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# SOME NEW EXAMPLES OF HOMOGENEOUS CURVES

**Piotr Minc and James T. Rogers, Jr.**[1]

## 1. Introduction

In the plane, the only aposyndetic, homogeneous continuum is the simple closed curve [3]. Outside the plane, things are more interesting. In particular, the second author [4 and 5] has proved the existence of uncountably many aposyndetic, homogeneous curves that are not locally connected. These continua were all constructed by spinning the Menger universal curve about one of its holes. These continua were termed Case continua, after J. H. Case, who first constructed such continua [2].

In the Special Session on Continua Theory at the AMS meeting at Baton Rouge in November, 1982, it was asked if these continua are the only aposyndetic, homogeneous curves that are not locally connected. In this paper we construct many others. The idea is to spin the Menger curve around several holes at the same time.

A Case continuum X has the property that $H^1(X) \cong H^1(S) \oplus \sum^{\infty} Z$, wbere S is a particular solenoid. We construct these new continua in two ways--a bundle construction and in inverse limit construction--and differentiate them from Case continua by calculating their cohomology.

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K. Kuperberg has asked if an arcwise-connected, homogeneous continuum must be locally connected. The first author has asked if a hereditarily decomposable, homogeneous curve must be a simple closed curve. The second author has asked if a pointed one-movable, aposyndetic homogeneous continuum must be locally connected.

Each of these new examples can be mapped onto some (nontrivial) solenoid. Hence none of them is arcwise-connected, hereditarily decomposable, or pointed one-movable. Whether an aposyndetic, homogeneous curve with one of these properties must be locally connected remains an open question.

Let X be a closed m-manifold, $m > 1$. If X retracts onto a finite wedge of circles, then the constructions of this paper show that associated with X is a collection of aposyndetic, homogeneous, m-dimensional continua that are not locally connected.

A continuum is a compact, connected, nondegenerate, metric space. A curve is a one-dimensional continuum. The first Čech cohomology group with integral coefficients of the continuum X is denoted $H^1(X)$.

## 2. A Bundle Construction

Let P be a Sierpiński curve "face" of the Menger curve M. Let $q: M \to P$ be the projection map.

Let $B_1$ be a bouquet of n circles embedded in P, and let $r: P \to B_1$ be a retraction.

Let $T = \Pi_{i=1}^{n} S^1$ be the n-dimensional torus, and let $v = (e,\cdots,e)$ be the identity of T. Let $i: B_1 \to T$ be the natural embedding of $B_1$ onto the set $\{(x_1,\cdots,x_n): x_i = e$ except for at most one coordinate$\}$. Often we write $B_1$ for $i(B_1)$.

Let $S = \Pi_{i=1}^n S_i$ be the product of n (possibly different) nontrivial solenoids. Let $f = (f_1,\cdots,f_n)$ be the product of n projection maps $f_i\colon S_i \to S^1$. Let $G_i = \ker f_i$, and let $G = \Pi_{i=1}^n G_i$.

It follows that $(S,f,T,G)$ is a principal bundle, and that G is an abelian group whose underlying space is a Cantor set.

This information enables us to construct the left column and bottom row of Diagram 1. Everything else in Diagram 1 arises from induced bundles. In particular, every horizontal level represents a principal bundle whose structure group is G.

$$\begin{tikzcd}
M \arrow[d, "q"'] & \tilde{M} \arrow[l] \arrow[d] \\
P \arrow[d, "r"'] & \tilde{P} \arrow[l] \arrow[d] \\
B_1 \arrow[d, "i"'] & B \arrow[l, "p"'] \arrow[d] \\
T & S \arrow[l, "f"']
\end{tikzcd}$$

Diagram 1

The space B may also be regarded as $f^{-1}(B_1)$, and the map p as the restriction of f. Our first task is to show that B is connected.

*Theorem* 1. *B is connected.*

*Proof.* It suffices to show that $p^{-1}(v)$ is contained in a connected subset of B, for if b is any point of B, there exists an arc from p(b) to v, which lifts to an arc from b to a point of $p^{-1}(v)$.

The set $p^{-1}(v) = \{(c_1,\cdots,c_n): f_i(c_i) = e, \text{ for all } i\}$. Let c and d be points of $p^{-1}(v)$ that differ in only one coordinate, say the first. The set $W = \{(x,c_2,c_3,\cdots,c_n): x \in S_1\}$, which is homeomorphic to the solenoid $S_1$, contains both c and d, and $f(W) = \{(x_1,e_2,\cdots,e_n): x_1 \in S^1\} \subset B_1$. Therefore, c and d belong to a connected subset of B.

For arbitrary points c and d in $p^{-1}(v)$, there exists a finite sequence of points of $p^{-1}(v)$ such that c is the first, d is the last, and two adjacent members of the sequence differ in at most one coordinate. It follows that c and d are points of a connected subset of B.

*Theorem* 2. *$\tilde{M}$ is connected.*

*Proof.* There is an arc from any point of $\tilde{M}$ to a point of B.

The next theorem follows from the methods of [4], in particular, Theorems 6 and 7.

*Theorem* 3. *$\tilde{M}$ is a colocally connected, homogeneous curve that is not locally connected. In particular, $\tilde{M}$ is aposyndetic.*

For n = 1, such continua were constructed in [2, 4, and 5] and termed Case continua. We wish to show that for

$n > 1$, the continuum $\tilde{M}$ is not homeomorphic to a Case continuum. Our method will be to calculate $H^1(\tilde{M})$ and show that it is not isomorphic to $H^1(X)$, for $X$ a Case continuum.

If $P = (p_1, p_2, \cdots)$ is a sequence of positive integers and $S_0$ is the solenoid associated with $P$, then $H^1(S_0)$ is isomorphic to the group of all rationals of the form $k/p_1 \cdots p_m$, where $k$ is an integer and $m$ is a positive integer. If $p_i > 1$, for infinitely many $i$, then $S_0$ is a non-trivial solenoid (i.e., not a simple closed curve), in which case $H^1(S_0)$ admits infinite division. This means that for each element $h_1$ of $H^1(S_0)$, there exist an integer $m > 1$ and an element $h_2$ of $H^1(S_0)$ such that $mh_2 = h_1$. Two solenoids $S_0$ and $S_1$ are homeomorphic if and only if $H^1(S_0)$ is isomorphic to $H^1(S_1)$.

Let $C_i$, $i = 1, \cdots, c$, and $D_i$, $i = 1, \cdots, d$, be groups each of which is isomorphic to the first cohomology group of a nontrivial solenoid (the solenoid may vary with the index). If

$$\Sigma_{i=1}^{c} C_i \oplus \Sigma Z \cong \Sigma_{i=1}^{d} D_i \oplus \Sigma Z,$$

then it can be shown [1] that $c = d$ and that, for some reindexing of $\{D_i\}$, $C_i$ is isomorphic to $D_i$, for all $i$.

*Theorem* 4. $H^1(S,B) \cong 0$ *and* $H^2(S,B) \cong \Sigma Z$.

*Proof*. Represent $S$ as the inverse limit of tori $\{T_m, g_m^{m+1}\}$. In the top row of Diagram 2, define $B_{m+1} = (g_m^{m+1})^{-1}(B_m)$ and the bonding maps to be restrictions. All vertical arrows are inclusions. Hence $B \cong \varprojlim B_m$, and $(S,B) \cong \varprojlim (T_m, B_m)$.

$$\begin{tikzcd}
B_1 \arrow[d] & B_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & B \arrow[d] \\
T_1 \arrow[d] & T_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & S \arrow[d] \\
(T_1,B_1) & (T_2,B_2) \arrow[l] & \cdots \arrow[l] & (S,B)
\end{tikzcd}$$

Diagram 2

Regard $T_m$ as a CW-complex with 1-skeleton $B_m$. Hence $(T_m,B_m)$ is 1-connected [6, p. 403] and so $H^1(T_m,B_m) \cong 0$. Hence $H^1(S,B) \cong \text{dir lim } H^1(T_m,B_m) \cong 0$.

Consider the following part of the cohomology sequence of the triple $(T_m,T_m^{(2)},B_m)$, where $T_m^{(2)}$ is the 2-skeleton of $T_m$.

$$H^2(T_m^{(2)},B_m) \xleftarrow{i_m^*} H^2(T_m,B_m) \longleftarrow H^2(T_m,T_m^{(2)})$$

Diagram 3

Since $(T_m T_m^{(2)})$ is 2 connected [6, p. 403], the relative Hurewicz isomorphism theorem [6, p. 397], together with [6, p. 373], implies that $H^2(T_m,T_m^{(2)}) \cong 0$.

The strong excision property implies that $H^2(T_m^{(2)},B_m) \cong H^2(T_m^{(2)}/B_m)$. Since $T_m^{(2)}/B_m$ is a wedge (or bouquet of 2-spheres, $H^2(T_m^{(2)},B_m) \cong \Sigma Z$. Hence Diagram 3 becomes

$$\Sigma Z \xleftarrow{i_m^*} H^2(T_m,B_m) \longleftarrow 0$$

and $i_m^*$ is an injection.

Since $g_m^{m-1}$ is a product of covering maps of circles, the map $\hat{g}$: $H^2(T_m^{(2)},B_m) \to H^2(T_{m+1}^{(2)},B_{m+1})$ induced by $g_m^{m+1}$ takes a generator of $H^2(T_m^{(2)},B_m)$ to a sum of generators of

$H^2(T^{(2)}_{m+1}, B_{m-1})$. Hence dir lim $H^2(T^{(2)}_m, B_m) \cong \tilde{\Sigma} Z$. It follows that the direct limit of Diagram 3 is

$$\Sigma Z \xleftarrow{i^*} H^2(S,B) \longleftarrow 0.$$

and hence $H^2(S,B) \cong \Sigma Z$.

*Theorem* 5. $H^1(\tilde{M}) = H^1(B) \oplus \overset{\infty}{\Sigma} Z$.

*Proof.* The proof of [4, Theorem 10] applies.

*Theorem* 6. $H^1(\tilde{M}) \cong (\Sigma_{i=1}^{n} H^1(S_i)) \oplus \overset{\infty}{\Sigma} Z$.

*Proof.* Consider the following part of the long exact sequence of the pair $(S,B)$.

$$H^1(S,B) \longrightarrow H^1(S) \longrightarrow H^1(B) \longrightarrow H^2(S,B).$$

Theorem 4 implies that this sequence is

$$0 \longrightarrow H^1(S) \longrightarrow H^1(B) \longrightarrow \Sigma Z.$$

Hence $H^1(B) \cong H^1(S) \oplus \Sigma Z$. The Kunneth formula implies $H^1(S) \cong \Sigma_{i=1}^{n} H^1(S_i)$. Thus, by Theorem 5,

$$H^1(\tilde{M}) = (\Sigma_{i=1}^{n} H(S_i)) \oplus \overset{\infty}{\Sigma} Z.$$

*Theorem* 7. *Let* $\tilde{M}_\alpha$ *and* $\tilde{M}_\beta$ *be two of the continua constructed in this section. Associated with* $\tilde{M}_\alpha$ *is a product of solenoids* $\Pi_{i=1}^{n_\alpha}(S_i^\alpha)$ *and with* $M_\beta$ *a product* $\Pi_{i=1}^{n_\beta}(S_i^\beta)$. *If the continua* $\tilde{M}_\alpha$ *and* $\tilde{M}_\beta$ *are homeomorphic, then* $n_\alpha = n_\beta$ *and, for some reindexing,* $S_i^\alpha = S_i^\beta$.

*Proof.* This follows immediately from Theorem 6 and the remarks preceding Theorem 4.

*Corollary* 8. *If* $\tilde{M}$ *is homeomorphic to a Case continuum, then* $n = 1$.

In general, we have proved the following theorem, which applies, in particular, to closed m-manifolds, $m > 1$, that retract onto a finite wedge of circles.

*Theorem* 9. *Suppose* X *is a strongly locally homogeneous,* m-*dimensional continuum such that points of* X *have arbitrarily small neighborhoods with connected boundaries and such that there exists a retraction of* X *onto a wedge of* n *circles. Then, for any* n *solenoids, there exists an aposyndetic, homogeneous,* m-*dimensional continuum* $\tilde{M}$ *that retracts onto each of these solenoids and that is the total space of a Cantor set bundle over* X.

## 3. An Inverse Limit Construction References

In this section we indicate how to construct the continuum $\tilde{M}$ as an inverse limit of universal curves and covering maps.

Let $T = \Pi_{i=1}^{n} S^1$ be the n-dimensional torus, $n \geq 3$. Let $Z_j = \Pi_{i=1}^{j-1}\{e\} \times S^1 \times \Pi_{i=j+1}^{n}\{e\}$, for $j = 1,2,\cdots,n$. One can assume that the Menger curve M is embedded in T such that $Z_j \subset M$ for every $j = 1,2,\cdots,n$.

For any sequence $\alpha$ of n positive integers $\alpha_1,\alpha_2,\cdots,\alpha_n$, let $f_\alpha\colon T \to T$ be the mapping defined by the formula

$$f_\alpha(z_1,z_2,\cdots,z_n) = (z_1^{\alpha_1},z_2^{\alpha_2},\cdots,z_n^{\alpha_n})$$

*Theorem* 10. *The space* $f_\alpha^{-1}(M)$ *is a Menger curve containing* $Z_j$, *for* $j = 1,2,\cdots,n$.

The proof is similar to the proof of [5, Lemma 6] and is omitted.

Let $\beta = (\alpha^k)_{k=1}^{\infty}$ be a sequence of finite sequences, each of n positive integers, say $\alpha^k = (\alpha_1^k, \alpha_2^k, \cdots, \alpha_n^k)$.

Define $M_0^\beta = M$ and $M_k^\beta = f_{\alpha_k}^{-1}(M_{k-1}^\beta)$.

Let $M^\beta = \varprojlim \{M_k^\beta, f_{\alpha_k} | M_k^\beta\}$.

*Theorem* 11. *The curve* $M^\beta$ *is homogeneous and colocally connected.*

The proof is the same as the proof of [5, Theorem 7].

The above examples were obtained by spinning the Menger curve M around a finite number of its holes. It remains an open question whether one can construct a similar example by spinning M around infinitely many of its holes. In particular, it is unknown whether there exists a colocally connected homogeneous curve containing an infinite number of different solenoids as its retracts.

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