TOPOLOGY PROCEEDINGS
Volume 11, 1986
Pages 25–27

http://topology.auburn.edu/tp/

A LOCALLY COMPACT, HOMOGENEOUS METRIC SPACE WHICH IS NOT BIHOMOGENEOUS

by

H. Cook

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

A LOCALLY COMPACT, HOMOGENEOUS METRIC SPACE WHICH IS NOT BIHOMOGENEOUS

H. Cook

Suppose S is a topological space. If, for each two points a and b of S, there is a homeomorphism of S onto S throwing a to b, then S is said to be homogeneous. If, for each two points a and b of S, there is a homeomorphism of S onto S throwing a to b and b to a, then S is said to be bihomogeneous. In [2], Kuratowski gave an example of a connected metric homogeneous space which is not bihomogeneous. His example is not locally compact (nor, indeed, is it topologically complete). Here, we give a locally compact example.

The Space X. Let P denote a pseudo-arc and C(P) denote the space of all subcontinua of P. Let X denote the subspace of C(P) comprising the nondegenerate proper subcontinua of P.

Theorem 0. *The space* X *is connected and locally compact.*

Proof. There is a continuous collection H of proper subcontinua of P filling up P, [1]. Now, H is a continuum in X and each point of X is connected to H by an arc, [1]. Thus, X is connected.

Since X is an open subset of the compact space C(P), X is locally compact.

Theorem 1. *The space* X *is homogeneous.*

Proof. Lehner has shown [4] that, if a and b are two proper subcontinua of P, then there is a homeomorphism h of P onto P that takes a onto b. Then h*, the homeomorphism of C(P) onto C(P) induced by h, throws X onto X and h*(a) = b. Thus, X is homogeneous.

Theorem 2. *The space* X *is not bihomogeneous.*

Proof. Let a and b be two points of X such that b is a proper subcontinuum of a. Further, let $b_1, b_2, b_3, \cdots$ be a sequence of proper subcontinua of a (points of X) converging to b such that no two of them lie in the same composant of a and no one of them lies in the composant of a containing b. There is only one arc ab in X from a to b; for each i, there is only one arc ab, from a to b_i; and if tb is an arc in X from a point $t \neq b$ of X to b such that $tb \cap ab = b$ then t is a proper subcontinuum of b (in P), [1].

Now, suppose that h is a homeomorphism from X onto X such that h(a) = b and h(b) = a. For each i, let $t_i = h(b_i)$. Now, h throws the arc ab onto itself, and, for each i, throws the arc ab_i onto an arc t_ib such that $t_ib \cap ab = b$. Then, for each i, t_i is a proper subcontinuum (in P) of b. But the sequence $t_1, t_2, t_3, \cdots$ converges to h(b) = a. But no sequence of proper subcontinua of b converges to a.

Thus, X is not bihomogeneous.

Notes. For a general exposition of the pseudo-arc, see [3]. For a thorough discussion of hyperspaces such as C(P), see [5].

Kuratowski's example, mentioned in the opening paragraph, is a 1-dimensional subspace of the plane. The space X is a 2-dimensional subspace of Euclidean 3-space, [6]. K. Kuperberg tells me that she has an example (to appear) of a 3-dimensional compact metric continuum which is homogeneous but not bihomogeneous.

References

[1] J. L. Kelley, *Hyperspaces of a continuum,* Trans. Amer. Math. Soc. 52 (1942), 22-36.

[2] K. Kuratowski, *Un probleme sur les ensembles homogenes,* Fund. Math. 3 (1922), 14-19.

[3] ________, *Topology,* Vol. II, Academic Press, New York and London, 1968.

[4] G. R. Lehner, *Extending homeomorphisms on the pseudo-arc,* Trans. Amer. Math. Soc. 98 (1961), 369-394.

[5] S. B. Nadler, Jr., *Hyperspaces of sets,* Marcel Dekker, Inc., New York, 1978.

[6] W. R. R. Transue, *On the hyperspace of continua of the pseudo-arc,* Proc. Amer. Math. Soc. 18 (1967), 1074-1075.

University of Houston

Houston, Texas 77004