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H-P A Künzi

0. Introduction

Let X be a (T_0-) space that admits a coarsest quasi-uniformity $\mathcal{U}$. In this paper we consider the bicompletion $(\tilde{X},\tilde{\mathcal{U}})$ of the quasi-uniform space $(X,\mathcal{U})$.

In [8] we have shown that the class of topological spaces that admit a coarsest quasi-uniformity generalizes the class of locally compact topological spaces (in fact, the class of core-compact spaces) in a natural way. Let us also observe that each topological space that admits a coarsest quasi-uniformity is locally bounded. For each point of a non-compact space X that admits a coarsest quasi-uniformity has a neighbourhood G that is relatively handy in $X\setminus\overline{\{y\}}$ for some $y \in X$ [8, Proposition 2] and, thus, G is relatively compact in X [8, Remark 2(d)]. (Recall that a topological space X is called *locally bounded* [7], if each of its points has a neighbourhood that is bounded (=relatively compact, in the sense of section 1.2) in X.)

In this note we show that for topological (T_0-) spaces admitting a coarsest quasi-uniformity, the bicompletion of the coarsest quasi-uniformity yields a super-sober locally compact compactification (Lemma 5), which is functorial for maps that are quasi-uniformly continuous with respect to the coarsest compatible quasi-uniformities on the topological spaces under consideration (Proposition 5). Hence the full subcategory $\mathcal{A}$ of locally compact strongly sober spaces of the category $\mathcal{B}$ of topological T_0-spaces admitting a coarsest quasi-uniformity and maps that are quasi-uniformly continuous with respect to these quasi-uniformities is epireflective in $\mathcal{B}$ (see also [10]; on the other hand compare with [4, Remark 6.10]).

If X is a locally compact (sober) space and $\mathcal{U}$ is the coarsest quasi-uniformity that X admits, then the *bicompletion* $(\tilde{X},\mathcal{T}(\tilde{\mathcal{U}}))$ yields the *pseudo-spectrum* ψX of the lattice of open sets of X (see [6]) and $(\tilde{X},\mathcal{T}(\tilde{\mathcal{U}}^*))$ is the so-called *Fell compactification* $\mathcal{H}(X)$ of X [1]. In particular, if the lattice of the open sets of X is algebraic, then $\mathcal{U}$ is transitive, and thus, ψX is a spectral space (compare [4, p. 114]; [9, Proposition 1]), because the bicompletion of a transitive quasi-uniformity is transitive.

We recall that each *perfect* continuous ([4, p. 101], compare [6, Remark 1.3]) map between two locally compact spaces is quasi-uniformly continuous in the sense mentioned above [9, Remark 2]. The converse obtains if the domain of the map is strongly sober, but it is wrong in general, as the canonical embedding of a locally compact non-compact T_0-space into the bicompletion of its coarsest quasi-uniformity shows (compare [9, Remark 2]).

The connections between the three different constructions mentioned above are rather transparent, because for a topological space X that admits a coarsest quasi-uniformity $\mathcal{U}$ the elements of the "standard" bicompletion $(\tilde{X},\tilde{\mathcal{U}})$ of $(X,\mathcal{U})$ (namely the minimal $\mathcal{U}^*$-Cauchy filters) are in an obvious one-to-one correspondence with the limit sets of the primitive filters on X (Corollary 1). Furthermore the pseudo-prime elements [4, p. 68 or 6, p. 291] of the lattice of open sets of a topological space X are exactly the (set-theoretical) complements of the limit sets of the open (proper) prime filters (equivalently, the complements of the limit sets of the primitive filters, see section 1.3; compare [4, Remark 1.17]) on X.

We remark that in [4] R.-E. Hoffmann studies $\mathcal{H}(X)$ using the methods of the theory of continuous lattices. Although we do not make any use of the theory of continuous lattices in this paper, readers familiar with this theory should have no problems in translating its content into a more lattice-theoretical terminology. Several of our results essentially show that some general facts about quasi-uniformities, when specialized to the coarsest compatible quasi-uniformity on a (sober) core-compact space, are equivalent to important results of the theory of (distributive) continuous lattices. In fact, this is not very surprising, because we can easily describe the coarsest compatible quasi-uniformity $\mathcal{U}$ on a core-compact space using its way-below relation [9, Lemma 5]. Furthermore, if X is locally compact, then, obviously, $\mathcal{T}(\mathcal{U}^{-1})$ is the cocompact and $\mathcal{T}(\mathcal{U}^*)$ is the patch topology of X [6, Definition 1.5]. The idea of the coarsest compatible quasi-uniformity (quasi-proximity) on a core-compact space also clarifies some obvious connections between the theory of the Fell compactification and the theory of Hausdorff compactifications (see [4, p. 114]). Of course, some readers may prefer, instead of considering the coarsest compatible quasi-proximity on a core-compact space, to study the corresponding notion of a coarsest approximating strong inclusion-relation ([2, Proposition 1.24], compare section 1.2) on a continuous frame. (Here a strong inclusion-relation $\ll$ on a frame L is said to be approximating if $\sup\{a \in L \mid a \ll b\} = b$ for each $b \in L$.)

Since several important results about locally compact and locally bounded spaces also hold for topological spaces that admit a coarsest quasi-uniformity (see [8]), it may be appropriate to close this introduction with a counterexample, which shows that an open continuous image of a space admitting a coarsest quasi-uniformity need not admit a coarsest quasi-uniformity. (Identify in the Example of [8] for each $\mathcal{J} \in \mathcal{P}$ the two points $(\mathcal{J},1)$ and $(\mathcal{J},2)$. The corresponding quotient space is a Hausdorff space that is not locally compact and, thus, does not admit a coarsest quasi-uniformity [8, Corollary 3], although the quotient map is open.)

In this paper a compact space need not be a Hausdorff space.

1. Definitions and Notation

In section 1.1 we recall several concepts of the theory of quasi-uniform spaces. Section 1.2 contains the facts that we will need about the class of topological spaces that admit a coarsest quasi-uniformity. Finally, in section 1.3 we discuss the basic notions of the theory of the Fell compactification.

1.1 We begin by recalling some basic results of the theory of quasi-uniform spaces. We refer the reader to [2] for the explanation of notions that are not explained here.

Let X be a set. A filter $\mathcal{U}$ on $X \times X$ is called a *quasi-uniformity* on X if (a) each member of $\mathcal{U}$ is a reflexive relation on X and (b) if $U \in \mathcal{U}$ then $V \circ V \subset U$ for some $V \in \mathcal{U}$. If $U \in \mathcal{U}$ and $x \in X$, then $U(x)$ denotes the set $\{y \in X | (x,y) \in U\}$. The topology $\mathcal{J}(\mathcal{U}) = \{G \subset X|$ for each $x \in G$ there is a $U \in \mathcal{U}$ such that $U(x) \subset G\}$ is said to be *induced* by $\mathcal{U}$ on X. If $\mathcal{U}$ is a quasi-uniformity on X, then $\mathcal{U}^{-1}$ denotes the quasi-uniformity $\{U^{-1}|U \in \mathcal{U}\}$ on X. Moreover, $\mathcal{U}^*$ denotes the uniformity generated by $\{U \cap U^{-1}|U \in \mathcal{U}\}$ on X. A quasi-uniformity $\mathcal{U}$ on X is said to be *bicomplete (totally bounded)* if $\mathcal{U}^*$ is complete (totally bounded). Observe that a filter $\mathcal{F}$ on X is a $\mathcal{U}^*$-*Cauchy filter* if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \times F \subset U$ [2, Proposition 3.2]. A $\mathcal{U}^*$-Cauchy filter is called *minimal* provided that it contains no $\mathcal{U}^*$-Cauchy filter other than itself. If $(X,\mathcal{U})$ is a quasi-uniform space, then a base $\mathcal{B}$ for a quasi-uniformity $\tilde{\mathcal{U}}$ can be defined on the set $\tilde{X}$ of all minimal $\mathcal{U}^*$-Cauchy filters as follows: If $U \in \mathcal{U}$ set $\tilde{U} = \{(\mathcal{F},\mathcal{G}) \in \tilde{X} \times \tilde{X}|$ there is an $F \in \mathcal{F}$ and a $G \in \mathcal{G}$ such that $F \times G \subset U\}$.

Note that $(U \cap V)^{\sim} = \tilde{U} \cap \tilde{V}$ if $U,V \in \mathcal{U}$. Set $\mathcal{B} = \{\tilde{U}|U \in \mathcal{U}\}$.

The quasi-uniform space $(\tilde{X},\tilde{\mathcal{U}})$ is bicomplete and $(\tilde{X},\mathcal{J}(\tilde{\mathcal{U}}))$ is a T_0-space [2, Proof of Theorem 3.33]. We recall also that $(\mathcal{U}^{-1})^{\sim} = \tilde{\mathcal{U}}^{-1}$ and that $(\mathcal{U}^*)^{\sim} = (\tilde{\mathcal{U}})^*$ [2, 3.37]. Furthermore the topology $\mathcal{J}(\tilde{\mathcal{U}}^*)$ is compact if and only if $\mathcal{U}$ is totally bounded [see 2, Proof of Proposition 3.36].

It is known that if $\mathcal{U}$ is totally bounded, then $(\tilde{X},\mathcal{T}(\tilde{\mathcal{U}}))$ is strongly sober and locally compact [9, see Proof of Theorem 1]. We recall that a topological space is called *strongly sober* if it is compact and super sober [4, p. 73]. A topological space is called *super sober* if the convergence set of each convergent ultrafilter is the closure of a unique singleton. A topological space is said to be *locally compact* if each point has a neighbourhood base consisting of compact sets.

If $(X,\mathcal{U})$ is a quasi-uniform space such that $(X,\mathcal{T}(\mathcal{U}))$ is a T_0-space, then $j\colon (X,\mathcal{U}) \to (\tilde{X},\tilde{\mathcal{U}})$ defined by $j(x) = \eta^*(x)$ for each $x \in X$ (where $\eta^*(x)$ denotes the $\mathcal{T}(\mathcal{U}^*)$-neighbourhood filter of x) is a quasi-uniform embedding [see 2, Theorem 3.33]. As usual, we will identify j(X) with X in this case and call $(\tilde{X},\tilde{\mathcal{U}})$ the bicompletion of $(X,\mathcal{U})$.

It is known that if $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are quasi-uniform spaces such that $(X,\mathcal{T}(\mathcal{U}))$ and $(Y,\mathcal{T}(\mathcal{V}))$ are T_0-spaces and $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a quasi-uniformly continuous map, then there is a (unique) quasi-uniformly continuous map $\tilde{f}\colon (\tilde{X},\tilde{\mathcal{U}}) \to (\tilde{Y},\tilde{\mathcal{V}})$ that extends f [see 2, Theorem 3.29 and Proposition 1.14].

1.2 In this section we recall some notions of the theory of the topological spaces that admit a coarsest quasi-uniformity.

Note first that since each compatible quasi-uniformity on a topological space contains a compatible totally bounded quasi-uniformity, a topological space admits a coarsest quasi-uniformity if and only if it admits a coarsest quasi-proximity.

Let A and B be subsets of a topological space X. We say that A is *relatively handy* in B (with respect to X) [8], if for each ultrafilter $\mathcal{U}$ on X that contains A there exists a finite collection $\mathcal{M}$ of open subsets of X such that $\cap\mathcal{M} \subset B$ and such that each member of $\mathcal{M}$ contains a limit point of $\mathcal{U}$. If A is relatively handy in B, we write $A < B$. If it is not clear from the context which space we are considering, we will write $A < B(X)$.

We will use the convention that $\cap\emptyset = X$.

Note that if X is a topological space, then we have that $X < X$ and that $\emptyset < \emptyset$.

It is known that a topological space X admits a coarsest quasi-proximity if and only if every open subset G of X is the union of open sets that are relatively handy in G [8, Proposition 2].

If a topological space X admits a coarsest quasi-proximity δ, then δ can be described as follows: For $A,B \subset X$ we have that $A\bar{\delta}B$ if and only if $A < X\backslash B$ [8, Corollary 1].

Note that if X is a topological space that admits a coarsest quasi-proximity, then $\{(X\backslash A) \times X \cup X \times B \mid A < B(X)\}$ is a subbase for the coarsest compatible quasi-uniformity on X (see [2, Theorem 1.33]).

A subset G' of a set G in a topological space X is called *relatively compact* in G, if each ultrafilter on X that contains G' has a limit point in G. Obviously, if G is open and G' is relatively compact in G, then $G' < G$. A topological space X is called *core-compact* [5], if each open set G of X is the union of open subsets that are relatively compact in G.

It is an immediate consequence of Lemma 5 of [9] that for a core-compact space X we have $A < B$ if and only if there is a finite collection of open sets $\mathcal{M}$ of X such that, for each $M \in \mathcal{M}$, A is relatively compact in M and such that $\cap\mathcal{M} \subset B$.

Clearly, each core-compact space admits a coarsest quasi-uniformity and each locally compact space is core-compact. Although none of these implications is reversible [see 5, 8], the three classes of topological spaces under consideration coincide in the class of super-sober spaces [8, Corollary 3].

It is known (and easy to see) that the coarsest compatible quasi-uniformity on a locally compact space X is generated by the subbase $\{(X\setminus K) \times X \cup X \times G \mid K \subset G$, K is compact and G is open in X$\}$ [9, Proof of Proposition 2].

Finally, let us mention that if $\mathcal{U}$ is a totally bounded quasi-uniformity on a set X, then $\tilde{\mathcal{U}}$ is the coarsest quasi-uniformity that the locally compact space $(\tilde{X},\mathcal{T}(\tilde{\mathcal{U}}))$ admits [9, Proposition 2].

1.3 In this section we discuss several notions of the theory of the Fell compactification of a locally compact topological space.

We will denote the convergence set of a filter base $\mathcal{F}$ on a topological space by lim $\mathcal{F}$.

Recall that a net (or a filter) $\mathcal{F}$ on a topological space is called *primitive*, if each cluster point of $\mathcal{F}$ is a limit point of $\mathcal{F}$ [1]. Clearly, each ultrafilter on a topological space is primitive.

In the following, $\mathcal{L}$ will denote the set of the convergence sets of the primitive filters on a topological space X.

Let X be a locally compact space. Fell studies in [1] the topology $\mathcal{J}$ on $\mathcal{L}$ that is generated by the subbase consisting of the union of the two collections $\{\langle X,G\rangle | G$ is open in $X\}$ (where $\langle X,G\rangle = \{F \in \mathcal{L} | G \cap F \neq \emptyset\}$) and $\{\langle X\backslash K\rangle | K$ is compact in $X\}$ (where $\langle X\backslash K\rangle = \{F \in \mathcal{L} | K \cap F = \emptyset\}$). He shows that $(\mathcal{L},\mathcal{J})$ is a compact Hausdorff space and calls $(\mathcal{L},\mathcal{J})$ the Hausdorff compactification $\mathcal{H}(X)$ of X. He observes that for a locally compact non-compact Hausdorff space X, $\mathcal{H}(X)$ is the Alexandroff one-point-compactification of X where each point x of X is identified with $\{x\} \in \mathcal{L}$ and $\emptyset$ is adjoined as the point-at-infinity.

We close this section with some remarks on the set $\mathcal{L}$ of the convergence sets of the primitive filters on a topological space X.

Note that if $\mathcal{U}$ is an ultrafilter on a topological space X and $\mathcal{F}$ is a primitive filter on X contained in $\mathcal{U}$, then $\mathcal{U}$ and $\mathcal{F}$ have the same limit points. Hence $\mathcal{L}$ is also the set of the convergence sets of the ultrafilters on X. Furthermore, note that if $\mathcal{U}$ is an ultrafilter on a topological space X, then, clearly, $\mathcal{R} = \{G \in \mathcal{U} | G$ is open in $X\}$ is an open prime filter on X and $\lim \mathcal{U} = \lim \mathcal{R}$. On the other hand, if $\mathcal{R}$ is an open prime filter on a topological space X, then $\mathcal{F} = \mathcal{R} \cup \{X\backslash G | G$ is open in X and $G \notin \mathcal{R}\}$ has the finite intersection property. If $\mathcal{U}$ is an ultrafilter on X that contains $\mathcal{F}$, then lim $\mathcal{R}$ = lim $\mathcal{U}$. We conclude that $\mathcal{L}$ is also the set of the convergence sets of the open prime filters on X.

These observations can be used to give different, but equivalent formulations of the definition of the relation $<$ on the power set of a topological space. Since one of the basic notions in Hoffmann's paper [4] is the notion of an open prime filter, we would like to mention the following result.

Lemma 0. *If* X *is a topological space and* G_1 *and* G_2 *are open subsets of* X, *then the following two conditions are equivalent:*

(i) $G_1 < G_2$.

(ii) *Whenever* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are two open prime filters on* X *such that* lim $\mathcal{R}_1 \subset$ lim $\mathcal{R}_2$ *and* $G_1 \in \mathcal{R}_1$, *then* $G_2 \in \mathcal{R}_2$.

Proof. Assume that $G_1 < G_2$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two open prime filters on X such that lim $\mathcal{R}_1 \subset$ lim $\mathcal{R}_2$ and $G_1 \in \mathcal{R}_1$. Consider the case that $G_2 \neq X$. Let $\mathcal{U}$ be an ultrafilter on X containing $\mathcal{R}_1$ such that lim $\mathcal{R}_1$ = lim $\mathcal{U}$. Then there is a nonempty finite collection $\mathcal{M}$ of open subsets of X such that $\cap\mathcal{M} \subset G_2$ and such that each member of $\mathcal{M}$ contains a limit point of $\mathcal{U}$. Since lim $\mathcal{R}_1 \subset$ lim $\mathcal{R}_2$, we have that $G_2 \in \mathcal{R}_2$.

In order to prove the converse, assume that G_1 and G_2 satisfy condition (ii), but that they do not satisfy condition (i). Then there is an ultrafilter $\mathcal{U}$ on X containing G_1 such that $\mathcal{F} = \{G \subset X | G$ is open in X and G contains a limit point of $\mathcal{U}\} \cup \{X \setminus G_2\}$ has the finite intersection property. If $\mathcal{V}$ is an ultrafilter on X containing $\mathcal{F}$, then $\mathcal{U}' = \{G \in \mathcal{U} | G$ is open in X$\}$ and $\mathcal{V}' = \{G \in \mathcal{V} | G$ is open in X$\}$ are open prime filters on X such that $G_1 \in \mathcal{U}'$, but $G_2 \notin \mathcal{V}'$. However, $\lim \mathcal{U}' \subset \lim \mathcal{V}'$. We have reached a contradiction.

2. Preliminaries

If $(X, \mathcal{U})$ is a quasi-uniform space, then $\leq = \cap \tilde{\mathcal{U}}$ is a partial order on $\tilde{X}$ and $(\tilde{X}, \mathcal{J}(\tilde{\mathcal{U}}^*), \cap \tilde{\mathcal{U}})$ is a partially ordered topological space [2, p. 81].

Let X be a topological space that admits a coarsest quasi-uniformity $\mathcal{V}$. After some preparation we will be able to characterize the partial order $\cap \tilde{\mathcal{V}}$ on $\tilde{X}$ in Lemma 4. Proposition 1, which is the main result in this section, will show the importance of this characterization in the theory of the topological spaces that admit a coarsest quasi-uniformity.

Lemma 1. *Let* X *be a topological space and let* $\mathcal{W}$ *be a compatible quasi-uniformity on* X. *If* $\mathcal{F}_1, \mathcal{F}_2 \in \tilde{X}$ *and* $(\mathcal{F}_1, \mathcal{F}_2) \in \cap \tilde{\mathcal{W}}$, *then* $\lim \mathcal{F}_1 \subset \lim \mathcal{F}_2$.

Proof. Consider the case that $\lim \mathcal{F}_1 \neq \emptyset$. Let $x \in \lim \mathcal{F}_1$ and let $U \in \mathcal{W}$. There is a $V \in \mathcal{W}$ such that $V^2 \subset U$. Since $(\mathcal{F}_1, \mathcal{F}_2) \in \tilde{V}$, there are $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1 \times F_2 \subset V$. Then $V^{-1}(F_1) \times F_2 \subset U$ and $x \in V^{-1}(F_1)$. Hence $F_2 \subset U(x)$ and $U(x) \in \mathcal{F}_2$. We conclude that $x \in \lim \mathcal{F}_2$. Thus $\lim \mathcal{F}_1 \subset \lim \mathcal{F}_2$.

Lemma 2. Let $\mathcal{V}$ be a totally bounded quasi-uniformity on a set X *and let $\mathcal{G}$ be a $\mathcal{V}^*$-Cauchy filter on* X. *Denote the quasi-proximity induced by $\mathcal{V}$ on* X *by* δ.

(a) *If* $A\bar{\delta}B$, *then* $X \setminus A \in \mathcal{G}$ *or* $X \setminus B \in \mathcal{G}$.

(b) $\mathcal{G}$ *is a* $\mathcal{T}(\mathcal{V})$*-primitive filter on* X.

Proof. If $A\bar{\delta}B$, then there is a $C \subset X$ such that $A\bar{\delta}C$ and $X \setminus C\bar{\delta}B$. Set $U = (X \setminus A) \times X \cup X \times (X \setminus C)$ and $H = C \times X \cup X \times (X \setminus B)$. Then U and H belong to $\mathcal{V}$ [2, Theorem 1.33]. Since $\mathcal{G}$ is a $\mathcal{V}^*$-Cauchy filter, there is an $x \in X$ such that $[(H \cap U) \cap (H \cap U)^{-1}](x) \in \mathcal{G}$. If $x \in C$, then $U^{-1}(x) \subset X \setminus A$. If $x \notin C$, then $H(x) \subset X \setminus B$. We conclude that $X \setminus A$ or $X \setminus B$ belong to $\mathcal{G}$.

In order to prove part (b) we assume that x is a $\mathcal{T}(\mathcal{V})$-cluster point of $\mathcal{G}$ that is not a $\mathcal{T}(\mathcal{V})$-limit point of $\mathcal{G}$. Then there is a $G \in \mathcal{T}(\mathcal{V})$ such that $x \in G \notin \mathcal{G}$. There is a $\mathcal{T}(\mathcal{V})$-open set D in X such that $x \in D$ and $D\bar{\delta}X \setminus G$. By part (a) we conclude that $X \setminus D \in \mathcal{G}$. Since $x \in D$, x is not a $\mathcal{T}(\mathcal{V})$-cluster point of $\mathcal{G}$, a contradiction.

We observe that if $(X, \mathcal{U})$ is a quasi-uniform space and $x \in X$, then the $\mathcal{T}(\mathcal{U})$-convergence set of the $\mathcal{T}(\mathcal{U}^*)$-neighbourhood filter $\eta^*(x)$ of x is $cl_{\mathcal{T}(\mathcal{U})}\{x\}$ [see e.g. 9, Lemma 1].

Lemma 3. Let $\mathcal{W}$ be a compatible totally bounded quasi-uniformity on a topological space X. *Let* $h\colon \tilde{X} \to \mathcal{L}$ *be defined by* $h(\mathcal{F}) = \lim \mathcal{F}$ *for each* $\mathcal{F} \in \tilde{X}$. *Then* h *is surjective.*

Proof. By Lemma 2 h is well-defined. Let $A \in \mathcal{L}$. Then $A = \lim \mathcal{F}$ for some primitive filter $\mathcal{F}$ on X. Let $\mathcal{U}$ be an ultrafilter on X that contains $\mathcal{F}$. Since $\mathcal{W}$ is totally bounded, $\mathcal{U}$ is a $\mathcal{W}$*-Cauchy filter on X [2, Proposition 3.14]. Hence it contains a minimal $\mathcal{W}$*-Cauchy filter $\mathcal{G}$ [2, Proposition 3.30]. Since $\mathcal{F}$ and $\mathcal{G}$ are primitive, lim $\mathcal{F}$ = lim $\mathcal{U}$ = lim $\mathcal{G}$. We conclude that h is surjective.

Lemma 4. *Let* X *be a topological space that admits a coarsest quasi-uniformity* $\mathcal{V}$. *If* $\mathcal{F}_1, \mathcal{F}_2 \in \tilde{X}$, *then* $(\mathcal{F}_1, \mathcal{F}_2) \in \cap\tilde{\mathcal{V}}$ *if and only if* lim $\mathcal{F}_1 \subset$ lim $\mathcal{F}_2$.

Proof. Because of Lemma 1 it remains to prove that if $\mathcal{F}_1, \mathcal{F}_2 \in \tilde{X}$ and lim $\mathcal{F}_1 \subset$ lim $\mathcal{F}_2$, then $(\mathcal{F}_1, \mathcal{F}_2) \in \cap\tilde{\mathcal{V}}$. Let $A < B(X)$ and let $H = (X\backslash A) \times X \cup X \times B$. If $X\backslash A \in \mathcal{F}_1$, then $(\mathcal{F}_1, \mathcal{F}_2) \in \tilde{H}$. If $X\backslash A \notin \mathcal{F}_1$, then $\mathcal{F}_1 \cup \{A\}$ has the finite intersection property. Let $\mathcal{U}$ be an ultrafilter that contains $\mathcal{F}_1 \cup \{A\}$. Then lim $\mathcal{U}$ = lim $\mathcal{F}_1$, because $\mathcal{F}_1$ is primitive. Since $A \in \mathcal{U}$, there is a finite collection $\mathcal{M}$ of open subsets of X such that each element of $\mathcal{M}$ contains a limit point of $\mathcal{U}$ and such that $\cap\mathcal{M} \subset B$. Since lim $\mathcal{F}_1 \subset$ lim $\mathcal{F}_2$, we conclude that $B \in \mathcal{F}_2$. Hence $(\mathcal{F}_1, \mathcal{F}_2) \in \tilde{H}$. Since $\mathcal{V}$ is generated by the subbase $\{(X\backslash A) \times X \cup X \times B | A < B(X)\}$, we have shown that $(\mathcal{F}_1, \mathcal{F}_2) \in \cap\tilde{\mathcal{V}}$.

Corollary 1. *Let* X *be a topological space that admits a coarsest quasi-uniformity* $\mathcal{V}$. *Then the map* h: $\tilde{X} \to \mathcal{L}$ *defined by* h($\mathcal{F}$) = lim $\mathcal{F}$ *for each* $\mathcal{F} \in \tilde{X}$ *is a bijection.*

Proof. By Lemma 3 it suffices to show that h is injective. Let $\mathcal{F}_1, \mathcal{F}_2 \in \tilde{X}$ such that lim $\mathcal{F}_1$ = lim $\mathcal{F}_2$. By Lemma 4 we have that $(\mathcal{F}_1, \mathcal{F}_2) \in \cap\tilde{\mathcal{V}}$ and that $(\mathcal{F}_2, \mathcal{F}_1) \in \cap\tilde{\mathcal{V}}$. Since $\cap\tilde{\mathcal{V}}$ is a partial order on $\tilde{X}$, we have that $\mathcal{F}_1 = \mathcal{F}_2$.

Corollary 2. *Let* X *be a topological space that admits a coarsest quasi-uniformity* $\mathcal{V}$. *A filter* $\mathcal{F}$ *on* X *is a* $\mathcal{V}^*$*-Cauchy filter if and only if it is* $\mathcal{T}(\mathcal{V})$*-primitive.*

Proof. By Lemma 2 every $\mathcal{V}^*$-Cauchy filter is $\mathcal{T}(\mathcal{V})$-primitive. In order to prove the converse we assume that $\mathcal{F}$ is a $\mathcal{T}(\mathcal{V})$-primitive filter on X. In the proof of Lemma 4 set $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$ (instead of assuming that $\mathcal{F}_1, \mathcal{F}_2 \in \tilde{X}$). Clearly, the proof shows that $\mathcal{F}$ is a $\mathcal{V}^*$-Cauchy filter.

In the following h will always denote the map defined in Corollary 1. Note that Corollary 1 says that if X is a topological space that admits a coarsest quasi-uniformity $\mathcal{V}$, then we can identify $\tilde{X}$ with $\mathcal{L}$ in a natural way. We will make use of this possibility by assuming in the following that the quasi-uniformity $\tilde{\mathcal{V}}$ is defined on $\mathcal{L}$ (instead of $\tilde{X}$).

The results mentioned in the introduction yield the following proposition.

Lemma 5. *Let* X *be a topological space that admits a coarsest quasi-uniformity* $\mathcal{V}$. *Let* $\tilde{\mathcal{V}}$ *be the quasi-uniformity on* $\mathcal{L}$ *that is generated by the sets* $\tilde{U} = \{(h(\mathcal{F}), h(\mathcal{G})) \mid \mathcal{F}, \mathcal{G} \in \tilde{X}$ *and there is an* $F \in \mathcal{F}$ *and a* $G \in \mathcal{G}$ *such that* $F \times G \subset U\}$ *where* $U \in \mathcal{V}$. *Then* $\mathcal{T}(\tilde{\mathcal{V}}^*)$ *is a compact Hausdorff topology on* $\mathcal{L}$. *Moreover, the bicomplete quasi-uniformity* $\tilde{\mathcal{V}}$ *is the coarsest quasi-uniformity that the strongly sober locally compact space* $(\mathcal{L}, \mathcal{T}(\tilde{\mathcal{V}}))$ *admits.*

Our last result in this section shows that the property considered in Lemma 4 can be used to characterize coarsest compatible quasi-uniformities on topological spaces.

Proposition 1. *Let* $\mathcal{W}$ *be a compatible totally bounded quasi-uniformity on a topological space* X. *Assume that* $\mathcal{W}$ *satisfies the following condition:*

$$(\mathcal{F}_1,\mathcal{F}_2) \in \cap\tilde{\mathcal{W}}, \textit{ whenever } \mathcal{F}_1,\mathcal{F}_2 \in \tilde{X} \textit{ such that } \lim \mathcal{F}_1 \subset \lim \mathcal{F}_2.$$

Then X *admits a coarsest quasi-uniformity and* $\mathcal{W}$ *is the coarsest quasi-uniformity that* X *admits.*

Proof. Let δ be the quasi-proximity that is induced by $\mathcal{W}$ on X. We want to show that δ is the coarsest compatible quasi-proximity on X. In [8, Lemma 2] it is proved that if $A < X\backslash B$ for subsets A and B of a topological space X, then $A\bar{\rho}B$ for each compatible quasi-proximity ρ on X. Hence it remains to show that if $A,B \subset X$ and $A\bar{\delta}B$, then $A < X\backslash B$.

Let $A,B \subset X$ such that $A\bar{\delta}B$. Then there is a $W \in \mathcal{W}$ such that $W^3(A) \cap B = \emptyset$. Hence if $H = (X\backslash W(A)) \times X \cup X \times (X\backslash W^{-1}(B))$, then $H \in \mathcal{W}$ by [2, Theorem 1.33]. If $A \not< X\backslash B$, then we construct as in the second part of the proof of Lemma 0 two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on X such that $A \in \mathcal{U}$, $B \in \mathcal{V}$ and $\lim \mathcal{U} \subset \lim \mathcal{V}$.

Since $\mathcal{W}$ is totally bounded, $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{W}^*$-Cauchy filters [2, Proposition 3.14]. By [2, Proposition 3.30] there are $\mathcal{F}_1,\mathcal{F}_2 \in \tilde{X}$ such that $\mathcal{F}_1 \subset \mathcal{U}$ and $\mathcal{F}_2 \subset \mathcal{V}$. Furthermore, $W(A) \in \mathcal{F}_1$ and $W^{-1}(B) \in \mathcal{F}_2$, because $A \in \mathcal{U}$ and $B \in \mathcal{V}$ [2, Proposition 3.30]. Hence $(\mathcal{F}_1,\mathcal{F}_2) \notin \tilde{H}$, although $\lim \mathcal{F}_1 = \lim \mathcal{U}$ and $\lim \mathcal{F}_2 = \lim \mathcal{V}$, because $\mathcal{F}_1$ and $\mathcal{F}_2$ are primitive filters on X. We have reached a contradiction. Hence $A < X\backslash B$. Thus $\mathcal{W}$ is the coarsest compatible quasi-uniformity on X.

3. Main Results

Let X be a topological space that admits a coarsest quasi-uniformity $\mathcal{V}$.

In this section we give an explicit description of the topology $\mathcal{T}(\tilde{\mathcal{V}})$ on $\mathcal{L}$, where $\tilde{\mathcal{V}}$ is the quasi-uniformity on $\mathcal{L}$ defined in Lemma 5. We will use this description in order to show that, if X is locally compact, then $\mathcal{T}(\tilde{\mathcal{V}}^*)$ is the topology $\mathcal{T}$ on $\mathcal{L}$ defined by Fell in [1]. Finally we will reconsider a problem studied by Hoffmann in [4].

Proposition 2. *Let* X *be a topological space and let* $\mathcal{L}$ *be the set of the convergence sets of the primitive filters on* X. *Let* $\mathcal{T}'$ *be the topology on* $\mathcal{L}$ *that is generated by the base* $\{[G] \mid G$ *is open in* $X\}$ *where* $[G] = \{F \in \mathcal{L} \mid$ *there is a finite collection* $\mathcal{M}$ *of open subsets of* X *such that* $\cap\mathcal{M} \subset G$ *and such that for each* $M \in \mathcal{M}$ *we have that* $M \cap F \neq \emptyset\}$.

Then $(\mathcal{L},\mathcal{T}')$ *is strongly sober and locally compact if and only if* X *admits a coarsest quasi-uniformity.*

If X *admits a coarsest quasi-uniformity* $\mathcal{V}$, *then* $\mathcal{T}(\tilde{\mathcal{V}}) = \mathcal{T}'$.

Proof. Let X be a topological space that admits a coarsest quasi-uniformity $\mathcal{V}$. By Lemma 5 $\mathcal{T}(\tilde{\mathcal{V}})$ is strongly sober and locally compact. Hence it suffices to show that the two topologies $\mathcal{T}(\tilde{\mathcal{V}})$ and $\mathcal{T}'$ on $\mathcal{L}$ coincide. Let us first show that $\mathcal{T}(\tilde{\mathcal{V}}) \subset \mathcal{T}'$.

Let $\mathcal{F} \in \tilde{X}$ and $U \in \mathcal{V}$. We assume without loss of generality that $U = (X\backslash A) \times X \cup X \times B$, where $A < B(X)$. Since X admits a coarsest quasi-uniformity, there exists $C \subset X$ such that $A < C < B$. If $X\backslash A \in \mathcal{F}$, then $\mathcal{L} = [X] = \tilde{U}(h(\mathcal{F}))$. If $X\backslash A \notin \mathcal{F}$, then by Lemma 2(a) we have that $C \in \mathcal{F}$. Since $C < B$ and $\mathcal{F}$ is a primitive filter on X, we get that $h(\mathcal{F}) \in [\text{int } B]$ by the definition of the relation $<$. If $h(\mathcal{H}) \in [\text{int } B]$ where $\mathcal{H} \in \tilde{X}$, then $B \in \mathcal{H}$ and, thus, $h(\mathcal{H}) \in \tilde{U}(h(\mathcal{F}))$. Hence $h(\mathcal{F}) \in [\text{int } B] \subset \tilde{U}(h(\mathcal{F}))$. Therefore $\mathcal{T}(\tilde{\mathcal{V}}) \subset \mathcal{T}'$.

Next we want to show that $\mathcal{T}' \subset \mathcal{T}(\tilde{\mathcal{V}})$. Let $h(\mathcal{F}) \in [G]$, where $\mathcal{F} \in \tilde{X}$ and $G \neq X$ is open in X. There is a nonempty finite collection $\mathcal{M}$ of open subsets of X such that for each $M \in \mathcal{M}$ we have that $h(\mathcal{F}) \cap M \neq \emptyset$ and such that $\cap\mathcal{M} \subset G$. If $M \in \mathcal{M}$, there are open subsets $P'(M), P''(M)$ of X such that $P'(M) < P''(M) < M$ and such that $P'(M) \cap h(\mathcal{F}) \neq \emptyset$, because X admits a coarsest quasi-uniformity. For each $M \in \mathcal{M}$ set $U_M := (X\backslash P'(M)) \times X \cup X \times P''(M)$. Then for each $M \in \mathcal{M}$ we have that $U_M \in \mathcal{V}$. Let $h(\mathcal{H}) \in \cap\{\tilde{U}_M(h(\mathcal{F})) \mid M \in \mathcal{M}\}$ where $\mathcal{H} \in \tilde{X}$. Let $M \in \mathcal{M}$. Then $P'(M) \in \mathcal{F}$, because $P'(M) \cap h(\mathcal{F}) \neq \emptyset$. Thus $P''(M) \in \mathcal{H}$, because $h(\mathcal{H}) \in \tilde{U}_M(h(\mathcal{F}))$. Since $\mathcal{H}$ is a primitive filter on X and $P''(M) < M$, there is a finite collection $\mathcal{N}(M)$ of open subsets of X such that for each $N \in \mathcal{N}(M)$, $N \cap h(\mathcal{H}) \neq \emptyset$, and such that $\cap\mathcal{N}(M) \subset M$. Since $\cap\mathcal{M} \subset G$, $h(\mathcal{H}) \in [G]$. Therefore, $\cap\{\tilde{U}_M(h(\mathcal{F})) \mid M \in \mathcal{M}\} \subset [G]$. Hence $\mathcal{T}' \subset \mathcal{T}(\tilde{\mathcal{V}})$ and, thus, $\mathcal{T}(\tilde{\mathcal{V}}) = \mathcal{T}'$.

In the second part of the proof we assume that X is a topological space such that $(\mathcal{L},\mathcal{T}')$ is strongly sober and locally compact. Since $(\mathcal{L},\mathcal{T}')$ is locally compact, $(\mathcal{L},\mathcal{T}')$ admits a coarsest quasi-uniformity.

Define i: $X \to \mathcal{L}$ by $i(x) = \overline{\{x\}}$ for each $x \in X$. We show that X admits a coarsest quasi-uniformity by proving that each open set G of X is the union of open subsets that are relatively handy in G. Let $G \neq X$ be an open neighbourhood of a point $x \in X$. Then $i(x) \in [G]$. Since $(\mathcal{L},\mathcal{T}')$ admits a coarsest quasi-uniformity, there is an open subset G' of X such that $i(x) \in [G'] < [G]$. Clearly, this implies that $x \in G' \subset G$. We show that $G' < G(X)$. Let $\mathcal{U}$ be an ultrafilter on X that contains G'. Then $[G'] \in i(\mathcal{U})$, where $i(\mathcal{U})$ denotes the ultrafilter on $\mathcal{L}$ that is generated by the filterbase $\{i(U) | U \in \mathcal{U}\}$. Since $[G'] < [G] \neq \mathcal{L}$ and $[G'] \in i(\mathcal{U})$, there is a nonempty finite collection $\mathcal{M}$ of open subsets of X such that $\cap\{[M] | M \in \mathcal{M}\} \subset [G]$ and such that if $M \in \mathcal{M}$, then [M] contains a limit point $\lim \mathcal{P}_M$ of $i(\mathcal{U})$, where $\mathcal{P}_M$ is a primitive filter on X. Let $M \in \mathcal{M}$. If there is a $y \in \lim \mathcal{P}_M \setminus \lim \mathcal{U}$, then there is an open neighbourhood G_y of y such that $G_y \notin \mathcal{U}$. Clearly $\lim \mathcal{P}_M \in [G_y]$. Since $\lim \mathcal{P}_M$ is a limit point of $i(\mathcal{U})$, we have that $i(U) \subset [G_y]$ for some $U \in \mathcal{U}$ and thus $U \subset G_y \in \mathcal{U}$, a contradiction. Therefore, for each $M \in \mathcal{M}$, $\lim \mathcal{P}_M \subset \lim \mathcal{U}$. Since for each $M \in \mathcal{M}$, $\lim \mathcal{P}_M \in [M]$, we conclude that $\lim \mathcal{U} \in [M]$ for each $M \in \mathcal{M}$. Since $\cap\{[M] | M \in \mathcal{M}\} \subset [G]$, we have shown that $\lim \mathcal{U} \in [G]$. Therefore, $G' < G(X)$ as desired. We have proved that X admits a coarsest quasi-uniformity.

Corollary 3. [8, Theorem 2'] *If* X *is a* T_0*-space that admits a coarsest quasi-uniformity* $\mathcal{V}$ *and each ultrafilter on* X *has an irreducible nonempty convergence set, then* $(\mathcal{L},\mathcal{T}(\tilde{\mathcal{V}}))$ *is the sobrification of* X.

Proof. By Proposition 2, $(\mathcal{L},\mathcal{T}(\tilde{\mathcal{V}})) = (\mathcal{L},\mathcal{T}')$. Since each ultrafilter on X has an irreducible nonempty convergence set, $\mathcal{L}$ is the set of the nonempty closed irreducible subsets of X and $[G] = \langle X,G \rangle$ for each open subset G of X. Hence $(\mathcal{L},\mathcal{T}')$ is the sobrification of X [see 4, p. 64].

Proposition 3. *Let* X *be a topological space that admits a coarsest quasi-uniformity* $\mathcal{V}$. *Then the topology* $\mathcal{T}(\tilde{\mathcal{V}})$ *on* $\mathcal{L}$ *is generated by the subbase* $\mathcal{S}_1 = \{\langle X,G \rangle | G$ *is open in* X$\}$ *if and only if* X *is core-compact.*

Proof. Recall that by Proposition 2 the topologies $\mathcal{T}'$ and $\mathcal{T}(\tilde{\mathcal{V}})$ on $\mathcal{L}$ coincide.

First we want to show that the topology $\mathcal{T}'$ on $\mathcal{L}$ is always coarser than the topology generated by $\mathcal{S}_1$.

Note that $[X] = \mathcal{L}$. Let $F \in \mathcal{L}$ and $F \in [G]$, where $G \neq X$ is open in X. There is a nonempty finite collection $\mathcal{M}$ of open subsets of X such that if $M \in \mathcal{M}$, then $F \cap M \neq \emptyset$, and such that $\cap\mathcal{M} \subset G$. Hence $F \in \cap\{\langle X,M \rangle | M \in \mathcal{M}\} \subset [G]$ and we are done.

Assume now that X is core-compact. Let $\lim \mathcal{P} \in \langle X,G \rangle$, where G is an open subset of X and $\mathcal{P}$ is a primitive filter on X. There is an open subset G' of X such that G' is relatively compact in G and $\lim \mathcal{P} \cap G' \neq \emptyset$. If $\lim \mathcal{P}' \in [G']$ where $\mathcal{P}'$ is a primitive filter on X, then $G' \in \mathcal{P}'$, and thus $\lim \mathcal{P}' \in \langle X,G \rangle$. Hence $\lim \mathcal{P} \in [G'] \subset \langle X,G \rangle$. Therefore we have proved that $\mathcal{S}_1$ generates the topology $\mathcal{T}(\tilde{\mathcal{V}})$ on $\mathcal{L}$, if X is core-compact.

In order to prove the converse we assume that $\mathcal{S}_1$ generates the topology $\mathcal{T}(\tilde{\mathcal{V}})$ on $\mathcal{L}$. We show that X is core-compact. Let G be an open neighbourhood of a point $x \in X$. Then $\overline{\{x\}} \in \langle X,G \rangle$. Hence there is an open set G' of X such that $\overline{\{x\}} \in [G'] \subset \langle X,G \rangle$. Thus $x \in G' \subset G$. Since X admits a coarsest quasi-uniformity, there is an open subset G" of X such that $x \in G'' < G' \subset G$. We show that G" is relatively compact in G. Let $\mathcal{U}$ be an ultrafilter on X that contains G". Since $G'' < G'$, we have that $\lim \mathcal{U} \in [G'] \subset \langle X,G \rangle$. Hence $\lim \mathcal{U} \cap G \neq \emptyset$. We conclude that X is core-compact.

Proposition 4. *If* X *is locally compact and* $\mathcal{V}$ *is the coarsest quasi-uniformity for* X*, then* $(\mathcal{L},\mathcal{J}(\tilde{\mathcal{V}}^*))$ *is Fell's compactification* $\mathcal{H}(X)$ *of* X.

Remark 1.

(a) It follows from Proposition 4 and Lemma 4 that $(\mathcal{L},\mathcal{J}(\tilde{\mathcal{V}}^*),\cap\tilde{\mathcal{V}})$ is the compact partially ordered topological space that Hoffmann calls the Fell compactification of X in [4].

(b) If X is a locally compact T_0-space, then X is strongly sober if and only if the coarsest compatible quasi-uniformity $\mathcal{V}$ on X is bicomplete [9, Proposition 2]. Hence if X is strongly sober and locally compact and $\mathcal{V}$ is the coarsest compatible quasi-uniformity on X, then Fell's compactification $\mathcal{H}(X)$ of X is $(X,\mathcal{J}(\mathcal{V}^*))$. In particular, if X is a compact Hausdorff space, then $\mathcal{J}(\mathcal{V}^*) = \mathcal{J}(\mathcal{V})$, because $\mathcal{V} = \mathcal{V}^*$ [2, Proposition 1.47].

(c) The elementary proof given below is self-contained. Readers familiar with the theory of continuous lattices may prefer a shorter proof: $(\mathcal{L},\mathcal{J})$ in its inclusion order is a compact Hausdorff pospace, whose upper topology is generated by $\mathcal{S}_1$ [see e.g. 6, section 2]. Hence $\mathcal{S}_1$ generates $\mathcal{T}(\mathcal{U})$ [2, Proposition 4.22], where $\mathcal{U}$ denotes the (unique) quasi-uniformity that determines $(\mathcal{L},\mathcal{T},\subset)$ [2, Theorem 4.21]. By Proposition 3, $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\tilde{\mathcal{V}})$. Since $\mathcal{U}$ and $\tilde{\mathcal{V}}$ are both totally bounded and bicomplete, $\mathcal{U} = \tilde{\mathcal{V}}$ by [9, Proposition 2] and, thus, $\mathcal{T}(\tilde{\mathcal{V}}^*) = \mathcal{T}$.

Proof of Proposition 4. We show that $\mathcal{S}_2 = \{\langle X\backslash K\rangle | K$ is compact in $X\}$ is a subbase for the strongly sober locally compact topology $\mathcal{T}(\tilde{\mathcal{V}}^{-1})$ on $\mathcal{L}$. Since $\mathcal{T}(\tilde{\mathcal{V}}^*) = \sup\{\mathcal{T}(\tilde{\mathcal{V}}),\mathcal{T}(\tilde{\mathcal{V}}^{-1})\}$ and since (by Proposition 3) $\mathcal{S}_1$ generates the topology $\mathcal{T}(\tilde{\mathcal{V}})$ on $\mathcal{L}$, this will complete the proof of Proposition 4.

Recall that the quasi-uniformity $\mathcal{V}$ on X is generated by the subbase $\mathcal{S} = \{(X\backslash K) \times X \cup X \times G | K$ is a compact subset of the open set G of X$\}$, because X is locally compact.

First let us show that $\mathcal{S}_2$ generates a topology on $\mathcal{L}$ that is finer than $\mathcal{T}(\tilde{\mathcal{V}}^{-1})$. Let $V \in \tilde{\mathcal{V}}$ and let $\mathcal{F} \in \tilde{X}$. We assume without loss of generality that $V = \tilde{U}$ where $U = (X\backslash K) \times X \cup X \times G \in \mathcal{S}$. Since X is locally compact, there exist an open subset G' of X and a compact subset K' of X such that $K \subset G' \subset K' \subset G$. If $h(\mathcal{F}) \cap G \neq \emptyset$, then $G \in \mathcal{F}$ and $\tilde{U}^{-1}(h(\mathcal{F})) = \mathcal{L}$. If $h(\mathcal{F}) \cap G = \emptyset$, then $h(\mathcal{F}) \in \langle X\backslash K'\rangle$. Let $h(\mathcal{H}) \in \langle X\backslash K'\rangle$ where $\mathcal{H} \in \tilde{X}$. Since $h(\mathcal{H}) \cap K' = \emptyset$ and K' is compact, $K' \notin \mathcal{H}$. Thus $G' \notin \mathcal{H}$. By Lemma 2(a) $X\backslash K \in \mathcal{H}$, because $K < G'$. Hence $h(\mathcal{H}) \in \tilde{U}^{-1}(h(\mathcal{F}))$ and thus $h(\mathcal{F}) \in \langle X\backslash K'\rangle \subset \tilde{U}^{-1}(h(\mathcal{F}))$.

Let us now show that $\mathcal{T}(\tilde{\mathcal{V}}^{-1})$ is finer than the topology generated by $\mathcal{S}_2$.

Assume that $h(\mathcal{J}) \in \langle X\backslash K \rangle$ where $\mathcal{J} \in \tilde{X}$ and K is compact in X. Since X is locally compact, there are open subsets G_1, G_2, G_3 of X and compact subsets K_1, K_2, K_3 of X such that $K \subset G_1 \subset K_1 \subset G_2 \subset K_2 \subset G_3 \subset K_3 \subset X\backslash h(\mathcal{J})$. Let $U = (X\backslash K_1) \times X \cup X \times G_2$. Then $U \in \mathcal{S}$. Let $h(\mathcal{H}) \in \tilde{U}^{-1}(h(\mathcal{J}))$ where $\mathcal{H} \in \tilde{X}$. Since $h(\mathcal{J}) \cap K_3 = \emptyset$, we have that $G_3 \notin \mathcal{J}$. By Lemma 2(a) we see that $X\backslash K_2 \in \mathcal{J}$. Hence $X\backslash K_1 \in \mathcal{H}$, because $h(\mathcal{H}) \in \tilde{U}^{-1}(h(\mathcal{J}))$. Thus $X\backslash G_1 \in \mathcal{H}$, which implies that $h(\mathcal{H}) \subset X\backslash G_1$. We have shown that $h(\mathcal{H}) \in \langle X\backslash K \rangle$. Hence $\tilde{U}^{-1}(h(\mathcal{J})) \subset \langle X\backslash K \rangle$. We conclude that $\mathcal{S}_2$ is a subbase for the topology $\mathcal{J}(\tilde{\mathcal{V}}^{-1})$ on $\mathcal{L}$.

Proposition 5. *If* X_i $(i = 1,2)$ *are two topological* T_0*-spaces that admit a coarsest quasi-uniformity* $\mathcal{V}_i$ *and* $f: (X_1, \mathcal{V}_1) \to (X_2, \mathcal{V}_2)$ *is a quasi-uniformly continuous map (equivalently,* f *is a* qp*-continuous map* [2, p. 23]*, i.e. if* $A, B \subset X_2$ *and* $A < B(X_2)$*, then* $f^{-1}A < f^{-1}B(X_1)$*), then there exists a unique quasi-uniformly continuous extension* $\tilde{f}: (\mathcal{L}_1, \tilde{\mathcal{V}}_1) \to (\mathcal{L}_2, \tilde{\mathcal{V}}_2)$ *of* f*. Here* $\mathcal{L}_i$ $(i = 1,2)$ *denotes the set of the convergence sets of the primitive filters on* X_i *and* X_i *is identified with* $j_i(X_i)$*, where* $j_i: (X_i, \mathcal{V}_i) \to (\mathcal{L}_i, \tilde{\mathcal{V}}_i)$ *is the quasi-uniform embedding defined by* $j_i(x) = \overline{\{x\}}$ *for each* $x \in X_i$.

Proof. This result is a special case of the corresponding result mentioned in the introduction.

Remark 2. We note that using a completely different method, Hoffmann proves (see [4, Theorem 6.8 and Remark 6.9]; [6] and [4, Remark 6.10]) similar results in the special case of core-compact spaces and perfect continuous maps (compare [9, Remark 2b]).

Since in [4] Fell's compactification is studied in the setting of topological partially ordered spaces, let us mention the following well-known result, which can be used to formulate several variants of Proposition 5.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be totally bounded quasi-uniform spaces such that $\mathcal{T}(\mathcal{U})$ and $\mathcal{T}(\mathcal{V})$ are T_0-topologies. If $\mathcal{U}$ is bicomplete, then a map $f: X \to Y$ is a quasi-uniformly continuous map from $(X,\mathcal{U})$ into $(Y,\mathcal{V})$ if and only if $f: (X,\mathcal{T}(\mathcal{U}^*),\cap\mathcal{U}) \to (Y,\mathcal{T}(\mathcal{V}^*),\cap\mathcal{V})$ is continuous and increasing [e.g. 11, Theorems 5 and 6].

Remark (added July 1986). In a recent paper [3] quasi-uniformities are used to study several hyperspace topologies. In particular Fell's topology [1] is considered.

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