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We construct an example of non-metric hereditarily indecomposable continuum that has many of the properties of the pseudo-arc. In particular, we construct a non-metric hereditarily indecomposable homogeneous hereditarily equivalent continuum.

Definitions. A continuum is defined to be a compact connected Hausdorff space. Suppose λ is an ordinal, X_α is a topological space for each $\alpha < \lambda$, and if $\alpha < \beta$ then h_α^β is a mapping from X_β to X_α. Then the space $X = \varprojlim_{\alpha<\beta<\lambda} \{X_\alpha, h_\alpha^\beta\}$ denotes the space which is the inverse limit of the inverse system $\{X_\alpha, h_\alpha^\beta\}_{\alpha<\beta<\lambda}$. Each point of X is a function $P\colon \lambda \to \cup_{\alpha<\lambda} X_\alpha$ such that for all $\alpha < \beta < \lambda$: $P(\alpha) = P_\alpha \in X_\alpha$ and $P_\alpha = h_\alpha^\beta(P_\beta)$. A basis for the topology is the collection to which the set U belongs if and only if there exists a $\beta < \lambda$ and an open set O_β of X_β so that $U = \{P \mid P_\beta \in O_\beta\}$. Let $\pi_\alpha\colon X \to X_\alpha$ be defined by $\pi_\alpha(P) = P_\alpha$.

Suppose that M is a continuum and $P \in M$. Then C is the composant of M at P means that C is the point set to which x belongs if and only if there is a proper subcontinuum of M containing x and P. The set C is a composant of M means that C is a composant of M at some point of M. A pseudo-arc is a nondegenerate hereditarily indecomposable metric chainable continuum. The pseudo-arc is homogeneous [Bi] and hereditarily equivalent [Ms].

We use the following results due to Wayne Lewis [L2].

Theorem A. *Suppose that* M *is a one-dimensional continuum. Then there exists a one dimensional continuum* $\hat{M}$ *and a continuous decomposition* G *of* $\hat{M}$ *into pseudo-arcs so that the decomposition space* $\hat{M}/G$ *is homeomorphic to* M. *Furthermore, if* $\pi\colon \hat{M} \to \hat{M}/G$ *is the mapping so that* $\pi(x)$ *is the element of* G *containing* x *then if* $h\colon \hat{M}/G \to \hat{M}/G$ *is a homeomorphism then there exists a homeomorphism* $\hat{h}\colon \hat{M} \to \hat{M}$ *so that* $\pi \circ \hat{h} = h \circ \pi$.

Theorem B. *Under the hypothesis of theorem* A *if* x *and* y *are elements of the same pseudo-arc in* G *then there exists a homeomorphism* $\hat{h}\colon \hat{M} \to \hat{M}$ *so that* $\hat{h}(x) = y$ *and* $\pi \circ \hat{h} = \pi$.

From the fact that the pseudo-arc of pseudo-arcs is unique [L1], we have the following:

Corollary B. *Suppose that* X *is a pseudo-arc and* G *is a continuous collection of pseudo-arcs filling* X, *so that for each* $x \in X$, $\pi(x)$ *is the element of* G *that contains* x, *and* $Y = X/G$. *Then* Y *is a pseudo-arc and if* $h\colon Y \to Y$ *is a homeomorphism then there exists a homeomorphism* $\hat{h}\colon X \to X$ *so that* $\pi \circ \hat{h} = h \circ \pi$.

Example 1. Let X_1 be a pseudo-arc, let X_2 be a pseudo-arc, and let G_2 be a continuous decomposition of X_2 into pseudo-arcs. Then X_2/G_2 is a pseudo-arc and is homeomorphic to X_1. Let f_1^2 be the open monotone map, $f_1^2\colon X_2 \to X_1$ so that $G_2 = \{f_1^{2-1}(x) \mid x \in X_1\}$. By induction, construct $\{X_\alpha\}_{\alpha<\omega_1}$ as follows. Suppose $\gamma < \omega_1$ and X_α and f_α^β have been constructed for all α and β such that if $\alpha < \beta < \lambda$ then X_α is a pseudo-arc and $f_\alpha^\beta\colon X_\beta \to X_\alpha$ is an open monotone map. Suppose $\lambda < \omega_1$ is not a limit ordinal. Then λ has a predecessor $\lambda - 1$. Then let X_λ be a pseudo arc and let G_λ be a continuous decomposition of X_λ into pseudo-arcs. Then X_λ/G_λ is homeomorphic to $X_{\lambda-1}$ so there is an open monotone map $f_{\lambda-1}^\lambda\colon X_\lambda \to X_{\lambda-1}$ so that $G_\lambda = \{f_{\lambda-1}^{\lambda}{}^{-1}(x) \mid x \in X_{\lambda-1}\}$. For $\alpha < \lambda-1$ let $f_\alpha^\lambda = f_\alpha^{\lambda-1} \circ f_{\lambda-1}^\lambda$. Suppose that λ is a limit ordinal. Then $\{X_\alpha, f_\alpha^\beta\}_{\alpha<\beta<\lambda}$ is an inverse system. Let $X_\lambda = \varprojlim_{\alpha<\beta}\{X_\alpha, f_\alpha^\beta\}$. If $\lambda < \omega_1$ then some countable set is cofinal in λ so X_λ is homeomorphic to an inverse limit of pseudo-arcs and hence must be a metric chainable hereditarily indecomposable continuum. So X_λ is a pseudo-arc. If $\alpha < \lambda$ then let $f_\alpha^\lambda\colon X_\lambda \to X_\alpha$ denote the projection of X_λ onto the $\alpha^{\underline{\text{th}}}$ coordinate space X_α.

Let M denote the space $X_{\omega_1} = \varprojlim_{\alpha<\beta<\omega_1} \{X_\alpha, f_\alpha^\beta\}$.

Theorem 1.1. *The space* M *is a non-metric chainable hereditarily indecomposable continuum.*

Proof. The chainability and hereditary indecomposability of M easily follows from the fact that each X_α is chainable and hereditarily indecomposable. The non-metrizability of M follows from the existence of an ω_1-long monotonic sequence of subcontinua of M which is constructed below.

Let $L_1 = X_1$, $I_1 = X_{\omega_1}$, and $P_1 \in L_1$. Let $I_2 = \{x \in M \mid x_1 = P_1\}$, $L_2 = \{x_2 \in X_2 \mid x \in I_2\} = \pi_2(I_2)$, and $P_2 \in L_2$. By the construction of X_α, L_2 is nondegenerate and in fact $L_2 = f_1^{2^{-1}}(P_1)$.

Let $\lambda < \omega_1$.

Suppose I_α, P_α, and L_α have been constructed for all $\alpha < \lambda$.

Case i: λ is not a limit ordinal and $\lambda = \lambda' + 1$. Then let $I_\lambda = \{x | x_{\lambda'} = P_{\lambda'}\}$, $L_\lambda = \{x_\lambda \in X_\lambda | x \in I_\lambda\} = \pi_\lambda(I_\lambda)$, and $P_\lambda \in L_\lambda$.

Case ii: λ is a limit ordinal. Then let $I_\lambda = \cap_{\alpha<\lambda} I_\alpha$, $L_\lambda = \pi_\lambda(I_\lambda)$, and $P_\lambda \in L_\lambda$.

Note that if $\alpha \neq \beta$ then $I_\alpha \neq I_\beta$ and if $\alpha < \beta$ then $I_\beta \subset I_\alpha$. So $\{I_\lambda\}_{\lambda<\omega_1}$ is the required monotonic collection.

Theorem 1.2. *The space* M *is homogeneous.*

Proof. Let x and y be two points of M. Since X_1 is homogeneous there exists a homeomorphism $h: X_1 \to X_1$ so that $h(x_1) = y_1$. By theorem A there is a homeomorphism $g: X_2 \to X_2$ so that $h \circ f_1^2 = f_1^2 \circ g$. Note that $f_1^2 \circ g(x_2) = h \circ f_1^2(x_2) = h(x_1) = y_1$ and $f_1^2(y_2) = y_1$. So $g(x_2)$ and y_2 both belong to the same element of G_2. So by theorem B there exists a homeomorphism $k: X_2 \to X_2$ so that $k \circ g(x_2) = y_2$ and $f_1^2 \circ k = f_1^2$. Thus $k \circ g: X_2 \to X_2$ is a homeomorphism with $f_1^2 \circ k \circ g = f_1^2 \circ g = h \circ f_1^2$ and $k \circ g(x_2) = y_2$. Define $\theta_1 = h$, and $\theta_2 = k \circ g$. Thus $\theta_1 \circ f_1^2 = f_1^2 \circ \theta_2$.

Proceeding by induction, suppose that $\lambda < \omega_1$ and θ_α has been defined for all $\alpha < \lambda$ so that if $\alpha < \beta < \lambda$ then $\theta_\alpha \circ f_\alpha^\beta = f_\alpha^\beta \circ \theta_\beta$.

Case i: λ is not a limit ordinal and $\lambda = \lambda' + 1$ for some λ'. Then using the same argument as above there exists

$\theta_\lambda: X_\lambda \to X_\lambda$ so that $\theta_{\lambda'} \circ f^\lambda_{\lambda'} = f^\lambda_{\lambda'} \circ \theta_\lambda$ and $\theta_\lambda(x_\lambda) = y_\lambda$. If $\alpha < \lambda$, then $\theta_\alpha \circ f^\lambda_\alpha = \theta_\alpha \circ f^{\lambda'}_\alpha \circ f^\lambda_{\lambda'_0} = f^{\lambda'}_\alpha \circ \theta_\lambda \circ f^\lambda_\lambda = f^{\lambda'}_\alpha \circ f^\lambda_{\lambda'} \circ \theta_\lambda = f^\lambda_\alpha \circ \theta_\lambda$.

Case ii: λ is a limit ordinal. Then, since X_λ is the inverse limit $\varprojlim_{\alpha<\beta<\lambda} \{X_\alpha, f^\beta_\alpha\}$, the collection $\{\theta_\lambda: X_\lambda \to X_\lambda\}_{\alpha<\lambda}$ induces a homeomorphism $\theta_\lambda: X_\lambda \to X_\lambda$ so that $\theta_\alpha \circ f^\lambda_\alpha = f^\lambda_\alpha \circ \theta_\lambda$.

Then since $M = X_{\omega_1}$ is the inverse limit $\varprojlim_{\alpha<\beta<\omega_1} \{X_\alpha, f^\beta_\alpha\}$. The collection $\{\theta_\alpha\}_{\alpha<\lambda}$ induces a homeomorphism $\theta: X_{\omega_1} \to X_{\omega_1}$ so that $f^{\omega_1}_\alpha \circ \theta = \theta_\alpha \circ f^{\omega_1}_\alpha$ and since $\theta_\lambda(x_\lambda) = y_\lambda$ we also have $\theta(x) = y$.

Definition. The continuum X is said to be *hereditarily equivalent* if it is homeomorphic to each of its nondegenerate subcontinua.

Theorem 1.3. *The space* M *is hereditarily equivalent.*

Proof. Let L be a nondegenerate subcontinuum of M. Let P and Q be two points of L. Then there exists $\lambda < \omega_1$ so that $P_\lambda \neq Q_\lambda$. Let L_α denote the projection of L into the $\alpha^{\underline{th}}$ coordinate. Thus $L_\alpha = \{x_\alpha | x \in L\} = f^{\omega_1}_\alpha(L)$. First we will show that if $\lambda < \gamma < \omega_1$ then

$$L_\gamma = f^{\gamma^{-1}}_\lambda(L_\lambda).$$

Clearly, $L_\gamma \subset f^{\gamma^{-1}}_\lambda(L_\lambda)$.

For each $x \in L_\lambda$ the set $f^{\gamma^{-1}}_\lambda(x)$ is a subcontinuum of X_λ. Since L_λ is nondegenerate, it follows that L_γ is not a subset of $f^{\gamma^{-1}}_\lambda(x)$. But by hereditary indecomposability one of L_γ and $f_\lambda^{\gamma^{-1}}(x)$ is a subset of the other. So $f_\lambda^{\gamma^{-1}}(L_\lambda) \subset L_\lambda$. Therefore we have $L_\gamma = f_\lambda^{\gamma^{-1}}(L_\lambda)$. Notice that this argument also verifies that $f_\lambda^\gamma|_{L_\gamma} : L_\gamma \to L_\lambda$ is a monotone map. Thus $L = \varprojlim_{\lambda<\alpha<\beta<\omega_1} \{L_\alpha, f_\alpha^\beta|_{L_\beta}\}$.

The set ω_1 is order isomorphic to the set $\{\gamma | \lambda < \gamma < \omega_1\}$. Let ψ be the isomorphism. Suppose $\lambda < \omega_1$ and $\{\theta_\alpha\}_{\alpha<\lambda}$ have been defined so that for all $\alpha < \beta < \lambda$

$$\theta_\alpha \circ f_\alpha^\beta = f_{\psi(\alpha)}^{\psi(\beta)}\Big|_{L\psi(\beta)} \circ \theta_\beta.$$

If λ is not a limit ordinal and $\lambda = \gamma + 1$ then using Wayne Lewis's results there exists a homeomorphism $\theta_{\gamma+1}: X_{\gamma+1} \to L_{\psi(\gamma+1)}$ so that the following diagram commutes

$$\begin{array}{ccccc}
X_\gamma & \xleftarrow{f_\gamma^{\gamma+1}} & X_{\gamma+1} & \leftarrow \cdots & X_\beta \\
\Big\downarrow \theta_\gamma & & \Big\downarrow \theta_{\gamma+1} & & \\
L_{\psi(\gamma)} & \xleftarrow[f_{\psi(\gamma)}^{\psi(\gamma+1)}|_{L_{\psi(\gamma+1)}}]{} & L_{\psi(\gamma+1)} & \leftarrow \cdots & L_{\psi(\beta)}.
\end{array}$$

If λ is a limit ordinal the maps $\{\theta_\gamma\}_{\gamma<\lambda}$ induce a homeomorphism θ_λ of X_λ onto $X_{\psi(\lambda)}$. Therefore for all $\alpha < \beta < \omega_1$

$$\theta_\alpha \circ f_\alpha^\beta = f_{\psi(\alpha)}^{\psi(\beta)}\Big|_{L_{\psi(\beta)}} \circ \theta_\beta$$ and the maps $\{\theta_\gamma\}_{\gamma<\omega_1}$ induce a homeomorphism of M onto L.

Theorem 1.4. *The continuum* M *is irreducible from the point* x *to the point* y *if and only if* X_1 *is irreducible from the point* x_1 *to the point* y_1.

Proof. Suppose that X_1 is not irreducible from x_1 to y_1. Then there is a proper subcontinuum L_1 of X_1 containing x_1 and y_1. Let $L_2 = f_1^{2^{-1}}(L_1)$; then, since f_1^2 is monotone, L_2 is a subcontinuum of X_2 and it must be a proper subcontinuum of X_2 because L_1 is proper in X_1. Let us construct a collection $\{L_\alpha\}_{\alpha<\omega_1}$ by induction so that L_α is a proper subcontinuum of X_α containing x_α and y_α. Suppose that L_α has been defined for all $\alpha \in \lambda$. If λ is not a limit ordinal then let $L_\lambda = f_{\lambda-1}^{\lambda^{-1}}(L_{\lambda-1})$. Since $f_{\lambda-1}^{\lambda}$ is monotone and $L_{\lambda-1}$ is a proper subcontinuum of $X_{\lambda-1}$ then L_λ is a proper subcontinuum of X_λ, and L_λ contains x_λ and y_λ. If λ is a limit ordinal then let $L_\lambda = \varprojlim_{\alpha<\beta<\lambda} \{X_\alpha, f_\alpha^\beta|_{L_\beta}\}$. Since L_1 is a proper subcontinuum of X_1 then L_λ is a proper subcontinuum of X_λ. Since x_α and y_α lie in L_α for $\alpha < \lambda$, and for $\alpha < \beta < \lambda$ $f_\alpha^\beta(x_\beta) = x_\alpha$ and $f_\alpha^\beta(y_\beta) = y_\alpha$, then x_λ and y_λ lie in L_λ. Therefore $L = \varprojlim_{\alpha<\beta<\omega_1} \{L_\alpha, f_\alpha^\beta|_{L_\beta}\}$ is a proper subcontinuum of M. Furthermore by construction for each $\alpha < \omega_1$ the points x_α and y_α both lie in L_α. So L contains x and y hence M is not irreducible from x to y.

Suppose that M is not irreducible from the point x to the point y. Let L be a proper subcontinuum of M containing x and y. Then for some $\lambda < \omega_1$, $f_\lambda^{\omega_1}(L) \neq X_\lambda$. Let $L_\lambda = f_\lambda^{\omega_1}(L)$. Then x_λ and y_λ both lie in L_λ. Since L_λ is a proper subcontinuum of X_λ there is a point $z_\gamma \in X_\lambda - L_\lambda$. Let $z_1 = f_1^\lambda(z_\lambda)$. Then $f_1^{\lambda^{-1}}(z_1)$ is a subcontinuum of X_λ. But $z_\lambda \in f_1^{\lambda^{-1}}(z_1)$ and $z_\lambda \notin L_\lambda$ also $x_\lambda \in L_\lambda$ so $x_1 \neq z_1$ and hence $x_\lambda \notin f_1^{\lambda^{-1}}(z_1)$. So by hereditary indecomposability, L_λ and $f_1^{\lambda^{-1}}(z_1)$ are disjoint continua. Thus $z_1 \notin f_1^\lambda(L_\lambda)$ but x_1 and y_1 are elements of $f_1^\lambda(L_\lambda)$. Therefore, $f_1^\lambda(L_\lambda)$ is a proper subcontinuum of X_1 containing x_1 and y_1.

The following corollary follows easily from the construction and theorem 1.4.

Corollary 1.5. *The continuum* M *has* c *composants.*

Example 2. In [S3] an example of a hereditarily indecomposable continuum with exactly two composants was constructed. The example was an inverse limit of pseudo-arcs indexed by ω_1 with special types of retractions as bonding maps.

We will use the following theorems from [S3].

Theorem C. *Suppose that* X *is a pseudo-arc,* X *is irreducible from the point* P *to the point* Q*,* Y *is a pseudo-arc,* $X \subset Y$*, and* Y *is the union of two closed sets* H *and* K *so that* X *is a component of* H*,* $X \cap K = \{Q\}$*, and* $\mathrm{Bd}(H) = \mathrm{Bd}(K) = K \cap H$. *Then there is a retraction* h *of* Y *onto* X *so that* $h(K) = Q$*,* $h^{-1}(P) = P$*, and* $h(Y-X)$ *lies in the composant of* X *at* Q.

Suppose X is a continuum. Let us use the following notation. If $H \subset X$, let $\mathrm{Bd}_X(H)$ denote the boundary of H in X, let $\mathrm{Int}_X(H)$ denote the interior of H with respect to X, and let $\mathrm{Cl}_X(H)$ denote the closure of H in X. If $Q \in X$, then let Cmps (X,Q) denote the composant of X at Q.

Theorem C was used to construct the example in [S3]. The example which we will denote by N was constructed so that $N = \varprojlim_{\alpha<\beta<\omega_1} \{X_\alpha, h_\alpha^\beta\}$ and for each $\alpha < \omega_1$:

1) X_α is a pseudo-arc with $X_\alpha \subset X_{\alpha+1}$,

2) X_α is irreducible from the point P to the point Q_α,

3) $X_{\alpha+1}$ is the union of two closed sets $H_{\alpha+1}$ and $K_{\alpha+1}$ so that X_α is a component of $H_{\alpha+1}$, $X_{\alpha+1} \cap K_{\alpha+1} = \{Q_\alpha\}$, $\mathrm{Bd}_{X_{\alpha+1}}(H_{\alpha+1}) = \mathrm{Bd}_{X_{\alpha+1}}(K_{\alpha+1}) = H_{\alpha+1} \cap K_{\alpha+1}$, $Q_{\alpha+1} \in \mathrm{Int}_{X_{\alpha+1}}(K_{\alpha+1})$, and $Q_{\alpha+1} \not\in \mathrm{Cmps}(X_{\alpha+1}, Q_\alpha)$,

4) $h_\alpha^{\alpha+1}\colon X_{\alpha+1} \to X_\alpha$ is a retraction so that $h_\alpha^{\alpha+1}(K_{\alpha+1}) = Q_\alpha$, ${h_\alpha^{\alpha+1}}^{-1}(P) = P$, and $h_\alpha^{\alpha+1}(X_{\alpha+1} - X_\alpha) \subset \mathrm{Cmps}(X_\alpha, Q_\alpha)$.

Conditions 1-4 were used to obtain the following theorem [S].

Theorem D. *The continuum* $N = \varprojlim_{\alpha<\beta<\omega_1} \{X_{\alpha_1} h_\alpha^\beta\}$ *is a hereditarily indecomposable continuum with exactly two composants.*

By Theorem D it follows that N is a non-metric continuum. By Theorem D and Corollary 1.5 the continua M and N are not homeomorphic. It would be of interest to determine if N is homogeneous or hereditarily equivalent. We will show that N is neither of these, and we will obtain a general theorem about non-metric hereditarily indecomposable continua.

The fact that N is not hereditarily equivalent easily follows from the following observation.

Theorem 2.1. *The continuum* N *contains a pseudo-arc.*

Proof. The proof easily follows from the construction. From condition 4 $h_\alpha^{\alpha+1}: X_{\alpha+1} \to X_\alpha$ is a retraction and $h_\alpha^{\alpha+1}(X_{\alpha+1} - X_\alpha) \subset \mathrm{Cmps}(X_\alpha, Q_\alpha)$. So if I is a proper subcontinuum of X_α that does not intersect $\mathrm{Cmps}(X_\alpha, Q_\alpha)$ then ${f_\alpha^{\alpha+1}}^{-1}(I) = I$. Therefore, if L is a nondegenerate subcontinuum of X_1 that does not intersect $\mathrm{Cmps}(X_1, Q_1)$, then ${f_1^\alpha}^{-1}(L) = L$. So $\hat{L} = \varprojlim_{\alpha<\beta<\omega_1} \{L, f_\alpha^\beta|_L\}$ is a pseudo-arc since $f_\alpha^\beta|_L$ is the identity on L.

Definitions. Suppose X is a space and $x \in X$. Then X is *first countable at* x means that there is a countable collection of open sets that forms a basis at x. The point x is a P-*point* of X means that if $\{O_i\}_{i=1}^\infty$ is a countable collection of open sets each containing x, then there exists an open set O containing x such that $O \subset \bigcap_{i=1}^\infty O_i$.

The fact that N is not homogeneous easily follows from Theorems D and 2.1 as well as from the following theorem.

Theorem 2.2. *The continuum* N *contains both a point at which it is first countable and a* P-*point.*

Proof. First we show that the point $Q = \{Q_\alpha\}$ is a P-point of N. Suppose $\alpha < \omega_1$ and R is an open set in X_α then let $\overleftarrow{R} = \{x \in N | x_\alpha \in R\}$, the set $\overleftarrow{R}$ is open in N. Suppose $\{O_i\}_{i=1}^\infty$ is a countable sequence of open sets in N each containing Q. Then for each i there is an ordinal α_i and an open set R_i in X_{α_i}, so that $Q_{\alpha_i} \in R_i$ and $Q \in \overleftarrow{R}_i \subset O_i$.

Since $\{\alpha_i\}_{i=1}^{\infty}$ is countable there exists $\lambda < \omega_1$ so that $\alpha_i < \lambda$ for all positive integers i and so that λ is not a limit ordinal. Let U be an open set containing Q_λ so that $Cl_{X_\lambda}(U) \subset K_\lambda$, this can be done by condition 3. Then by condition 4, $f^{\lambda}_{\lambda-1}(Cl_X U) = Q_{\lambda-1}$ and hence $\overleftarrow{U} \subset O_{\alpha_1}$ for all α_i.

Now we prove that if $x \in X_1 - Cmps(X_1,Q_1)$ then the point $z \in N$ so that $z_\alpha = x$ for all $\alpha < \omega_1$ is a point of first countability of N. Let $\{U_i\}_{i=1}^{\infty}$ be a countable local basis of open sets of X_1 at x. We claim that $\{\overleftarrow{U}_i\}_{i=1}^{\infty}$ is a local basis for z in N. Suppose on the other hand that $\{\overleftarrow{U}_i\}_{i=1}^{\infty}$ is not a local basis for z. Then there is a point $y \neq z$ so that $y \in \cap_{i=1}^{\infty}\overleftarrow{U}_i$. Since $y \neq z$ there is a first λ so that $y_\lambda \neq z_\lambda = x$. Clearly λ is not a limit ordinal and $\lambda \neq 1$ since $\{U_i\}_{i=1}^{\infty}$ is a local basis for $x \in X_1$. Therefore, $f^{\lambda}_{\lambda-1}(y_\lambda) = x$. But $x \in X_1 \subset X_{\lambda-1} \subset X_\lambda$ and $f^{\lambda}_{\lambda-1}(X_\lambda - X_{\lambda-1}) \subset Cmps(X_{\lambda-1},Q_{\lambda-1})$. Also, $x \not\in Cmps(X_{\lambda-1}, Q_{\lambda-1})$ because for $\lambda = 2$ x was chosen so that $x \not\in Cmps(X_1,Q_1)$ and for $\lambda > 2$, X_1 is a proper subcontinuum of $X_{\lambda-1}$ that contains P and hence cannot intersect $Cmps(x_{\lambda-1},Q_{\lambda-1})$. Therefore, the only point of X_λ that is mapped onto x by $f^{\lambda}_{\lambda-1}$ is x. But this contradicts the fact that $y_\lambda \neq x$. So N is first countable at x. Similarly it can be shown that if $\lambda < \omega_1$ and $x \in X_\lambda - Cmps(X_\lambda,Q_\lambda)$ then N is first countable at the point z so that $z_\alpha = x$ for all $\lambda < \alpha < \omega_1$.

The next theorem shows that, in terms of the existence of points of first countability and P-points in hereditarily indecomposable continua example 2 is as complicated as it can get.

Theorem 3. *If* X *is a hereditarily indecomposable continuum then no proper subcontinuum of* X *can contain a* P*-point of* X *and a point at which* X *is first countable.*

Proof. Suppose X is a hereditarily indecomposable continuum, x is a P-point of X, y is a point of X at which X is first countable, and L is a proper subcontinuum of X containing both x and y. Let $\{R_i\}_{i=1}^{\infty}$ be a countable local basis at y so that $R_{i+1} \subset R_i$. Let $z \in X - L$.

Let I_n be the component of $X - R_n$ containing z. Then $I_n \cap \mathrm{Bd}_X(R_n) \neq \emptyset$, and since $y \notin I_n$ by hereditary indecomposability $I_n \cap L = \emptyset$. Thus $x \notin I_n$. Let K be the limiting set of $I_1, I_2, \cdots$. Since x is a P-point then $x \notin K$. Since y is the sequential limit of $\{I_n \cap \mathrm{Bd}_X(R_n)\}_{n=1}^{\infty}$ and $I_n \subset I_{n+1}$ for each n then K is a continuum that contains y. Thus $y \in L$, $y \in K$, $z \in K$, $z \notin L$, $x \notin K$, and $x \in L$; but this contradicts the hereditary indecomposability of X.

The following questions arise naturally from our discussion.

Question 1. Are there other non-metric hereditarily equivalent continua?

Question 2. Are there other non-metric homogeneous chainable continua? In particular, is there an inverse limit on a larger index set of chainable continua which is homogeneous?

Question 3. How many different inverse limits of pseudo-arcs indexed by ω_1 are there?

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