TOPOLOGY PROCEEDINGS
Volume 10, 1985
Pages 377–384

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

In the study of the fixed point properties of a continuous map $f: X \to X$ from a compact, connected ANR X into itself, a number of interesting numbers are associated with f. In particular, we are interested in the Nielsen number $N(f)$, fixed point index $i\ (X,f,U)$, and the Lefschetz number $L(f)$.

An advantage of studying the Nielsen number is that every map g homotopic to f has at least $N(f)$ fixed points while the Lefschetz fixed point theorem says that if $L(f) \neq 0$ then $g \simeq f$ has a fixed point. Furthermore, in many cases, for example, a manifold $X = M^n$, $n \geq 3$, there is a map g homotopic to f that has exactly $N(f)$ fixed points. Unfortunately, this number is rather hard to compute in general. In 1964, Jiang [6] has shown that if X satisfies the so called Jiang condition, then every essential fixed point class F of f has the same fixed point index $i(f) = i(X,f,U)$, $U \supset F$, such that $L(f) = i(f)N(f)$. This theorem strengthens the Lefschetz fixed point theorem and has a number of interesting applications. Thus it is very interesting to know exactly what kind of spaces satisfy the Jiang condition.

Let $M(X)$ be the space of all continuous maps from X into itself with the compact open topology. For a given base point $x \in X$, the evaluation map $\gamma: M(X) \to X$ defined by $\gamma(g) = g(x)$ for every $g \in M(X)$ becomes a continuous map.

Then γ induces a homomorphism $\gamma_{\#}: \pi_1(M(X),id) \to \pi_1(X,x)$ of their fundamental groups. The image group $\gamma_{\#}(\pi_1(M(X),id))$ in $\pi_1(X,x)$ is called a Jiang subgroup $T(X)$ of $\pi_1(X,x)$. It is well known that $T(X)$ lies in the center of $\pi_1(X,x)$, and thus if $\gamma_{\#}$ is an onto homomorphism then $\pi_1(X,x)$ must be an abelian group. If $T(X) = \pi_1(X,x)$ then we say that X satisfies the Jiang condition (J-condition), and X is called the Jiang space (J-space). The well known examples are the lens spaces and the H-spaces.

In 1967, Brown [2] studied the product properties of Nielsen numbers for a fiber-preserving maps triple $(f,f_b,\tilde{f})$ of a locally trivial fiber space $\mathcal{J} = \{E,p,B\}$ into itself in order to devise a new method of computing the Nielsen numbers. He has shown $N(f) = N(f_b) \cdot N(f)$ in many situations. Note that on $\mathcal{J}$ the Euler characteristic χ satisfies the relations $\chi(E) = \chi(p^{-1}(b))\chi(B)$ and furthermore, the Lefschetz numbers satisfy $L(f) = L(f_b)L(\tilde{f})$. In 1975, Pak [10] has shown that if the spaces in $\mathcal{J}$ satisfy the J-condition, then there is a constant $P(f)$ depending on f such that $N(f)P(f) = N(f_b)N(\tilde{f})$ for each $b \in B$. Thus it became an interesting question, to ask under the what conditions on $\mathcal{J}$ if two spaces satisfy the J-condition then the other space satisfies the J-condition. In [8] and [10], we have shown that if $\mathcal{J}$ is a principal torus bundle over a lens space or a complex projective space then the total space E satisfies the J-condition. In this paper, we generalize the above result. Theorem 2 says that the total space of a (principal) fiber bundle over a simply connected space satisfies the J-condition. We also show that if the total space E is simply connected then the fiber has to satisfy the J-condition. The last theorem deals with aspherical manifolds. If $\mathcal{J}$ is a principal fiber bundle and if E is an aspherical manifold satisfying the J-condition then B satisfies the J-condition. An open problem is to eliminate some of the conditions we imposed to prove these theorems.

2. Main Theorems

In order to prove our first theorem let us introduce the notion of cyclic homotopy from [4] and [7]. Let $[\alpha] \in \pi_1(X,x_0)$. The loop α is called the *trace* of a cyclic homotopy H if there is a homotopy $H\colon X \times I \to X$ such that $H(x_0 \times I) = \alpha$ and $H(x \times 0) = x = H(x \times 1)$ for all $x \in X$. The homotopy classes of the traces of cyclic homotopies form a subgroup of $\pi_1(X,x_0)$ and it is well known that this subgroup is isomorphic to $T(X)$, the Jiang subgroup of $\pi_1(X,x_0)$. Note that if X is path-connected then at any point $x \in X$, the loop $H(x \times I)$ is homotopic to $\alpha = H(x_0 \times I)$.

Theorem 1. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable, locally trivial fiber space. If the total space* E *is simply connected, then the fibers* $p^{-1}(b)$, $b \in B$, *satisfy the* J-*condition.*

Proof. Let $[\alpha] \in \pi_1(p^{-1}(b),e)$. We want to show that $[\alpha]$ is the homotopy class of the trace of a cyclic homotopy. From the fiber-homotopy exact sequence

$$- - - \to \pi_2(B,b) \overset{d_\#}{\to} \pi_1(p^{-1}(b),e) \overset{i_\#}{\to} \pi_1(E,e) = 0$$

we have an element $[\beta] \in \pi_2(B,b)$ such that $d_\#[\beta] = [\alpha]$. We examine $d_\#$ more carefully in terms of [5].

We have the following commutative diagram

$$\begin{array}{ccccc} p^{-1}(b) X (I^2,I^1,J^1) & \overset{\overline{g}}{- - \to} & (E,p^{-1}(b),e) & \overset{j}{\leftarrow} & E \\ \text{incl} \nwarrow & & q \downarrow & \swarrow p & \\ (I^2,I^1,J^1) & \overset{\beta}{\to} & (B,b) & & \end{array}$$

where $\beta: I^2 \to B$ is such that $\beta(\partial I^2) = b$ and $[\beta] \in \pi_2(B,b)$. Let q be a map such that $p = qj$ and q induces an isomorphism $q_\#: \pi_n(E,p^{-1}(b),e) \to \pi_n(B,b)$ for all n. Then we have the isomorphism $q_\#^{-1}: \pi_n(B,b) \to \pi_n(E,p^{-1}(b),e)$ for all n. Let $\partial: \pi_n(E,p^{-1}(b),e) \to \pi_{n-1}(p^{-1}(b),e)$. Then $d_\#$ is defined to be $d_\# = \partial q_\#^{-1}$. In our case, let $q_\#^{-1}[\beta] \in \pi_2(E,p^{-1}(b),e)$ such that $d_\#[\beta] = \partial q_\#^{-1}[\beta] = [\alpha]$. Let $f: (I^2,I^1,J^1) \to (E,p^{-1}(b),e)$ such that $q_\#^{-1}[\beta] = [f]$. Then $\partial[f]$ is the homotopy class of $f|I^1$ such that $qf = \beta$, and $\partial[f] = [f|I^1] \in \pi_1(p^{-1}(b),e)$. We extend f to $\tilde{f}: p^{-1}(b) \times (I^2,I^1,J^1) \to (E,p^{-1}(b))$ by running $e \in p^{-1}(b)$ all over $p^{-1}(b)$. That is, at any $e \in p^{-1}(b)$, $\tilde{f}(e \times I^2,I^1,J^1) \to (E,p^{-1}(b),e)$ is such that $q\tilde{f} = \beta$, defined by $\tilde{f}|p^{-1}(b) = i(p^{-1}(b))$, where $i: p^{-1}(b) \to E$ is the inclusion map, and $\tilde{f}|(I^2,I^1,J^1) = f$. Then the restriction of $\tilde{f}$ to $p^{-1}(b) \times I^1$ becomes a desired cyclic homotopy. Thus $p^{-1}(b)$ satisfies the J-condition.

In order to prove the next theorem we study the group of homeomorphisms on X from [9]. Let G(X) be the group of all homeomorphisms of X onto itself with the compact open topology. Then G(X) becomes a subspace of M(X). Let $G_0(X)$ be the arc component of $id \in G(X)$, the identity map. It is a normal subgroup of G(X), and let $G(X)/G_0(X) = H(X)$

be the arc component group. Let $G(X,x) = \{g \in G(X) | g(x) = x\}$ and $G_0(X,x)$ be the arc component of $id \in G(X,x)$. Let $G(X,x)/G_0(X,x) = H(X,x)$ be the arc component group of $G(X,x)$.

If X is a homogeneous space or a manifold, $\{G(X), G(X,x), \gamma, X\}$ becomes a principal fiber bundle over X with the evaluation map $\gamma: G(X) \to X$ at $x \in X$ as the projection map. Let $\gamma_\#: \pi_1(G,id) \to \pi_1(X,x)$ be the induced homomorphism. If $\gamma_\#$ is an onto homomorphism then we say that X satisfies the P-*condition*. From the following commutative diagram

$$\begin{array}{ccc} & \pi_1(G,id) & \\ {}^{incl_\#}\swarrow & & \searrow^{r_\#} \\ \pi_1(Map(X),id) & \xrightarrow{\gamma_\#} & \pi_1(X,x) \end{array}$$

We can see that P-condition is a stronger condition on the space X than the J-condition. It would be interesting to find an example where X satisfies the J-condition but does not satisfy the P-condition.

Theorem 2. *Let* $\mathcal{J} = \{E,p,B\}$ *be a fiber bundle, where all the spaces in* $\mathcal{J}$ *are compact, connected manifolds. If* $\pi_1(B) = 0$ *and* $p^{-1}(b)$, $b \in B$, *satisfies* P-*condition, then the total space* E *satisfies the* J-*condition.*

Proof. Let $G(E)$, $G(p^{-1}(b))$, and $G(B)$ be the groups of homeomorphisms on the spaces E, $p^{-1}(b)$, and B respectively. Then we have the following diagram:

$$\begin{array}{ccccccc} & & & & \downarrow & & \\ & & & \tilde{\gamma}_\# & & & \\ - - \to & \pi_1(G(p^{-1}(b),id) & & \to & \pi_1(p^{-1}(b),e) & \to H(p^{-1}(b),e) \to & H(p^{-1}(b)) \to 0 \\ & & & \gamma_\# & \downarrow i_\# & & \\ - - \to & \pi_1(G(E),id) & & \to & \pi_1(E,e) & \to H(E,e) \to & H(E) \to 0 \\ & & & \gamma_\# & \downarrow p_\# & & \\ - - \to & \pi_1(G(B),id) & & \to & \pi_1(B,b) & \to H(B,b) \to & H(B) \to 0 \end{array}$$

where both the vertical and horizontal sequences are fiber homotopy exact sequences. Since $\pi_1(B,b) = 0$, $i_\#$ is an onto homomorphism. What we need to show is that $\gamma_\#$ is an onto map or $H(E,e) \simeq H(E)$. Let $\alpha \in \pi_1(E,e)$. Since $i_\#$ is onto we have an element $\beta \in \pi_1(p^{-1}(b),e)$ such that $i_\#(\beta) = \alpha$. Since $p^{-1}(b)$ satisfies the P-condition we have $H(p^{-1}(b),e) \simeq H(p^{-1}(b))$ and $\hat{\gamma}_\#$ is onto. Let $\hat{\gamma}_\#(\sigma) = \beta$, where $\sigma \in \pi_1(G(p^{-1}(b)),id)$. Since B is simply connected we can extend any homeomorphism of $p^{-1}(b)$ onto itself to a homeomorphism of E onto itself [11], i.e., we have an inclusion map incl: $G(p^{-1}(b)) \to G(e)$, and the following cummutative diagram:

$$\begin{array}{ccc} G(p^{-1}(b)) & \xrightarrow{\hat{\gamma}} & p^{-1}(b) \\ \text{incl}\downarrow & & \downarrow i \\ G(E) & \xrightarrow{\gamma} & E \end{array}$$

This diagram induces

$$\begin{array}{ccc} \pi_1(G(p^{-1}(b)),id) & \xrightarrow{\hat{\gamma}_\#} & \pi_1(p^{-1}(b),e) \\ \text{incl}_\#\downarrow & & \downarrow i_\# \\ \pi_1(G(E),id) & \xrightarrow{\gamma_\#} & \pi_1(E,e) \end{array}$$

Now we have $\gamma_\# \text{incl}_\#(\sigma) = i_\#\hat{\gamma}_\#(\sigma) = i_\#(\beta) = \alpha$. Thus $\gamma_\#$ is an onto homomorphism and this completes the proof.

Note. This theorem generalizes theorem 3.4 in [9], where McCarty shows that if $\mathcal{J}$ is a principal fiber bundle over a simply connected manifold B, then $H(E,e) \cong H(E)$ i.e., $\gamma_\#$ is an onto homomorphism.

Finally, I state a theorem which follows directly from various well-known theorems.

Theorem 3. *Let* $\mathcal{J} = \{E,p,B\}$ *be a principal torus* T^k, *bundle, where* E *is an aspherical manifold. If* E *satisfies the* J-*condition then* B *satisfies the* J-*condition.*

Proof. All we need to show is that B is aspherical and has an abelian fundamental group. Asphericity follows from [3], and the abelianess follows from the fiber homotopy exact sequence $- - - \to \pi_1(T^k,id) \xrightarrow{i_\#} \pi_1(E,e) \xrightarrow{p_\#} \pi_1(B,b) \to 0$, where $\pi_1(E,e)$ is an abelian group since it satisfies the J-condition. Now the theorem follows, since if X is aspherical and $\pi_1(X,x)$ is abelian then X satisfies the J-condition [1], [4].

Note. In this theorem, if the fiber homotopy exact sequence splits, then we can state that E satisfies the J-condition if and only if B satisfies the J-condition.

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