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DECOMPOSITIONS FOR CLOSED MAPS

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Introduction

Well-known decomposition theorems for closed maps are given by the following type (L):

(L) For spaces X, Y and a closed map $f: X \to Y$, $Y = Y_0 \cup (\cup_{n=1}^{\infty} Y_n)$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_n is closed discrete in Y for each $n \geq 1$.

First, Morita [10] showed that the following special type (M) of (L) holds for any paracompact and locally compact space X.

(M) For spaces X, Y and a closed map $f: X \to Y$, $Y = Y_0 \cup Y_1$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_1 is closed discrete in Y.

Lašnev [7] proved that (L) holds for any metric space X. Subsequently, Filippov [4] extended this result, showing that (L) holds for any paracompact M-space X. Moreover, many mathematicians extended Lašnev's "decomposition theorem" for several generalized metric spaces (for example, see [14] and [15] etc.). In particular, recently, Chaber [2] proved that (L) holds for any regular σ-space X.

Now, let us consider the following modifications (wL) and (wL)' of (L), which are the weakening of the compactness of $f^{-1}(y)$ in (L):

(wL) In (L), the $f^{-1}(y)$ is Lindelöf for each $y \in Y_0$.

(wL)' In (L), the $f^{-1}(y)$ is ω_1-compact for each $y \in Y_0$.

Here, we say that a space X is ω_1-*compact* if every uncountable subset of X has a cluster point in X. Every Lindelöf space is ω_1-compact. It should be remarked that the condition (wL) was considered in [2], where it was labeled (*'). The modifications (wM) and (wM)' of (M) are to be made similarly. That is,

(wM) In (M), the $f^{-1}(y)$ is Lindelöf for each $y \in Y_0$.

(wM)' In (M), the $f^{-1}(y)$ is ω_1-compact for each $y \in Y_0$.

In Section 1, we investigate the decomposition types (wL), (wL)' and some variations of these tyeps. Nagami [12] introduced the notion of Σ-spaces as a generalization of σ-spaces and M-spaces. We prove that (wL)' holds for any Σ-space X, thus (wL) holds for any strong Σ-space X. We shall remark that it is impossible to replace (wL) with (L) for every regular Lindelöf Σ-space X.

In Section 2, we discuss the decomposition types (M), (wM) and (wM)'. We prove that (M) holds for any space X dominated by compact sets X_α, or determined by a point-countable cover of compact sets X_α. If the X_α's are Lindelöf; ω_1-compact, then (wM); (wM)' holds respectively.

We assume that all spaces are T_1, and all maps are continuous and onto.

1. Decomposition Types (wL) and (wL)'

Let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of a space X. We say that $\mathcal{A}$ is *hereditarily closure-preserving* (abbreviated by HCP) if any collection $\{B_\lambda : \lambda \in \Lambda\}$ with $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$ is closure-preserving (that is, $\overline{\cup\{B_\lambda : \lambda \in \Lambda'\}} = \cup\{\overline{B}_\lambda : \lambda \in \Lambda'\}$ for any $\Lambda' \subset \Lambda$).

Lemma 1.1 ([15, *Lemma* 5.4]). *Let* Y *be a space, and* $\mathcal{F}$ *a* HCP *collection of closed sets in* Y. *For each* $n \geq 1$, *let*

$Y_n = \cup\{F_1 \cap \cdots\cdots \cap F_n\colon F_1, \cdots\cdots, F_n \in \mathcal{F}$ *and* $F_1 \cap \cdots\cdots \cap F_n$ *is a non-empty finite set*$\}$.

Then each Y_n *is a closed discrete subset of* Y.

Proof. It is routinely verified that each collection

$$\{F_1 \cap \cdots\cdots \cap F_n\colon F_i \in \mathcal{F}\ (i = 1, \cdots\cdots, n)\}$$

is HCP. Thus each subset of Y_n is closed in Y. Hence each Y_n is closed discrete in Y.

Let $\mathcal{K}$ be a cover of a space X. A cover $\mathcal{F}$ is called a *(mod* $\mathcal{K}$*)-net* for X [9], if, whenever $K \subset U$ with $K \in \mathcal{K}$ and U open in X, there is some $F \in \mathcal{F}$ such that $K \subset F \subset U$.

Lemma 1.2. *Let* X *be a space, and* $\mathcal{K}$ *a cover of* X *by countably compact sets. If* X *has a (mod* $\mathcal{K}$*)-net* $\mathcal{F}$ *which is countable, then it is* ω_1*-compact*.

Proof. Assume the contrary. Then there is an uncountable closed discrete subset D of X. For each $K \in \mathcal{K}$, since K is countably compact, $D \cap K$ is at most finite. Thus there is some $F_K \in \mathcal{F}$ such that $K \subset F_K$ and $D \cap F_K$ is at most finite. But, since $\mathcal{F}$ is countable and $\mathcal{K}$ is a cover of X, we may assume that $\{F_K\colon K \in \mathcal{K}\}$ is a countable cover of X. Thus there is some $F_K \in \mathcal{F}$ such that $D \cap F_K$ is infinite. This is a contradiction.

A space X is called a Σ*-space* [12] (Σ^**-space* [13]) if it has a σ-locally finite (σ-HCP) closed (mod $\mathcal{K}$)-net $\mathcal{F} = \cup_{n=1}^{\infty}\mathcal{F}_n$ for some closed cover $\mathcal{K}$ by countably compact sets. Here we can assume that $\mathcal{F}_n$ is a locally finite (HCP) closed cover of X and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each $n \geq 1$. It should be noted that any sequence $\{x_n\}$ such that

$$x_n \in \cap\{F \in \mathcal{F}_n : x \in F\} \text{ for some } x \in X$$

has a cluster point in X. Such a sequence $\{\mathcal{F}_n\}$ is called a Σ-*net* (Σ^*-*net*) of X.

Theorem 1.3. *If* X *is a* Σ-*space, then* (wL)' *holds.*

Proof. Let $\{\mathcal{F}_n\}$ be a Σ-net of X. We may assume that each $\mathcal{F}_n$ is finitely multiplicative. For each $n \geq 1$, put

$Y_n = \cup\{f(F) \cap f(F') : F, F' \in \mathcal{F}_n$ and $f(F) \cap f(F')$ is a non-empty finite set$\}$.

Since each $\mathcal{F}_n$ is locally finite in X and f is a closed map, $\{f(F) : F \in \mathcal{F}_n\}$ is HCP. It follows from Lemma 1.1 that Y_n is closed discrete in Y for each $n \geq 1$. Put $Y_0 = Y \setminus \cup_{n=1}^{\infty} Y_n$. Pick any $y \in Y_0$. Let us show that $f^{-1}(y)$ is ω_1-compact. It suffices to show from Lemma 1.2 that the subcollection $\{F \in \mathcal{F}_n : F \cap f^{-1}(y) \neq \emptyset\}$ of $\mathcal{F}_n$ is finite for each $n \geq 1$. Assume the contrary. Then there are some $m \geq 1$ and a sequence $\{F_n\}$ of distinct members of $\mathcal{F}_m$ such that each F_n meets $f^{-1}(y)$. Pick an $x_0 \in f^{-1}(y)$. By the choice of $\{\mathcal{F}_n\}$, $E_n = \cap\{F \in \mathcal{F}_n : x_0 \in F\}$ and F_n belong to $\mathcal{F}_n$ for each $n \geq m$. Since $y \in f(E_n) \cap f(F_n) \setminus Y_n$, $f(E_n) \cap f(F_n)$ is an infinite set. So we can choose a sequence $\{y_n\}_{n \geq m}$ of distinct points in Y such that $y_n \in f(E_n) \cap f(F_n)$. For each $n \geq m$, pick two points p_n and q_n from $E_n \cap f^{-1}(y_n)$ and $F_n \cap f^{-1}(y_n)$, respectively. Since $\{p_n\}_{n \geq m}$ has a cluster point in X, $\{y_n\}_{n \geq m}$ has also a cluster point in Y. On the other hand, $\{q_n : n \geq m\}$ is closed discrete in X, because $\{F_n : n \geq m\}$ is locally finite in X. Thus, since f is a closed map, $\{y_n : n \geq m\}$ is also closed discrete in Y. This contradiction completes the proof.

A space is called a *strong Σ-spaces* [12] if it satisfies the definition of a Σ-space for some closed cover $\mathcal{C}$ by compact sets instead of $\mathcal{K}$. By [6, Proposition 4.4], an ω_1-compact, strong Σ-space is Lindelöf. So Theorem 1.3 yields

Corollary 1.4. *If* X *is a strong Σ-space, then* (wL) *holds.*

Remark. In the previous corollary, by [15, Example 5.12] or [2, Example 1.2], we cannot replace "Lindelöf" with "compact" even if X is regular σ-compact. Chaber [2] showed that Corollary 1.4 holds under the assumption of X being a k-space.

Next, we proceed with some variations of (wL) or (wL)'.

Theorem 1.5. *Let* $f: X \to Y$ *be a closed map. If* X *is a Σ^*-space, then* $Y = Y_0 \cup (\cup_{n=1}^{\infty} Y_n)$, *where* $f^{-1}(y)$ *is* ω_1*-compact for each* $y \in Y_0$ *and* Y_n *is a discrete set such that* $\cup_{i=1}^{n} Y_i$ *is closed in* Y *for each* $n \geq 1$.

Proof. Let $\{\mathcal{F}_n\}$ be a Σ^*-net of X. For each $n \geq 1$, put

$\mathcal{C}_n = \{f(\cap\mathcal{E}) \cap f(F): \mathcal{E} \subset \mathcal{F}_n, F \in \mathcal{F}_n$ and $f(\cap\mathcal{E}) \cap f(F)$ is a non-empty finite set$\}$.

Since each $\mathcal{F}_n$ is a closure-preserving closed cover of X and f is a closed map, each $\mathcal{C}_n$ is a closure-preserving collection by finite sets. Then $Y_n' = \cup\mathcal{C}_n$ is a closed set in Y with a closure-preserving cover by finite sets for each $n \geq 1$. By [16, Theorem 1], we have $Y_n' = \cup_{k=1}^{\infty} Y_{nk}'$, where Y_{nk}' is a discrete subset and $\cup_{i=1}^{k} Y'_{ni}$ is closed in Y'_n for each $k \geq 1$. Hence $\cup_{n=1}^{\infty} Y'_n$ can be represented as $\cup_{n=1}^{\infty} Y_n$ described in the theorem. Put $Y_0 = Y \setminus \cup_{n=1}^{\infty} Y_n$. Pick any $y \in Y_0$. By a similar way as in the proof of Theorem 1, we can show that the subcollection $\{F \in \mathcal{F}_n : F \cap f^{-1}(y) \neq \emptyset\}$ of $\mathcal{F}_n$ is finite for each $n \geq 1$. Thus, it follows from Lemma 1.2 that $f^{-1}(y)$ is ω_1-compact. The proof is complete.

Let $\{F_n\}$ be a sequence of subsets of a space X. We say that $\{F_n\}$ *converges to* $E \subset X$ if for any open set V with $E \subset V$ there is some $m \geq 1$ such that V contains F_n for each $n \geq m$. The following lemma is a modification of Lašnev's lemma in [7].

Lemma 1.6. *Let* $\{F_n\}$ *be a sequence of non-empty closed sets in a regular space* X, *and let* E *be a closed Lindelöf subspace of* X. *If* $\{F_n\}$ *converges to* E *and each* F_n *is disjoint from* E, *then* $K = E \cap \overline{\cup_{n=1}^{\infty} F_n}$ *is compact.*

Proof. Suppose that K is not countably compact. Since X is regular and K is Linedlöf, there is an increasing sequence $\{U_k\}$ of open sets in X such that $K \subset \cup_{k=1}^{\infty} U_k$ and $(U_k \setminus \overline{U}_{k-1}) \cap K \neq \emptyset$ for each $k \geq 1$, where $U_0 = \emptyset$ (This choice is seen in the proof of [2, Theorem 1.1]). Since each $U_k \setminus \overline{U}_{k-1}$ meets infinitely many F_n's, there are two sequences $\{n_k\}$ and $\{x_k\}$ such that $n_k < n_{k+1}$ and $x_k \in (U_k \setminus \overline{U}_{k-1}) \cap F_{n_k}$ for each $k \geq 1$. Put $G = X \setminus \overline{\{x_k : k \geq 1\}}$. Then G is an open set in X. If $x \in E \setminus K$, then

$$x \notin \overline{\cup_{n=1}^{\infty} F_n} \supset \overline{\cup_{k=1}^{\infty} F_{n_k}} \supset \overline{\{x_k : k \geq 1\}} = X \setminus G.$$

Let $x \in E \cap K$. Take some $m \geq 1$ with $x \in U_m$. Since

$x_k \notin \overline{U}_{k-1} \supset U_m$ for each $k > m$, $x \notin \overline{\{x_k: k > m\}}$. Since $x \in E$ and $x_k \in F_{n_k}$, $x \neq x_k$ for each $k \geq 1$. These imply $x \notin \overline{\{x_k: k \geq 1\}} = X \backslash G$. Thus, we have $E \subset G$. But G does not contain any F_{n_k}'s. This contradicts to the fact that $\{F_n\}$ converges to E. So we conclude that K is countably compact. Therefore, K is compact.

Using Lemma 1.6 instead of [14, Lemma 2.1], the proof of the following theorem is quite parallel to that of [14, Theorem 1.2]. So the details are left to the reader.

Theorem 1.7. *Let* $f: X \to Y$ *be a closed map. If* X *is a regular semi-stratifiable space* [3], *then*

$$\{y \in Y: f^{-1}(y) \text{ is Lindelöf}\} = Y_0 \cup (\cup_{n=1}^{\infty} Z_n),$$

where $f^{-1}(y)$ *is compact for each* $y \in Y_0$ *and* Z_n *is closed discrete in* Y *for each* $n \geq 1$.

Since σ-spaces are semi-stratifiable, strong Σ-spaces, the following is an immediate consequence of Corollary 1.4 and Theorem 1.7.

Corollary 1.8 ([2, *Theorem* 1.1]). *If* X *is a regular* σ-*space, then* (L) *holds.*

2. Decomposition Types (M), (wM) and (wM)'

Let us recall basic definitions concerning weak topologies. Let X be a space, and $\mathcal{C}$ be a cover of X. We say that X *is determined by* $\mathcal{C}$ [5], or X has the weak topology with respect to $\mathcal{C}$, if $A \subset X$ is closed (open) in X whenever $A \cap C$ is relatively closed (relatively open) in C for each $C \in \mathcal{C}$. Every space is determined by any open cover of it.

Let $\mathcal{C}$ be a closed cover of X. We say that X *is dominated by* $\mathcal{C}$ if for any subcollection $\mathcal{C}'$ of $\mathcal{C}$ the union $\cup\mathcal{C}'$ is closed in X and is determined by $\mathcal{C}'$. Every space is dominated by any HCP closed cover of it. We remark that if X is dominated by $\mathcal{C}$, then it is determined by $\mathcal{C}$, but the converse is not true.

The following elementary facts will be often used later on. The proof is straightforwards, so we omit it.

Lemma 2.1. (1) *Let* $f: X \to Y$ *be a quotient map. If* X *is determined by a cover* $\mathcal{C}$, *then* Y *is determined by a cover* $f(\mathcal{C}) = \{f(C): C \in \mathcal{C}\}$.

(2) *Let* X *be determined by a cover* $\{X_\alpha\}$. *If* $X_\alpha \subset X'_\alpha$ *for each* α, *then* X *is determined by* $\{X'_\alpha\}$.

Recall that a collection $\mathcal{C}$ of subsets of X is *point-countable* if each $x \in X$ is in at most countably many $C \in \mathcal{C}$.

Theorem 2.2. (1) *If a space* X *is dominated by a cover* $\mathcal{C}$ *of compact (Lindelöf;* ω_1*-compact) sets, then* (M) ((wM); (wM)') *holds.*

(2) *If a space* X *is determined by a point-countable cover* $\mathcal{C}$ *of compact (Lindelöf;* ω_1*-compact) sets, then* (M) ((wM); (wM)') *holds.*

Proof. (1): Let $\mathcal{C} = \{X_\alpha\}$ with the index set well-ordered. Suppose that each X_α is ω_1-compact (for the other cases, the proofs are similar). Let $L_\alpha = X_\alpha \setminus \cup_{\beta<\alpha} X_\beta$ for each α. Since $\{L_\alpha\}$ is a cover of X and $L_\alpha \subset X_\alpha$, it suffices to show that

$$Y_1 = \{y \in Y\colon f^{-1}(y) \text{ meets uncountably many } L_\alpha\text{'s}\}$$

is closed discrete in Y.

Claim 1. For any $D \subset Y_1$ with cardinality $\leq \omega_1$, D is closed in Y:

Let $D = \{y_\beta\colon \beta < \kappa\}$, where $\kappa \leq \omega_1$. For each $\beta < \kappa$, we can choose some x_β and $\alpha(\beta)$ such that $x_\beta \in f^{-1}(y_\beta) \cap L_{\alpha(\beta)}$ and $\alpha(\beta) \neq \alpha(\beta')$ for $\beta \neq \beta'$. Let $E = \{x_\beta\colon \beta < \kappa\}$. We show that E is closed in X. First, $E \cap X_0$ is at most one point, hence closed in X_0. Assume that $E \cap X_\lambda$ is closed in X_λ for each $\lambda < \eta$. Let $E_\eta = (E \cap (\cup_{\lambda<\eta} X_\lambda)) \cap X_\eta$. Then $E_\eta \subset \cup_{\lambda<\eta} X_\lambda$, and $E_\eta \cap X_\lambda = (E \cap X_\lambda) \cap X_\eta$ is closed in X_λ for each $\lambda < \eta$. Thus E_η is closed in X. But, $E \cap X_\eta = (E \cap L_\eta) \cup E_\eta$, and $E \cap L_\eta$ is at most one point. Hence $E \cap X_\eta$ is closed in X_η. Thus E is closed in X. Since f is a closed map and $D = f(E)$, D is closed in Y.

Claim 2. For any $Y' \subset Y_1$ and for any ω_1-compact set $K \subset Y$, $Y' \cap K$ is closed in K:

If $Y' \cap K$ is countable, by Claim 1, $Y' \cap K$ is closed in Y. Hence it is closed in K. If $Y' \cap K$ is not countable, then there is a subset D of $Y' \cap K$ such that the cardinality of D is ω_1 and D has a cluster point in $K \backslash D$. But, by Claim 1, D is closed in Y. This contradiction implies that $Y' \cap K$ is countable, hence closed in K.

Now, X is determined by a cover $\mathcal{C}$ of ω_1-compact sets. Since f is quotient, by Lemma 2.1 (1), Y is determined by a cover $f(\mathcal{C})$ of ω_1-compact sets. Thus it follows from Claim 2 that Y_1 is closed discrete in Y. Hence (wM)' holds.

(2): Let $\mathcal{C} = \{X_\alpha\}$. Suppose that each X_α is ω_1-compact (for the case of X_α being Lindelöf, the proof is similar). Let

$$Y_1 = \{y \in Y: \text{ no countable } \mathcal{C}' \subset \mathcal{C} \text{ covers } f^{-1}(y)\}.$$

Let $D = \{y_\beta: \beta < \kappa\}$, where $\kappa \leq \omega_1$, be a subset of Y_1. Then there is some

$$x_\beta \in f^{-1}(y_\beta)\setminus\cup\{X_\alpha: x_\gamma \in X_\alpha \text{ for some } \gamma < \beta\}$$

for each $\beta < \kappa$. Let $E = \{x_\beta: \beta < \kappa\}$. Since each $E \cap X_\alpha$ is at most one point, E is closed in X. Then $D = f(E)$ is closed in Y. Thus Claim 1 in (1) is also valid.

Next, suppose that each X_α is compact. Let

$$Y_1^* = \{y \in Y: \text{ no finite } \mathcal{C}' \subset \mathcal{C} \text{ covers } f^{-1}(y)\}.$$

Then we can show that any countable $D = \{y_n: n \geq 1\} \subset Y_1^*$ is closed in Y. Indeed, there is some

$$x_n \in f^{-1}(y_n)\setminus\cup\{X_{ij}: i, j \leq n\} \text{ for each } n \geq 1,$$

where $\{X_{ij}: j \geq 1\} = \{X_\alpha: x_i \in X_\alpha\}$ for each $i \geq 1$. Let $E = \{x_n: n \geq 1\}$. Since each $E \cap X_\alpha$ is at most finite, E is closed in X. Then $D = f(E)$ is closed in Y. Thus the modification of Claim 1 in (1) where "ω_1" is replaced with "ω_0" is valid.

As in the proof of (1), if the X_α's are respectively ω_1-compact; compact, we can show that the set Y_1; Y_1^* is closed discrete in Y, hence (wM)'; (M) holds. The proof of Theorem 2.2 is complete.

As is well-known, every CW-complex is dominated by a cover of compact (metric) sets. So, by Theorem 2.2 (1), we have

Corollary 2.3. *If* X *is a* CW-*complex (more generally, a chunk-complex in the sense of* [1]*), then* (M) *holds.*

A space is called a k_ω-*space* [8] (Morita [11] calls it a space of the class $\mathfrak{G}'$) if it is determined by a countable cover of compact sets. Such a space is characterized as a quotient image of a locally compact Lindelöf space [11].

By Theorem 2.2 (2), if X is a k_ω-space, then (M) holds. It should be noted by Remark to Corollary 1.4 that we cannot replace "k_ω-space" with "σ-compact space."

Let us call a space *locally* k_ω *(locally Lindelöf)* if each point has a neighborhood which is k_ω (Lindelöf), where the neighborhood is not necessarily open. Every locally compact space is locally k_ω, and every locally k_ω-space is locally Lindelöf. Recall that a space X is *meta-Lindelöf* if every open cover of X has a point-countable open refinement.

Morita [10] showed that if X is a paracompact and locally compact space, then (M) holds. We can extend this result as follows.

Proposition 2.4. *If* X *is a meta-Lindelöf and locally* k_ω-*space, then* (M) *holds.*

Proof. By the assumptions of X, X is determined by a point-countable open cover $\{V_\alpha\}$, where each V_α is contained in a space determined by a countable cover $\{K_{\alpha n}: n \geq 1\}$ of compact sets. Let $\mathcal{C} = \{V_\alpha \cap K_{\alpha n}\}$. Then it is routinely verified that X is determined by the point-countable cover $\mathcal{C}$. But, by Lemma 2.1 (1), Y is determined by the cover

$f(\mathcal{C})$. Since $f(V_\alpha \cap K_{\alpha n}) \subset f(K_{\alpha n})$ for each α and n, by Lemma 2.1 (2), Y is determined by the cover $\{f(K_{\alpha n})\}$ of compact sets. Let

$$Y_1^* = \{y \in Y: \text{ no finite } \mathcal{C}' \subset \mathcal{C} \text{ covers } f^{-1}(y)\}.$$

By the proof of Theorem 2.2 (2) for the compact case, we can show that Y_1^* is closed discrete in Y. Hence (M) holds.

Proposition 2.5. *Let* X *be a locally Lindelöf space. If* X *is subparacompact (meta-Lindelöf), then* (wL) ((wM)) *holds.*

Proof. If X is subparacompact, then X has a σ-locally finite closed cover $\mathcal{F} = \cup_{n=1}^{\infty}\mathcal{F}_n$ of Lindelöf sets, where each $\mathcal{F}_n$ is locally finite in X. Let $X_n = \cup\mathcal{F}_n$, and $Y_n = f(X_n)$ for each $n \geq 1$. Then each X_n is dominated by $\mathcal{F}_n$ and $g_n = f|X_n$ is closed. Thus, by Theorem 2.2 (1), $Y_n = Y_{n0} \cup Y_{n1}$, where $g_n^{-1}(y)$ is Lindelöf for each $y \in Y_{n0}$ and Y_{n1} is closed discrete in Y_n, hence in Y. Let $Y_0 = Y \setminus \cup_{n=1}^{\infty} Y_{n1}$ and $Y_n = Y_{n1}$ for each $n \geq 1$. Then the sets Y_n, $n \geq 0$, satisfy the desired property in (wL).

If X is meta-Lindelöf, then X is determined by a point-countable open cover $\{V_\alpha\}$ such that each V_α is contained in a Lindelöf set L_α. But, by Lemma 2.1 (1), Y is determined by a cover $\{f(V_\alpha)\}$. Since $f(V_\alpha) \subset f(L_\alpha)$ for each α, by Lemma 2.1 (2), Y is determined by a cover $\{f(L_\alpha)\}$ of Lindelöf sets. Thus, in view of the proof of Theorem 2.2 (2), the assertion for the parenthetic part holds.

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