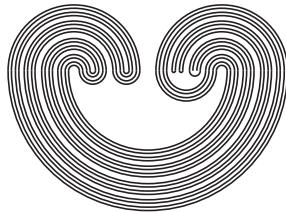

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Suppose S is a topological space. If, for each two points a and b of S , there is a homeomorphism of S onto S throwing a to b , then S is said to be homogeneous. If, for each two points a and b of S , there is a homeomorphism of S onto S throwing a to b and b to a , then S is said to be bihomogeneous. In [2], Kuratowski gave an example of a connected metric homogeneous space which is not bihomogeneous. His example is not locally compact (nor, indeed, is it topologically complete). Here, we give a locally compact example.

The Space X. Let P denote a pseudo-arc and $C(P)$ denote the space of all subcontinua of P . Let X denote the subspace of $C(P)$ comprising the nondegenerate proper subcontinua of P .

Theorem 0. *The space X is connected and locally compact.*

Proof. There is a continuous collection H of proper subcontinua of P filling up P , [1]. Now, H is a continuum in X and each point of X is connected to H by an arc, [1]. Thus, X is connected.

Since X is an open subset of the compact space $C(P)$, X is locally compact.

Theorem 1. The space X is homogeneous.

Proof. Lehner has shown [4] that, if a and b are two proper subcontinua of P, then there is a homeomorphism h of P onto P that takes a onto b. Then h^* , the homeomorphism of $C(P)$ onto $C(P)$ induced by h, throws X onto X and $h^*(a) = b$. Thus, X is homogeneous.

Theorem 2. The space X is not bihomogeneous.

Proof. Let a and b be two points of X such that b is a proper subcontinuum of a. Further, let b_1, b_2, b_3, \dots be a sequence of proper subcontinua of a (points of X) converging to b such that no two of them lie in the same composant of a and no one of them lies in the composant of a containing b. There is only one arc ab in X from a to b; for each i, there is only one arc $a b_i$ from a to b_i ; and if $t b$ is an arc in X from a point $t \neq b$ of X to b such that $t b \cap ab = b$ then t is a proper subcontinuum of b (in P), [1].

Now, suppose that h is a homeomorphism from X onto X such that $h(a) = b$ and $h(b) = a$. For each i, let $t_i = h(b_i)$. Now, h throws the arc ab onto itself, and, for each i, throws the arc $a b_i$ onto an arc $t_i b$ such that $t_i b \cap ab = b$. Then, for each i, t_i is a proper subcontinuum (in P) of b. But the sequence t_1, t_2, t_3, \dots converges to $h(b) = a$. But no sequence of proper subcontinua of b converges to a.

Thus, X is not bihomogeneous.

Notes. For a general exposition of the pseudo-arc, see [3]. For a thorough discussion of hyperspaces such as $C(P)$, see [5].

Kuratowski's example, mentioned in the opening paragraph, is a 1-dimensional subspace of the plane. The space X is a 2-dimensional subspace of Euclidean 3-space, [6]. K. Kuperberg tells me that she has an example (to appear) of a 3-dimensional compact metric continuum which is homogeneous but not bihomogeneous.

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