

Since our assumption leads to a contradiction, the lemma is true.

Theorem 5. *Suppose* $\mathcal{J}$ *is one of the following classes of functions. Then* X *has the* p.f.p.p. *for* $\mathcal{J}$, *if and only if, for every positive number* ε, *there is an* ε*-map that is a proximate near homeomorphism from* X *onto some compactum with the* p.f.p.p. *for* $\mathcal{J}$.

(1) *Functions.*

(2) *One-to-one functions.*

(3) *Onto functions.*

(4) *One-to-one, onto functions.*

Corollary 3. *Suppose* $\mathcal{J}$ *is one of the classes of functions in Theorem* 5, X *is a continuum, and* $\mathcal{P}$ *is the class of continua that have the* p.f.p.p. *for* $\mathcal{J}$. *Then* X $\in \mathcal{P}$, *provided*

(1) *there is a proximately refinable map from some member of* $\mathcal{P}$ *onto* X, *or* (2) *for each positive number* ε, *there is a proximately refinable* ε*-map from* X *onto some member of* $\mathcal{P}$.

Proof. By [5, Th. 1], a proximately refinable map on a continuum is a proximate near homeomorphism. Hence (1) follows from Theorem 4 and (2) follows from Theorem 5.

4. The Fixed-Point Property

In the proof of Theorem 1, the Function F is continuous if G and g are continuous. In the proof of Theorem 3, the function G is continuous if F and g are continuous. Also, a compactum Z has the f.p.p., if, for every positive number ε, every continuous function $f: Z \to Z$ moves some point less than ε. Theorems 6 and 7 follow from those observations.

Definition. $f: X \twoheadrightarrow Y$ is *continuously weakly refinable*, if f is continuous and, for every positive number ε, there is a continuous function $g: Y \to X$ such that g^{-1} is ε-near $f|g[Y]$.

Theorem 6. *If* $f: X \twoheadrightarrow Y$ *is continuously weakly refinable and* X *has the* f.p.p., *then* Y *has the* f.p.p.

Theorem 6 is a generalization of the well-known (and easily proved) result that any retract of a continuum with the f.p.p. also has the f.p.p.

Theorem 7. X *has the* f.p.p., *if and only if, for every positive number* ε, *there is a continuously weakly refinable* ε*-map from* X *onto some compactum with the* f.p.p.

The following example (suggested by the referee) shows that the ε-map condition cannot be omitted in Theorem 3 or in Theorem 7 even if the function is required to be refinable.

Example. A refinable, continuously weakly refinable map f from a continuum X that does not have the p.f.p.p., or even the f.p.p., onto a continuum Y with the p.f.p.p. and therefore with the f.p.p.

Let Z_0 be a tree-like continuum that does not have the f.p.p. [1]. Since Z_0 is tree-like, for each positive integer n there is a tree Z_n and a $\frac{1}{n}$-map $f_n\colon Z_0 \twoheadrightarrow Z_n$. Let p_0 be any point of Z_0, and let $p_n = f_n(p_0)$, for $n = 1,2,\cdots$. Let $Z_0, Z_1, Z_2, \cdots$ be chosen so that (1) $Z_i \cap Z_j = \{p_i\} = \{p_j\}$, for i and j any distinct nonnegative integers, (2) diam $(Z_i) < \frac{1}{i}$, for $i = 1,2,\cdots$, and, therefore, (3) $Z_1, Z_2, \cdots$ converges to $\{p_0\} = \{p_1\} = \{p_2\} = \cdots$. Let $f\colon X \twoheadrightarrow Y$ be the constant map p_0 on Z_0 and be the identity map on $X \setminus Z_0$. For $n = 1,2,\cdots$, let $g_n\colon X \twoheadrightarrow Y$ be the identity map on $X \setminus (Z_0 \cup Z_n)$, be the constant map p_0 on Z_n, and equal f_n on Z_0. Then g_n is a $\frac{1}{n}$-map $\frac{1}{n}$-near f. Hence f is refinable. Let g be the identity map on Y. Then g is a continuous function from Y into X such that g^{-1} is ε-near $f|g[Y]$. Hence f is continuously weakly refinable. Z_0 is a retract of X, and Z_0 does not have the f.p.p. Therefore X does not have the f.p.p., since any retract of a continuum with the f.p.p. also has the f.p.p. Since Y is a dendrite, it is an absolute retract, and therefore has the p.f.p.p. [6, Th. 6, p. 45].

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