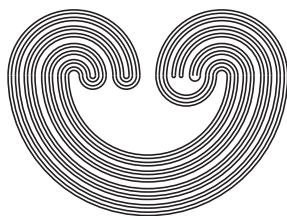

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CARDINAL FUNCTIONS ON HYPERSPACES AND FUNCTION SPACES

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CARDINAL FUNCTIONS ON HYPERSPACES AND FUNCTION SPACES

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I. Introduction

Throughout this paper all topological spaces $X, Y, \dots$ are Tychonoff spaces (i.e., T_1 - and completely regular). We denote the reals with the usual topology by R , $N = \{1, 2, 3, \dots\}$ and, for a set X , $|X|$ will be the cardinality of X . Every subset of R carries its subspace topology and $|N| = \omega$. We use [8] as a reference book.

For spaces X and Y we put:

$K(X)$ = the collection of all non-empty compact subsets of X ;

$F(X)$ = the collection of all non-empty finite subsets of X ;

$F_n(X)$ = the collection of all non-empty finite subsets of X of cardinality less than or equal to n .

$C(X, Y)$ = the collection of all continuous functions from X into Y .

$C(X) = C(X, R)$.

We topologize $K(X)$ with the Vietoris (i.e., the finite) topology and $C(X, Y)$ with the compact-open topology. A basic open set of $K(X)$ is of the form $\langle V_1, V_2, \dots, V_n \rangle = \{A \in K(X) : A \subset \bigcup_{i=1}^n V_i \text{ and } A \cap V_i \neq \emptyset \text{ for each } i = 1, 2, \dots, n\}$, where $n \in N$ and each V_i is an open set in X . Note that each $F_n(X)$ is a closed subspace of $K(X)$ and $F(X) = \bigcup_{n < \omega} F_n(X)$. If $A \in K(X)$ and V is open in Y then $[A, V] = \{f \in C(X, Y) : f(A) \subset V\}$ is a subbasic open set for the compact-open topology on $C(X, Y)$. In this paper we will be concerned with $C(X)$.

Generally speaking, since the compact-open topology on $C(X)$ is defined via the compact sets of X , one would expect that a given topological property on $C(X)$ will translate on X into a property involving compact subsets of X . The behavior of these sets determines the properties of the hyperspace $K(X)$. The converse is also true. It is in fact sometimes useful to go back and forth between $C(X)$ and $K(X)$ or X . In this paper we use cardinal functions to study the connection between the three spaces.

II. Main Results

An important tool when working with hyperspaces and function spaces is the concept of induced function. Let $f: X \rightarrow Y$ be a continuous function. We define the induced function of f :

On hyperspaces by $\hat{f}: K(X) \rightarrow K(Y)$, where $\hat{f}(A) = f(A)$
for all $A \in K(X)$;

On function spaces by $f^*: C(Y) \rightarrow C(X)$, where $f^*(g) = g \circ f$ for each $g \in C(Y)$.

The properties of f^* are summarized in [24] (see also [23]). Theorem 0.1 below outlines the main properties of $\hat{f}$. Recall that a continuous function $f: X \rightarrow Y$ is *almost onto* (or *almost surjective*) if $f(X)$ is a dense subspace of Y . We say that f is *compact-covering* if each compact set in Y is the image under f of some compact set of X .

Theorem 0.1. If $f: X \rightarrow Y$ is continuous and $\hat{f}: K(X) \rightarrow K(Y)$ is its induced function, then

(1) $\hat{f}$ is continuous;

- (2) $\hat{f}$ is one-to-one if and only if f is one-to-one;
- (3) $\hat{f}$ is onto if and only if f is compact-covering;
- (4) $\hat{f}$ is bijective if and only if f is both one-to-one and compact covering;
- (5) $\hat{f}$ is almost onto if and only if f is almost onto;
- (6) $\hat{f}$ is perfect if and only if f is perfect;
- (7) $\hat{f}$ is an embedding if and only if f is an embedding;
- (8) $\hat{f}$ is a homeomorphism if f is a homeomorphism.

Proof. A proof for (6) is in [7]. The converse to part (8) is not true in general. Indeed, [27] gives two non-homeomorphic compact metric spaces with homeomorphic hyperspaces. The remaining part of theorem 0.1 is straightforward.

The idea of characterizing the compact subspaces of $K(X)$ may be compared to Ascoli's theorem describing the compact subsets of $C(X)$. Since $C(X)$ contains a closed copy of $\mathbb{R}$, it cannot be compact. For $K(X)$, Michael [19] has the following. " $K(X)$ is compact if and only if X is compact." In fact, if Λ is a compact subspace of $K(X)$ then $U\Lambda = U\{A \in K(X) : A \in \Lambda\}$ is a compact subset of X . To be more precise, we state:

Theorem 0.2. A subspace Λ of $K(X)$ is compact if and only if Λ is closed in $K(X)$ and $U\Lambda$ is compact in X .

The main core of this paper deals with cardinal functions on $K(X)$, $C(X)$ and X . We follow Juhász [10] and Hodel [9] when using the standard cardinal functions.

The reader may not be familiar with the following concepts on a space X .

The *pseudo-density* of $X = \psi d(X) = \omega + \min\{|D| : D \text{ is a subset of } X \text{ such that } D \cap V \neq \emptyset \text{ for each open set } V \text{ of } X \text{ whose complement } X - V \text{ is compact}\};$

The *k-netweight* of $X = knw(X) = \omega + \min\{|\beta| : \beta \text{ is a k-network of } X\};$

The *k-k-netweight* of $X = kknw(X) = \omega + \min\{|\beta| : \beta \text{ is a k-network of } X \text{ with compact members}\};$

The *Arens number* of $X = a(X) = \omega + \min\{|\partial| : \partial \text{ is a subset of } K(X) \text{ that is a c-cover of } X\}.$

By a *c-cover* for X , we mean any collection ∂ of subsets of X with the property that if $A \in K(X)$ then $A \subset U$ for some $U \in \partial$.

The *compact-covering number* of $X = kc(X) = \omega + \min\{|\partial| : \partial \text{ is a subset of } K(X) \text{ that is a cover of } X\};$

The *weak compact-covering number* of $X = wkc(X) = \omega + \min\{|\partial| : \partial \text{ is a subset of } K(X) \text{ such that } \cup \partial \text{ is dense in } X\};$

The *weak weight* of $X = ww(X) = \omega + \min\{w(Y) : \text{there is a continuous bijection from } X \text{ onto } Y\};$

The *q-ness* of $X = q(X) = \sup\{q(x, X) : x \in X\};$ where $q(x, X) = \omega + \min\{|\theta| : \theta \text{ is a collection of neighborhoods of } x \text{ in } X \text{ such that if } x_0 \in O \text{ for each } O \in \theta \text{ then the set } \{x_0 : O \in \theta\} \text{ clusters in } X\}.$

The above concepts generalize well-known ideas. Indeed, one has the following.

X is pseudoseparable ([14]) if and only if $\psi d(X) = \omega$.

Note that for a compact space X , $\psi d(X) = d(X)$.

X is an $\aleph_0$ -space ([18]) if and only if $k\omega(X) = \omega$.

X is a hemicompact $\aleph_0$ -space if and only if $kknw(X) = \omega$.

X is hemicompact ([1]) if and only if $a(X) = \omega$.

X is almost σ -compact if and only if $wkc(X) = \omega$.

X is σ -compact if and only if $kc(X) = \omega$.

X is subcosmic (i.e., has a coarser separable metrizable topology) if and only if $ww(X) = \omega$.

X is a q -space ([18]) if and only if $q(X) = \omega$.

The following inequalities are easily obtained:

$$\begin{aligned} q(X) &\leq \chi(X); kknw(X) = a(X)knw(X); wkc(X) \leq d(X) \leq nw(X) \leq \\ knw(X) &\leq w(X); wkc(X) \leq kc(X) \leq a(X) \text{ and } \psi(X) \leq ww(X) \leq \\ nw(X) &\leq |X|. \end{aligned}$$

We subdivide the remaining part of the paper into six sections.

1. Main Cardinal Functions on $K(X)$

We will need the following definitions. Let $\Phi \in \{\chi, \psi, d, hd, \pi w\}$ where $\chi, \psi, d, hd, \pi w$ denote the character, the pseudocharacter, the density, the hereditary density, and the π -weight respectively. Define $\Phi_c(X)$ by

$$\Phi_c(X) = \begin{cases} \sup\{\Phi(A, X) : A \in K(X)\} & \text{if } \Phi = \chi \text{ or } \psi; \\ \sup\{\Phi(A) : A \in K(X)\} & \text{if } \Phi = d, hd, \text{ or } \pi w. \end{cases}$$

Theorem 1.1.

1. $d(K(X)) = d(X)$ ([20])
2. $\psi d(K(X)) \leq \psi d(X)$
3. $w(K(X)) = w(X)$ ([20])
4. $\pi w(K(X)) = \pi w(X)$ ([20])

$$5. \quad \chi(K(X)) = \chi_C(X) d_C(X) = \chi_C(X) \cdot hd_C(X) \quad ([20])$$

$$6. \quad \psi(K(X)) = \psi_C(X) \cdot \pi_{W_C}(X) \quad ([20])$$

$$7. \quad nw(K(X)) = knw(K(X)) = knw(X)$$

$$8. \quad ww(K(X)) = ww(X)$$

Proof. For part (2), let $\tau = \psi d(X)$. Then there is $D \subset X$ such that $|D| \leq \tau$ and D meets every open set of X with compact complement. Take $\partial = \{\{d\} : d \in D\}$ and let W be an open set in $K(X)$ such that $K(X) - W$ is compact. Then $UW^C = U\{A : A \in K(X) - W\}$ is compact in X . Therefore $(X - UW^C) \cap D \neq \emptyset$. Now if $d \in (X - UW^C) \cap D$ then $d \notin UW^C$ so that $\{d\} \notin W^C$. Clearly, $\{d\} \in W \cap \partial$. It follows that $\psi d(K(X)) \leq |\partial| \leq \tau$.

A proof of (7) may be obtained by a generalization of Michael's result in [18].

We prove part (8): clearly $ww(X) \leq ww(K(X))$ so that only the reverse inequality needs proof. To this end let $\tau = ww(X)$ and choose Y a space of weight $\leq \tau$ and a continuous bijection $f: X \rightarrow Y$. Then $\hat{f}: K(X) \rightarrow K(Y)$ is a continuous injection so that $ww(K(X)) \leq w(\hat{f}(K(X))) \leq w(K(Y)) = w(Y) \leq \tau$.

The next theorem is the countable version of Theorem 1.1. The concepts of cosmic and $\aleph_0$ -spaces are used as in [18] and $\aleph$ -spaces are due to O'Meara [25].

Theorem 1.2.

1. $K(X)$ is separable if and only if X is separable;
2. $K(X)$ is pseudoseparable whenever X is pseudoseparable;
3. $K(X)$ is second countable if and only if X is second countable;

4. $K(X)$ has a countable π -base if and only if X has a countable π -base;

5. $K(X)$ is (sub)metrizable if and only if X is (sub)metrizable;

6. $K(X)$ is first countable if and only if each compact subset of X is (hereditarily) separable and of countable character;

7. $K(X)$ is subcosmic if and only if X is subcosmic;

8. Each point of $K(X)$ is G_δ if and only if each compact subset of X is G_δ and has countable π -base;

9. $K(X)$ is cosmic if and only if $K(X)$ is $\aleph_0$ -space, if and only if X is $\aleph_0$ -space;

10. $K(X)$ is (paracompact) $\aleph$ -space if and only if X is (paracompact) $\aleph$ -space ([11]);

11. $K(X)$ is a space of point countable type if and only if $K(X)$ is a space of countable type, if and only if X is a space of countable type ([7]).

We close this section with an observation on the q -ness of $K(X)$. It is well known that if $n < \omega$ then the function $\rho_n: X^n \rightarrow F_n(X)$ defined by $\rho_n(x_1, x_2, \dots, x_n) = \{x_1, x_2, \dots, x_n\}$ for each $(x_1, x_2, \dots, x_n) \in X^n$ is a perfect map. Since $F_n(X)$ is closed in $K(X)$, then $q(F_n(X)) \leq q(K(X))$. In fact, more is true.

Theorem 1.3.

1. $q(X^n) \leq q(K(X))$ for each $n < \omega$;

2. $K(X)$ is a paracompact q -space if and only if X is a paracompact q -space.

We now turn to those cardinal functions that will enable us to connect $K(X)$ to $C(X)$. Roughly speaking, $K(X)$ behaves nicely with respect to the global topological properties, whereas $C(X)$ is suitable for studying local properties.

2. Cardinal Functions of Compact-Type on $K(X)$

Theorem 2.1.

$$(1) \ a(K(X)) = kc(K(X)) = a(X);$$

$$(2) \ wkc(K(X)) = wkc(X).$$

Proof. (1) We already have $kc(K(X)) \leq a(K(X))$. To show that $a(X) \leq kc(K(X))$, let $\tau = kc(K(X))$. Choose $\Omega = \{\Lambda_\alpha : \alpha \in \Gamma\}$ a cover of $K(X)$ such that $|\Gamma| \leq \tau$ and each Λ_α is compact in $K(X)$. Now, if A is compact in X then $A \in \Lambda_\alpha$ for some α , so that $A \subset U\Lambda_\alpha$. It then follows that the collection $U\Omega = \{U\Lambda_\alpha : \alpha \in \Gamma\}$ is a c -cover of X . Since each $U\Lambda_\alpha$ is compact in X , then $a(X) \leq |\Gamma| \leq \tau$.

It remains to prove that $a(K(X)) \leq a(X)$. If so, then the sequence of inequalities $a(X) \leq kc(K(X)) \leq a(K(X)) \leq a(X)$ will yield the needed equalities. Now, let $\tau = a(X)$. Choose $\vartheta = \{C_\alpha : \alpha \in \Gamma\} \subset K(X)$ such that ϑ is a c -cover for X with $|\Gamma| \leq \tau$. Put $K(\vartheta) = \{K(C_\alpha) : \alpha \in \Gamma\}$. Then each $K(C_\alpha)$ is a compact subset of $K(X)$. If Λ is any compact subspace in $K(X)$, then $U\Lambda$ is compact in X so that $U\Lambda \subset C_\alpha$ for some $\alpha \in \Gamma$. But then $\Lambda \subset K(C_\alpha)$. Therefore, $a(K(X)) \leq |K(\vartheta)| \leq a(X)$.

(2) To see that $wkc(X) = wkc(K(X))$, let first $\tau = wkc(K(X))$. Then there is $\Omega = \{\Lambda_\alpha : \alpha \in \Gamma\}$ such that each Λ_α is compact in $K(X)$, $U\{\Lambda_\alpha : \alpha \in \Gamma\}$ is dense in $K(X)$

and $|\Gamma| \leq \tau$. If we put $A_\alpha = \bigcup \Lambda_\alpha$ then each $A_\alpha \in K(X)$ and $\bigcup \{A_\alpha : \alpha \in \Gamma\}$ is dense in X . Therefore, $\text{wkc}(X) \leq \tau$.

For the reverse inequality let $\tau = \text{wkc}(X)$. Choose ∂ a subset of $K(X)$ such that ∂ is closed under finite unions with $|\partial| \leq \tau$ and $\bigcup \partial$ dense in X . Take $K(\partial) = \{K(A) : A \in \partial\}$ and $D = \bigcup \partial$. Then $F(D) \subset \bigcup \{K(A) : A \in \partial\} \subset K(D) \subset K(X)$. Since D is dense in X , then so is $F(D)$ in $K(X)$. It then follows that $\bigcup \{K(A) : A \in \partial\}$ is dense in $K(X)$. Thus, $\text{wkc}(K(X)) \leq |K(\partial)| \leq \tau$.

Corollary 2.1.

(1) *The following are equivalent:*

- (i) $K(X)$ is σ -compact
- (ii) $K(X)$ is hemicompact
- (iii) X is hemicompact

(2) $K(X)$ is almost σ -compact if and only if X is almost σ -compact.

The next result follows from part (7) of Theorem 1.1, part (1) of Theorem 2.1, and the equality $\text{kknw} = \text{knw.a}$.

Theorem 2.2. $\text{kknw}(K(X)) = \text{kknw}(X)$

Corollary 2.2. $K(X)$ is a hemicompact $\aleph_0$ -space if and only if X is a hemicompact $\aleph_0$ -space.

3. Cardinal Functions on $C(X)$

The concept of a *diagonal degree* for a space X used in the next theorem is defined by $\Delta(X) = \omega + \min\{\tau : X \text{ has sequence } \{\Lambda_\alpha : \alpha < \tau\} \text{ of open covers with } \bigcap_{\alpha < \tau} \text{st}(p, \Lambda_\alpha) = \{p\} \text{ for all } p \in X\}$ (see [9]).

Theorem 3.1.

- (1) $d(C(X)) = ww(X) = ww(K(X));$
- (2) $nw(C(X)) = knw(C(X)) = knw(X) = knw(K(X)) = nw(K(X));$
- (3) $\psi(C(X)) = \Delta(C(X)) = wkc(X) = wkc(K(X));$
- (4) $q(C(X)) \leq \chi(C(X)) = \pi\chi(C(X)) = a(X) = a(K(X));$
- (5) $w(C(X)) = \pi w(C(X)) = kknw(X) = kknw(K(X));$
- (6) $ww(C(X)) \leq \tau$ if and only if $wkc(X) \leq \tau$ and $knw(X) \leq 2^\tau$.

Proof. (1) See Theorem 1 of [22].

(2) Generalize Michael's proof of the countable version in [18].

(3) The pseudocharacter and the diagonal degree are identical for any topological group (see [2], p. 153). Note that $C(X)$ is a locally convex topological vector space (hence, is a topological group). Therefore, $\psi(C(X)) = \Delta(C(X))$. For a proof of the equality $\psi(C(X)) = wkc(X)$, see [24].

For (4) and (5), see Theorem 2 and Theorem 3 of [24].

Proof of (6): Suppose that $ww(C(X)) \leq \tau$. Since $\psi(C(X)) \leq ww(C(X))$, then $wkc(X) \leq \tau$ by part (3). Let M be a space of weight $\leq \tau$ and $f: C(X) \rightarrow M$ a continuous bijection. Then $|C(X)| = |M| \leq 2^{w(M)} \leq 2^\tau$. Therefore, $knw(X) = nw(C(X)) \leq |C(X)| \leq 2^\tau$.

Conversely, suppose that $wkc(X) \leq \tau$ and $knw(X) \leq 2^\tau$. Then $\psi(C(X)) \leq \tau$ and $|C(X)| \leq [nw(C(X))]^{\psi(C(X))}$ (see Theorem 4.1 of [9]). Therefore, $|C(X)| \leq [knw(X)]^\tau \leq 2^\tau$. Now, a generalization of a result in Vidossich [30] insures that $ww(C(X)) \leq \tau$.

Corollary 3.1. $C(X)$ admits a coarser separable metrizable topology if and only if X is almost σ -compact and $\text{knw}(X) \leq 2^\omega$.

4. Covering Properties

We introduce the concept of a *compact-Lindelöf degree* on a space X as follows:

$$kL(X) = \omega + \min\{\tau: \text{every open } c\text{-cover of } X \text{ has a } c\text{-subcover of cardinality } \leq \tau\}.$$

Theorem 4.1. $kL(X) \leq L(K(X))$

Proof. Let $\tau = L(K(X))$. If Θ is an open c -cover of X , then $\langle \Theta \rangle = \{\langle \theta \rangle: \theta \in \Theta\}$ is an open cover of $K(X)$. Let $\langle \theta' \rangle$ be a subcover of $\langle \Theta \rangle$ such that $|\langle \theta' \rangle| \leq \tau$. Then $\{\theta: \theta \in \theta'\}$ is a c -subcover for Θ of cardinality $\leq \tau$. Therefore, $kL(X) \leq |\theta'| \leq \tau$.

The compact-Lindelöf degree can be used to characterize the tightness of $C(X)$.

Theorem 4.2. $kL(X) = t(C(X))$

A proof of Theorem 4.2 may be obtained by a simple generalization of Theorem 4.1.1 in [23]. (See also [16].)

[26] gives an example of a cosmic space X whose $K(X)$ is not paracompact. We improve this example in the next theorem.

Theorem 4.3. There exists a countable space X whose $K(X)$ is not paracompact.

Proof. Let X be the space of example 15 of [15]. Then X is a countable space whose $C(X)$ is not of countable

tightness. By Theorems 4.1 and 4.2 its hyperspace $K(X)$ is not Lindelöf. Since $K(X)$ is separable, then it cannot be paracompact.

Lemma 3 of [32] may be used to study the hereditary density and the hereditary Lindelöf degree of $K(X)$ and $C(X)$.

Theorem 4.4.

1. $\text{hd}([C(X)]^\omega) \leq \text{hL}([K(X)]^\omega)$
2. $\text{hL}([C(X)]^\omega) \leq \text{hd}([K(X)]^\omega)$

We now turn to the cellularity of $K(X)$ and $C(X)$. Let us define the *compact-weight* of X by $w_c(X) = \sup\{w(A) : A \in K(X)\}$.

Theorem 4.5.

1. $c(X) \leq c(K(X)) \leq \sup\{c(X^n) : n < \omega\}$
2. $w_c(X) = w_c(K(X))$
3. $w_c(X) \leq c(C(X))$.

Proof. For (1), $c(X) \leq c(K(X))$ is easy. For the inequality $c(K(X)) \leq \sup\{c(X^n) : n < \omega\}$ note first that $c(K(X)) = c(F(X))$. This is because $F(X)$ is dense in $K(X)$. The inequality $c(F(X)) \leq \sup\{c(X^n) : n \in \omega\}$ is a result of J. Ginsburg (see Bell [3], p. 18).

In (2) the inequality $w_c(X) \leq w_c(K(X))$ follows from the fact that $K(X)$ contains a (closed) copy of X . For the reverse inequality, let Λ be a compact subspace of $K(X)$. Since $U\Lambda$ is a compact subspace in X and $\Lambda \subset K(U\Lambda)$, then $w(\Lambda) \leq w(K(U\Lambda)) = w(U\Lambda) \leq w_c(X)$. Therefore, $w_c(K(X)) \leq w_c(X)$.

For a proof of (3), let A be compact in X and $j: C(X) \rightarrow C(A)$ be defined by $j(f) = f|_A$ for all f in $C(X)$. Then j is continuous and onto and, since $C(A)$ is metrizable, $w(A) = w(C(A)) = c(C(A)) \leq c(C(X))$. Clearly then $w_c(X) \leq c(C(X))$.

Recall that a space X is ccc if $c(X) = \omega$. The next result is due to G. Vidossich.

Theorem 4.6. If X is submetrizable, then $C(X)$ is ccc.

Corollary 4.6. If X is σ -compact, then $C(X)$ is ccc if and only if $C(X)$ is separable.

We close this section with a linear property on $C(X)$ that is characterized by the compact-weight of X . Let τ be an infinite cardinal and $(G, *)$ a topological group with an identity element e . $(G, *)$ is τ -bounded if, for every neighborhood W of e in G there exists a subset A_W of G such that $|A_W| \leq \tau$ and $G = A_W * W = \{a * w: a \in A_W \text{ and } w \in W\}$. τ -boundedness is weaker than the cellularity and the (weak) Lindelöf degree. In fact, Arhangel'skii [2] has characterized τ -bounded groups as being groups topologically isomorphic to a sub-group of a group of cellularity $\leq \tau$.

Theorem 4.7. $C(X)$ is τ -bounded if and only if $w_c(X) \leq \tau$.

Proof. Suppose that $C(X)$ is τ -bounded. If A is a compact subspace of X , then the function $\phi: C(X) \rightarrow C(A)$ defined by $\phi(f) = f|_A$ for each $f \in C(X)$ is a group homomorphism that is a continuous surjection. It then follows

that $C(A)$ is a metrizable τ -bounded group (τ -boundedness is preserved by group homomorphisms). It is easy to see that $d(C(A)) \leq \tau$. Since $d(C(A)) = w(A)$, then $w_C(X) \leq \tau$.

For the converse, assume that $w_C(X) \leq \tau$ and let Y be the topological sum of the non-empty compact subsets of X . If $p: Y \rightarrow X$ is the natural map, then p is compact-covering so that the induced function $p^*: C(X) \rightarrow C(Y)$ is an embedding of $C(X)$ into $C(Y)$. We are through if we show that $C(Y)$ has cellularity $\leq \tau$. To see this, note first that $C(Y)$ is homeomorphic to the product space $\prod\{C(A) : A \in K(X)\}$. Now, since $d(C(A)) = w(A) \leq \tau$ for each A in $K(X)$, then $c(\prod\{C(A) : A \in K(X)\}) \leq \tau$ by Corollary 11.3 of [9]. It follows that $c(C(Y)) \leq \tau$.

Corollary 4.7. The following are equivalent:

- (1) $C(X)$ is $\aleph_0$ -bounded;
- (2) each compact subset of X is metrizable;
- (3) X is a compact-covering image of a metrizable space.

5. Completeness Properties

This section is devoted to the completeness properties of $K(X)$ and $C(X)$.

A completely metrizable separable space is usually called a *Polish space*.

Theorem 5.1.

1. $K(X)$ is completely metrizable if and only if X is completely metrizable.

2. $K(X)$ is Čech-complete if and only if X is Čech-complete ([7]).

3. $K(X)$ is Polish if and only if X is Polish.

4. $K(X)$ is Dieudonné-complete if and only if X is Dieudonné-complete ([31]).

5. $K(X)$ is real compact if and only if X is real compact ([31]).

6. If $K(X)$ is Baire (respectively, of second category in itself), then so is X .

7. If X is pseudocomplete, then so is $K(X)$.

Theorem 5.2. ([17, [23]]). *The following are equivalent:*

- (i) $C(X)$ is completely metrizable;
- (ii) $C(X)$ is Čech-complete;
- (iii) $C(X)$ is almost Čech-complete;
- (iv) X is a hemicompact k -space.

Theorem 5.3. ([23]). *The following are equivalent:*

- (i) $C(X)$ is Polish;
- (ii) $C(X)$ contains a dense Polish subspace;
- (iii) $C(X)$ contains a dense copy of $\mathbb{R}^\omega$;
- (iv) $C(X)$ is homeomorphic to $\mathbb{R}^\omega$;
- (v) X is a hemicompact cosmic k -space.

Theorem 5.4. ([29]). *If $C(X)$ is a Baire space, then every infinite pairwise disjoint family in $K(X)$ contains an infinite subfamily Λ such that $U\Lambda$ is discrete and C -embedded in X .*

6. Analyticity

A Tychonoff space X will be called an *analytic space* if it is a continuous image of a Polish space, or equivalently, a continuous image of the space of irrational numbers. Note that the latter is homeomorphic to the product space $\mathbb{N}^\omega$.

Every Polish space is analytic and analytic spaces are cosmic.

Theorem 6.1. ([5], [23]). *If X is a quasi- k -space, then $C(X)$ is analytic if and only if X is a σ -compact $\aleph_0$ -space.*

Corollary 6.1. ([23]). *If X is a q -space, then $C(X)$ is analytic if and only if X is a σ -compact metrizable space.*

Theorem 6.2. *If X is a compact-covering image of a Polish space, then $K(X)$ is analytic.*

Is the converse to Theorem 6.2 true?

In the next theorem we denote the Cantor set by K . Note that every compact metric space is a continuous image of K .

Theorem 6.3. *The following statements are all equivalent for a q -space X :*

- (1) $K(X)$ is analytic;
- (2) $K(X)$ is Polish;
- (3) $C(K, X)$ is Polish;
- (4) $C(K, X)$ is analytic;
- (5) X is Polish.

Proof. (5) implies (2): follows from Theorem 5.1.

(2) implies (1) is clear. (1) implies (5): Suppose $K(X)$ analytic and let $\phi: N^\omega \rightarrow K(X)$ be a continuous surjection. X is an $\aleph_0$ -space since $K(X)$ is cosmic. Therefore X is separable metrizable. By Theorem 3.3 of [6], X will become Polish if we can find a Polish space Y and a function ξ from $K(Y)$ onto $K(X)$ satisfying the following conditions:

- (i) If $A \subset B$ then $\xi(A) \subset \xi(B)$ for all A, B in $K(Y)$;
- (ii) If $A \in K(X)$, then there is $B \in K(Y)$ such that $A \subset \xi(B)$.

Now, take $Y = N^\omega$ and define $\xi: K(N^\omega) \rightarrow K(X)$ by $\xi(\{m \in N^\omega: m \leq n\}) = \bigcup \{\phi(m): m \leq n\}$ for each $n \in N^\omega$. Then ξ satisfies the required conditions. So X is Polish. (5) implies (3): If X is Polish, then $C(K, X)$ is completely metrizable and separable. (3) implies (4) is obvious. (4) implies (1): Suppose $C(K, X)$ analytic. Define $\gamma: C(K, X) \rightarrow K(X)$ by $\gamma(f) = f(K)$ for each $f \in C(K, X)$. Then γ is a continuous surjection. It follows that $K(X)$ is analytic, being a continuous image of an analytic space.

From Theorem 6.3, one concludes that in Michael's result " $K(X)$ $\aleph_0$ -space if and only if X is $\aleph_0$ -space," we cannot replace " $\aleph_0$ -space" by "analytic space." As an example, take X to be the space of rational numbers.

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