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1. Hyperspace Retractions

For X a metric continuum, let 2^X be the hyperspace of all nonempty subcompacta, with the Hausdorff metric topology, and let $C(X) \subset 2^X$ be the hyperspace of subcontinua. If X is locally connected, both C(X) and 2^X are absolute retracts [9], and in particular C(X) is a retract of 2^X. In the non-locally connected case, neither hyperspace is an absolute retract, but we may still ask whether C(X) is a retract of 2^X. Until now, this question has been answered in only two specific cases. In 1977, Goodykoontz [2] constructed a 1-dimensional continuum X in E^3 such that C(X) is *not* a retract of 2^X. And in 1983, Goodykoontz [3] showed that for X the cone over a convergent sequence, C(X) *is* a retract of 2^X. Thus, for X non-locally connected, C(X) is not necessarily a retract of 2^X, but it may be. (Nadler [6] had earlier shown the existence of surjections from 2^X to C(X), in all cases.)

At present, a completely general answer for the hyperspace retraction question seems out of reach. In this paper, we answer the question for a certain class of non-locally connected continua, large enough to be of interest, but sufficiently delimited so as to be manageable. This class will consist of those half-line compactifications with locally connected remainder which are "regular" in the following sense. Let $X = [0,\infty) \cup K$ denote an arbitrary half-line compactification with a nondegenerate locally connected remainder K (which is therefore a Peano continuum). In this situation, there always exists a retraction $X \to K$. We say that X is a *regular* compactification if there exists a retraction $r: X \to K$ such that, for some homeomorphism $\phi: [0,\infty) \to [0,\infty)$, the map $r \circ \phi: [0,\infty) \to K$ is a *periodic* surjection, i.e., there exists $p > 0$ such that $r(\phi(t)) = r(\phi(t + p))$ for all t. Our main result is that the only regular half-line compactifications for which there exist hyperspace retractions $2^X \to C(X)$ are the following: the topologist's sine curve; the circle with a spiral; and a sequence of other regular compactifications with a circle as remainder, to be described below.

The case of the circle with a spiral (labelled below as X_1) is of particular interest. It is known that Cone X_1 does not have the fixed point property [5], and that $C(X_1)$ is homeomorphic to Cone X_1 [8]. Noting this, Nadler [7] conjectured that 2^{X_1} does not have the fixed point property (which would make it the first such example to be known), and that the way to prove this is to construct a retraction from 2^{X_1} to $C(X_1)$. Our result confirms his conjecture.

Every periodic surjection $\pi: [0,\infty) \to K$ onto a Peano continuum induces a regular compactification $X(\pi)$, which may be defined as follows:

$$X(\pi) = \{(t,\pi(t)): t \geq 0\} \cup \{(\infty,k): k \in K\} \subset [0,\infty] \times K.$$

Alternatively, we may consider $X(\pi)$ to be the disjoint union $[0,\infty) \cup K$, with the topology defined by the open base

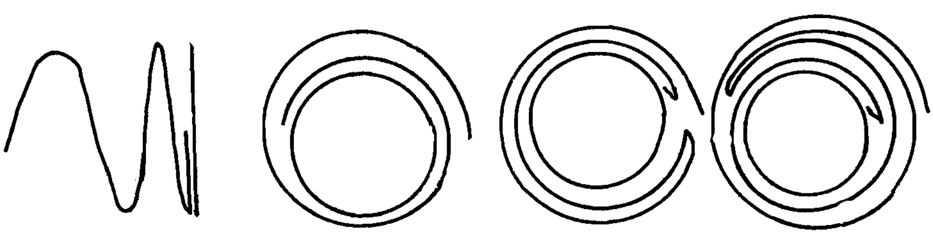
$$\{U\colon U \text{ open in } [0,\infty)\} \cup \{V \cup (\pi^{-1}(V) \cap (N,\infty))\colon V \text{ open in } K \text{ and } N < \infty\}.$$

Clearly, every regular half-line compactification is homeomorphic to some $X(\pi)$.

Let $I = [-1,1]$, and $S = \{z\colon |z| = 1\}$, the unit circle in the complex plane. Define $\pi_0\colon [0,\infty) \to I$ by $\pi_0(t) = \sin \pi t$; define $\pi_1\colon [0,\infty) \to S$ by $\pi_1(t) = e^{i\pi t}$; and for $n > 1$, define $\pi_n\colon [0,\infty) \to S$ by the formulas

$$\pi_n(t) = \begin{cases} e^{in\pi t}, & 0 \le t \le 1 \pmod 2, \\ e^{-in\pi t}, & 1 \le t \le 2 \pmod 2. \end{cases}$$

Then $X_0 = X(\pi_0)$ is the topologist's sine curve; $X_1 = X(\pi_1)$ is the circle with a spiral; and for $n = 2,3,\cdots,X_n = X(\pi_n)$ is the regular compactification obtained by alternately "wrapping" and "unwrapping" subintervals of $[0,\infty)$ about S, with each subinterval covering S $n/2$ times. Note that the spaces $X_0,X_1,X_2,\cdots$ are topologically distinct.



X_0 X_1 X_2 X_3

Theorem. For X *a regular half-line compactification, there exists a hyperspace retraction* $2^X \to C(X)$ *if and only if* X *is homeomorphic to some* X_n, $n = 0,1,2,\cdots$.

Of course, no hyperspace retraction $2^X \to C(X)$ for non-locally connected X can be quite as nice as those which may be constructed in the locally connected case. For locally connected X, we may use a convex metric d, and define a retraction R: $2^X \to C(X)$ by taking $R(A) = \overline{N}_d(A;t)$, where $t \geq 0$ is the smallest value for which $\overline{N}_d(A;t) \in C(X)$. Such a retraction has the property that $R(A) \supset A$ for each $A \in 2^X$. Clearly, this is impossible for non-locally connected X. However, there may exist a retraction R: $2^X \to C(X)$ such that $R(A) \cap A \neq \emptyset$ for each A (we say that R is *conservative*). In the course of proving the above theorem, it will be shown that only for X_0 and X_1 do there exist conservative hyperspace retractions.

In the final section of the paper, we note the connection between the existence of a hyperspace retraction $2^X \to C(X)$ and the existence of a mean for C(X), and we give examples of continua X (from the class of regular half-line compactifications) for which C(X) does not admit a mean, thereby answering a question of Nadler [7].

2. A Necessary Condition

Let X be any metric continuum, and let ρ denote the Hausdorff metric on 2^X. We say that X has the *subcontinuum approximation property* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $L,M \in C(X)$ with $\rho(L,M) < \delta$, and for every subcontinuum $P \subset M$, there exist $P', M' \in C(X)$ with $\rho(P,P') < \varepsilon$, $\rho(M,M') < \varepsilon$, and $L \cup P' \subset M'$. (In the locally connected case we may of course choose M' such that $L \cup M \subset M'$, but in general M and M' will be disjoint.) We will show that this property is a necessary condition for the existence of a hyperspace retraction $2^X \to C(X)$, and that a regular half-line compactification has the property if and only if the remainder is either an arc or a simple closed curve.

In what follows, we shall have occasion to use order arcs and segments in the hyperspaces 2^X and $C(X)$. An arc $\alpha \subset 2^X$ is an *order arc* if for each $E,F \in \alpha$, either $E \subset F$ or $F \subset E$. For elements $A,B \in 2^X$, there exists an order arc α with $\cap\alpha = A$ and $\cup\alpha = B$ if and only if $A \subset B$ and each component of B intersects A. Every order arc α can be uniquely parametrized as a *segment* $\alpha\colon [0,1] \to 2^X$ with respect to a given Whitney map $\omega\colon 2^X \to [0,\infty)$, i.e., $\alpha = \{\alpha(t)\colon 0 \leq t \leq 1\}$, with $\alpha(0) = \cap\alpha$, $\alpha(1) = \cup\alpha$, and $\omega(\alpha(t)) = (1 - t)\omega(\alpha(0)) + t\omega(\alpha(1))$ for each t. (Order arcs were first used by Borsuk and Mazurkiewicz [1] to show that $C(X)$ and 2^X are arcwise connected. Segments were introduced by Kelley [4], who also formulated the necessary and sufficient conditions given above for the existence of an order arc, or segment, from A to B.) Let $\Gamma(X) = \{\alpha \in C(2^X)\colon \alpha$ is an order arc or $\alpha = \{A\}$ for $A \in 2^X\}$, and let $S(\omega)$ be the function space of all segments $\alpha\colon [0,1] \to 2^X$ (including the constant maps), with the topology of uniform convergence. Then the spaces $\Gamma(X)$ and

$S(\omega)$ are compact, and the natural correspondence $\alpha \to \{\alpha(t): 0 \leq t \leq 1\}$ is a homeomorphism from $S(\omega)$ to $\Gamma(X)$ (for a complete discussion, see [7]). Henceforth, we implicitly use this correspondence wherever convenient. Without confusion, we let ρ denote both the Hausdorff metric on 2^X and the sup metric on $S(\omega)$.

2.1. *Lemma. Let* $P, M \in C(X)$, *with* $P \subset M$. *Then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for every* $L \in C(X)$ *with* $\rho(L,M) < \delta$, *there exist order arcs* $\alpha \subset 2^X$ *and* $\beta \subset C(X)$ *with* $\alpha(1) = L$, $\beta(0) = P$, $\beta(1) = M$, *and* $\rho(\alpha,\beta) < \varepsilon$.

Proof. Suppose that for some $\varepsilon > 0$ there exists a sequence $\{L_i\}$ in $C(X)$ converging to M, with no L_i satisfying the required condition. Choose a finite subset $F \subset P$ such that $\rho(F,P) < \varepsilon$. For each $x \in F$ and each i, choose $x_i \in L_i$ and an order arc $\alpha_{x_i} \subset C(X)$ such that $x_i \to x$, $\alpha_{x_i}(0) = \{x_i\}$, and $\alpha_{x_i}(1) = L_i$. Then for each i let α_i be the order arc in 2^X defined by $\alpha_i(t) = \cup\{\alpha_{x_i}(t): x \in F\}$. Thus $\alpha_i(0) = \{x_i: x \in F\}$ and $\alpha_i(1) = L_i$. Since the space $\Gamma(X)$ is compact, some subsequence of $\{\alpha_i\}$ must converge to an order arc λ in 2^X with $\lambda(0) = F$ and $\lambda(1) = M$. Define an order arc β in $C(X)$ by $\beta(t) = P \cup \lambda(t)$. Thus $\beta(0) = P$ and $\beta(1) = M$. Since $\rho(\lambda,\beta) < \varepsilon$, we have $\rho(\alpha_i,\beta) < \varepsilon$ for some large i, contradicting our supposition about the sequence $\{L_i\}$.

2.2. *Proposition. Let* X *be any continuum for which there exists a hyperspace retraction* $2^X \to C(X)$. *Then* X *has the subcontinuum approximation property.*

Proof. Suppose X does not have the property. Then by compactness of $C(X)$, there exist $P, M \in C(X)$ with $P \subset M$, and a sequence $\{L_i\}$ in $C(X)$ converging to M such that, for some $\varepsilon > 0$, there do *not* exist $P', M' \in C(X)$ with $\rho(P,P') < \varepsilon$, $\rho(M,M') < \varepsilon$, and $L_i \cup P' \subset M'$ for some i. Let $R: 2^X \to C(X)$ be a retraction. Choose $0 < \eta < \varepsilon$ such that, for every $A \in 2^X$ with $\rho(A, M_0) < \eta$ for some subcontinuum $M_0 \subset M$, $\rho(R(A), M_0) < \varepsilon$. By (2.1), for sufficiently large i there exist order arcs $\alpha \subset 2^X$ and $\beta \subset C(X)$ with $\alpha(1) = L_i$, $\beta(0) = P$, $\beta(1) = M$, and $\rho(\alpha,\beta) < \eta$. Then the continua $P' = R(\alpha(0))$ and $M' = \cup\{R(\alpha(t)): 0 \leq t \leq 1\}$ satisfy the conditions $\rho(P,P') < \varepsilon$, $\rho(M,M') < \varepsilon$, and $L_i \cup P' \subset M'$, contradicting our supposition.

Note. The example constructed by Goodykoontz in [2] does not have the subcontinuum approximation property; our proof for (2.2) is a generalization of his argument for the non-existence of a hyperspace retraction.

2.3. *Lemma. Let* $\pi: I \to K$ *be a map of an arc onto a Peano continuum which is neither an arc nor a simple closed curve. Then for some subarc* $J \subset I$, $\pi(J)$ *is a proper subcontinuum of* K *containing a simple triod.*

Proof. Let $\mathcal{L}$ denote the collection of all proper subcontinua of K which are of the form $\pi(J)$ for some subarc J. Since K is neither an arc nor a simple closed curve, there must be some $L \in \mathcal{L}$ which is not an arc. Then the Peano continuum L either contains a simple triod or is a simple closed curve. In either case there exists $\tilde{L} \in \mathcal{L}$ properly containing L, and therefore containing a simple triod.

2.4. *Lemma.* *Let* $\pi: I \to T$ *be a map of an arc onto a simple triod. Then there exists a subcontinuum* $P \subset T$ *such that* $P \neq \pi(J)$ *for any subarc* $J \subset I$.

Proof. Choose a sequence $\{T_n\}$ of triods in T such that $T_n \subset \text{int } T_{n+1}$. Suppose that for each n there exists a subarc $J_n \subset I$ with $\pi(J_n) = T_n$. We may assume that each endpoint of J_n is mapped to an endpoint of T_n. Since for $m < n$, $T_m \subset \text{int } T_n$, we must have either $J_m \cap J_n = \emptyset$ or $J_m \subset J_n$. Choose $\delta > 0$ such that for each $A \subset I$ with diam $A < \delta$ and each n, $\pi(A)$ contains at most one endpoint of T_n. Since one of the endpoints of T_n can be the image only of interior points of J_n, it follows that diam $J_n \geq 2\delta$ for each n. Also, if $m < n$ and $J_m \subset J_n$, then diam $J_n \geq$ diam $J_m + \delta$. The sequence $\{J_n\}$ in C(I) clusters at some nondegenerate J. But for any pair of distinct arcs J_m, J_n sufficiently close to J, it's impossible that either $J_m \cap J_n = \emptyset$ or $J_m \subset J_n$. Thus some T_n must satisfy the conclusion of the lemma.

2.5. *Proposition. A regular half-line compactification has the subcontinuum approximation property if and only if the remainder is either an arc or a simple closed curve.*

Proof. Let $X = [0,\infty) \cup K$ be the regular half-line compactification corresponding to a periodic surjection $\pi: [0,\infty) \to K$, and let $I \subset [0,\infty)$ be a subarc such that π goes through at least two complete cycles over I.

Suppose first that K is neither an arc nor a simple closed curve. Applying (2.3) to the restriction π/I, we obtain a proper subcontinuum $M \subset K$ such that M contains a simple triod T and $M = \pi(J)$ for some subarc $J \subset I$. Thus, there exists a sequence $\{J_i\}$ of subarcs in $[0,\infty)$ converging to M, and since $M \neq K$, every $M' \in C(X)$ sufficiently close to M and containing some J_i must itself be a subarc of $[0,\infty)$. Let $r: K \to T$ be any retraction, and apply (2.4) to the map $r \circ \pi: I \to T$. We obtain a subcontinuum $P \subset T$ such that $P \neq \pi(I_0)$ for any subarc $I_0 \subset I$. Thus, every $P' \in C(X)$ sufficiently close to P must lie in K. It follows that X does not have the subcontinuum approximation property with respect to the pair (M,P).

Now suppose that K is either an arc or a simple closed curve, and consider any $P,M \in C(X)$ with $P \subset M$. It suffices to verify the subcontinuum approximation property with respect to this pair (see the proof of (2.2)). The property is obvious if either $M \subset [0,\infty)$ or $M \supset K$, so we may suppose that M is a proper subcontinuum of K (and therefore an arc). Each $L \in C(K)$ which is close to M intersects M, so in this case we may take $M' = L \cup M$ and $P' = P$. And for any arc $L \subset [0,\infty)$ close to M, there is a subarc $L_0 \subset L$ close to P, so we may take $M' = L$ and $P' = L_0$. This completes the argument that X has the subcontinuum approximation property.

It may be of interest to note that the subcontinuum approximation property is implied by property [K], which was introduced by Kelley [4] in the study of hyperspace contractibility and which has been used extensively in recent years (see [7]). In the class of regular half-line compactifications, the only spaces with property [K] are the spaces X_0 and X_1 which admit conservative hyperspace retractions. Thus, the spaces X_n for $n > 1$ show that property [K] is *not* necessary for the existence of hyperspace retractions. Whether there is any general relationship between property [K] and the existence of conservative hyperspace retractions remains an open question.

3. A Monotonicity Requirement

Let $X = [0,\infty) \cup K$ be the regular half-line compactification corresponding to a periodic surjection $\pi\colon [0,\infty) \to K$, and suppose there exists a hyperspace retraction $2^X \to C(X)$. By (2.2) and (2.5), the remainder K is either an arc or a simple closed curve. In the case that K is an arc, we say that π is *interior monotone* if, for each arc $J \subset [0,\infty)$ such that $\pi(J) \cap \partial K = \phi$, the restriction π/J is monotone (perhaps nonstrictly). A similar definition is made in the case that K is a simple closed curve, using a covering projection $(-\infty,\infty) \to K$. Specifically, let $\tilde{\pi}\colon [0,\infty) \to (-\infty,\infty)$ be a lift of π, and set $\tilde{K} = \text{im}\ \tilde{\pi}$. We say that $\tilde{\pi}$ is *interior monotone* if $\tilde{\pi}/J$ is monotone for each arc $J \subset [0,\infty)$ such that $\tilde{\pi}(J) \cap \partial\tilde{K} = \phi$. We will show that π, or $\tilde{\pi}$, must be interior monotone. It follows easily that either $X \approx X_0$ (if K is an arc), or $X \approx X_1$ (if K is a simple closed curve and $\tilde{K}$ is unbounded), or $X \approx X_n$ for some $n > 1$ (if $\tilde{K}$ is bounded).

We will need the following result concerning the composition semigroup $\mathcal{S}$ of all self-maps of the interval [0,1] which are fixed on the endpoints.

3.1. *Proposition. For every* $f_1, f_2 \in \mathcal{S}$ *and* $\varepsilon > 0$, *there exist* $g_1, g_2 \in \mathcal{S}$ *such that* $d(f_1 \circ g_1, f_2 \circ g_2) < \varepsilon$.

Proof. For each pair (m,n) of positive integers with $m \geq n$, let $P(m,n)$ denote the finite set of piecewise-linear maps f in $\mathcal{S}$ satisfying the following conditions:

1) for each $0 \leq j \leq m$, $f(j/m) = k/n$ for some $0 \leq k \leq n$; and

2) for each $0 \leq j < m$, $|f((j + 1)/m) - f(j/m)| \leq 1/n$, and f is linear over the interval $[j/m, (j + 1)/m]$.

Choose n such that $1/n < \varepsilon/4$, and choose m_1, m_2 such that $|f_i(s) - f_i(t)| \leq 1/n$ whenever $|s - t| \leq 1/m_i$, $i = 1,2$. Then there exist maps $\phi_i \in P(m_i,n)$ with $d(f_i,\phi_i) \leq 1/n + 1/2n + 1/2n < \varepsilon/2$, $i = 1,2$. We show that, for some $m \geq \max\{m_1,m_2\}$, there exist $g_1 \in P(m,m_1)$ and $g_2 \in P(m,m_2)$ with $\phi_1 \circ g_1 = \phi_2 \circ g_2$ (note that the compositions are members of $P(m,n)$). It then follows that $d(f_1 \circ g_1, f_2 \circ g_2) < \varepsilon$.

The proof is by induction on $m_1 + m_2$. If $m_1 + m_2 = 2n$ (the least possible value), then $m_1 = m_2 = n$ and $\phi_1 = \phi_2 = \mathrm{id}$. In this case take $m = n$ and $g_1 = g_2 = \mathrm{id}$.

Now assume $m_1 + m_2 > 2n$. Suppose first that for some $j < m_1$, $\phi_1(j/m_1) = \phi_1((j + 1)/m_1)$. Then we may consider the corresponding $\tilde{\phi}_1 \in P(m_1 - 1,n)$, obtained topologically by collapsing to a point the arc $[j/m_1, (j + 1)/m_1] \times \phi_1(j/m_1)$ on the graph of ϕ_1. Application of the inductive hypothesis to the pair $\tilde{\phi}_1,\phi_2$ gives maps $\gamma_1 \in P(m_0,m_1 - 1)$ and $\gamma_2 \in P(m_0,m_2)$, for some $m_0 \geq \max\{m_1 - 1,m_2\}$, such that $\tilde{\phi}_1 \circ \gamma_1 = \phi_2 \circ \gamma_2$. It's not difficult to see that this implies the corresponding result for the pair ϕ_1,ϕ_2. Of course, the same argument works if $\phi_2(j/m_2) = \phi_2((j + 1)/m_2)$ for some $j < m_2$.

Thus, we may suppose that neither ϕ_i is constant on any subinterval. Then there exists a least integer k for which $\phi_i(j/m_i) = k/n$ and $\phi_i((j - 1)/m_i) = \phi_i((j + 1)/m_i) = (k - 1)/n$, for some $1 \leq j < m_i$ and $i = 1,2$; suppose this holds for $i = 1$. Consider the corresponding $\tilde{\phi}_1 \in P(m_1 - 2,n)$, obtained topologically by identifying the points $((j - 1)/m_1, (k - 1)/n)$ and $((j + 1)/m_1, (k - 1)/n)$ of the restriction $\phi_1/[0,(j - 1)/m_1] \cup [(j + 1)/m_1,1]$. Applying the inductive hypothesis to the pair $\tilde{\phi}_1,\phi_2$, we obtain maps $\gamma_1 \in P(m_0,m_1 - 2)$ and $\gamma_2 \in P(m_0,m_2)$, for some $m_0 \geq \max\{m_1 - 2,m_2\}$, such that $\tilde{\phi}_1 \circ \gamma_1 = \phi_2 \circ \gamma_2$. Note that by the choice of k, if $\phi_2(i/m_2) = (k - 1)/n$, then either $\phi_2((i - 1)/m_2) = k/n$ or $\phi_2((i + 1)/m_2) = k/n$. Clearly, the above implies the corresponding result for the pair ϕ_1,ϕ_2. This completes the proof of the proposition.

3.2. *Remark*. If $\sup f_i^{-1}(0) < \inf f_i^{-1}(1)$ for each $i = 1,2$, then there exists $\delta > 0$ (independent of ε) such that the maps g_1,g_2 may be chosen so that $\sup(f_i \circ g_i)^{-1}([0,\delta]) < \inf(f_i \circ g_i)^{-1}([1 - \delta,1])$, $i = 1,2$.

3.3. *Theorem*. *Let* $X = [0,\infty) \cup K$ *be a regular half-line compactification for which there exists a hyperspace retraction* $2^X \to C(X)$. *Then* $X \approx X_n$ *for some* $n = 0,1,2,\cdots$.

Proof. As observed at the beginning of this section, K is either an arc or a simple closed curve. We consider first the case that K is an arc. Suppose π is *not* interior monotone. Then it's not difficult to see that there exists a proper subarc σ of K, with endpoints v and w, and points $t_0,\cdots,t_n$ in $(0,\infty)$, with $t_0 < t_1 < \cdots < t_n$ and $n \geq 3$, such that:

1) $\pi(t_0) = \pi(t_2) = \cdots = v$;

2) $\pi(t_1) = \pi(t_3) = \cdots = w$;

3) $\pi([t_0,t_n]) = \sigma$, and $[t_0,t_n]$ is a maximal subinterval in $[0,\infty)$ with respect to this property; and

4) for each $i = 1,\cdots,n$, the subsets $\pi^{-1}(v) \cap [t_{i-1},t_i]$ and $\pi^{-1}(w) \cap [t_{i-1},t_i]$ lie in disjoint subintervals.

An application of (3.1) to the maps $\pi|[t_0,t_1]$ and $\pi|[t_1,t_2]$, suitably re-parametrized, shows that for every $\varepsilon > 0$ there exist maps $g_1: [0,1] \to [t_0,t_1]$ and $g_2: [0,1] \to [t_1,t_2]$ such that $g_1(0) = t_1 = g_2(0)$, $g_1(1) = t_0$, $g_2(1) = t_2$, and $d(\pi g_1(t), \pi g_2(t)) < \varepsilon$ for all $0 \leq t \leq 1$. Furthermore, we may assume by (3.2) and the above property 4) that, independently of ε, there exist neighborhoods N(v) and N(w) in σ of v and w such that for each $i = 1,2$, $\sup(\pi \circ g_i)^{-1}(N(w)) < \inf(\pi \circ g_i)^{-1})(N(v))$.

For maps g_1 and g_2 as above, consider the path $\alpha: [0,1] \to 2^X$ between $\{t_1\}$ and $\{t_0,t_2\}$, defined by $\alpha(t) = \{g_1(t),g_2(t)\}$. Let $R: 2^X \to C(X)$ be a retraction. If $\varepsilon > 0$ is sufficiently small and t_0 sufficiently large (use the periodicity of π), then for each $0 \leq t \leq 1$, $\pi R(\alpha(t))$ is a small diameter continuum lying in some neighborhood of σ which is a proper subset of K. Since $\cup\{R(\alpha(t)): 0 \leq t \leq 1\}$ is a continuum containing $R(\alpha(0)) = \{t_1\}$, this implies that $\cup\{R(\alpha(t))\} \subset [0,\infty)$. Moreover, since $\sup(\pi \circ g_i)^{-1}(N(w)) < \inf(\pi \circ g_i)^{-1}(N(v))$, we may assume

ε sufficiently small and t_0 sufficiently large so that $\cup\{R(\alpha(t))\} \subset [0,t_3)$. Thus $R(\{t_0,t_2\}) = R(\alpha(1)) \subset [0,t_3)$. In fact, we claim that $R(\{t_0,t_2\}) \subset [0,t_1)$ for all sufficiently large t_0. Otherwise, the small diameter continuum $R(\{t_0,t_2\})$ would lie in the interval (t_1,t_3), hence $R([t,t_0] \cup \{t_2\}) \subset (t_1,t_3)$ for some $t < t_0$. But by the maximal nature of $[t_0,t_n]$, $\pi([t,t_0]) \neq \sigma$, and since $R([t,t_0] \cup \{t_2\})$ is arbitrarily close to $\pi([t,t_0])$ for sufficiently large t_0, this leads to a contradiction.

By another application of (3.1) we obtain maps $h_1\colon [0,1] \to [t_0,t_1]$ and $h_2\colon [0,1] \to [t_2,t_3]$ with $h_1(0) = t_0$, $h_1(1) = t_1$, $h_2(0) = t_2$, $h_2(1) = t_3$, and such that the maps $\pi \circ h_1$ and $\pi \circ h_2$ are arbitrarily close. As before, we may also assume that $\sup(\pi \circ h_i)^{-1}(N(v)) < \inf(\pi \circ h_i)^{-1}(N(w))$. Consideration of the path β in 2^X between $\{t_0,t_2\}$ and $\{t_1,t_3\}$, defined by $\beta(t) = \{h_1(t), h_2(t)\}$, shows that $R(\{t_1,t_3\}) \subset [0,t_2)$. Continuing in this fashion we obtain $R(\{t_{n-2},t_n\}) \subset [0,t_{n-1})$. But an argument analogous to that given above for $R(\{t_0,t_2\})$ shows that $R(\{t_{n-2},t_n\}) \subset (t_{n-1},\infty)$. This contradiction shows that π must be interior monotone. Clearly, this implies that $X \approx X_0$.

In the case that K is a simple closed curve, the same type of arguments show that the lift $\tilde{\pi}\colon [0,\infty) \to \tilde{K}$, defined at the beginning of this section, must be interior monotone. If $\tilde{K} = \operatorname{im} \tilde{\pi}$ is unbounded, then in fact $\tilde{\pi}$ is monotone and $X \approx X_1$. And if $\tilde{K}$ is bounded, then $X \approx X_n$ for some $n > 1$. Specifically, $X \approx X_{2n}$ if the interval $\tilde{K}$ wraps around K exactly n times, while $X \approx X_{2n+1}$ if $\tilde{K}$ wraps around K n times plus a fraction.

4. Conservative Hyperspace Retractions

Recall that a retraction $R: 2^X \to C(X)$ is *conservative* if $R(A) \cap A \neq \emptyset$ for each $A \in 2^X$. We show that the topologist's sine curve and the circle with a spiral are the only regular half-line compactifications admitting conservative hyperspace retractions.

4.1. *Theorem. Let* X *be a regular half-line compactification for which there exists a conservative retraction* $R: 2^X \to C(X)$. *Then either* $X \approx X_0$ *or* $X \approx X_1$.

Proof. We assume that $X = X(\pi)$, with $\pi = \pi_n$ for some $n > 1$, and show that this leads to a contradiction; the result then follows from (3.3).

Suppose first that n is even. Then for every large integer k, $R(\{k,k + 1\})$ is a small diameter continuum containing either k or k + 1, and therefore contained in a small neighborhood in $[0,\infty)$ of either k or k + 1. If k is sufficiently large, then $\pi R([k - \varepsilon, k + \varepsilon] \cup \{k + 1\})$ must be arbitrarily close to $\pi([k - \varepsilon, k + \varepsilon])$, for each $\varepsilon > 0$. Since for all sufficiently small ε, $\pi([k - \varepsilon, k + \varepsilon]) \cap \pi([k + 1 - \varepsilon, k + 1 + \varepsilon]) = \{p\}$, where $p = (1,0) \in S$, consideration of an order arc in 2^X between the elements $\{k, k + 1\}$ and $[k - \varepsilon, k + \varepsilon] \cup \{k + 1\}$ shows that $R(\{k, k + 1\})$ cannot lie in a small neighborhood of k + 1. An analogous argument involving an order arc between $\{k, k + 1\}$ and $\{k\} \cup [k + 1 - \varepsilon, k + 1 + \varepsilon]$ shows that

$R(\{k, k+1\})$ cannot lie in a small neighborhood of k. Thus n cannot be even.

Now suppose n is odd. For any large integer k, set $k_1 = \inf\{t: t > k \text{ and } \pi(t) = \pi(k)\}$ and $k_2 = \sup\{t: t < k+1 \text{ and } \pi(t) = \pi(k+1)\}$. Clearly, $k < k_i < k+1$ for each $i = 1,2$. Since π is locally 1-1 at each k_i, but not at k or k + 1, arguments analogous to those above show that, for sufficiently large k, $R(\{k,k_1\})$ must lie in a small neighborhood of k_1, and $R(\{k_2, k+1\})$ must lie in a small neighborhood of k_2. Let $\alpha: [0,1] \to 2^X$ be the path between $\{k,k_1\}$ and $\{k_2, k+1\}$ defined by $\alpha(t) = \{(1-t)k + tk_2, (1-t)k_1 + t(k+1)\}$. Note that for each $0 \leq t \leq 1$, $\pi(\alpha(t))$ is a singleton, and therefore $R(\alpha(t))$ must lie in a small neighborhood of one of the points of $\alpha(t)$. But since for each t the points of $\alpha(t)$ remain a constant distance apart, this is inconsistent with the noted properties of $R(\alpha(0))$ and $R(\alpha(1))$. Thus n cannot be odd, and this completes the proof that X is homeomorphic to either X_0 or X_1.

5. Construction of Hyperspace Retractions

From this point through section 8, $X = [0,\infty) \cup K$ will denote one of the regular compactifications X_n, $n \geq 0$, described in section 1. Thus, K is either the interval I or the circle S. Let $\pi: X \to K$ be the retraction defined by the periodic surjection $\pi_n: [0,\infty) \to K$. The construction of a retraction $R: 2^X \to C(X)$ is based on the two propositions stated next, whose proofs will be given in sections 7 and 8.

5.1. *Proposition. There exists a map* $G: 2^X \to C(X)$ *with the following properties:*

i) $G|C(K) = \mathrm{id}$;

ii) *either* $G(A) \supset \pi(A)$ *or* $G(A) \subset [0,\infty)$;

iii) $G(A) \subset K$ *if* $A \cap K \neq \emptyset$;

iv) $G(A) \supset K$ *if* $A \subset [0,\infty)$ *and* $G(A) \supset \pi([\inf A, \sup A])$; *and*

v) $G(A) \cap (K \cup A) \neq \emptyset$.

Remark. In the cases n = 0,1, the above property v) may be strengthened by requiring that $G(A) \cap A \neq \emptyset$.

For a given subset $\mathcal{N}$ of C(K), let $\mathcal{D}$ be the subset of $C(X) \times C(X)$ defined by $\mathcal{D} = \{(M,N): (M \cup K) \cap N \neq \emptyset$, and either $M \not\supseteq K \supset N \in \mathcal{N}$ or $M \cap K = \emptyset\}$.

5.2. *Proposition. For some neighborhood* $\mathcal{N} \subset C(K)$ *of* K, *there exists a map* $H: \mathcal{D} \times [0,1] \to C(X)$ *satisfying the following conditions, for every* $(M,N) \in \mathcal{D}$ *and* $0 \leq t \leq 1$:

i) $H(M,N,0) = M$ *and* $H(M,N,1) = N$;

ii) *either* $H(M,N,t) \supset M$ *or* $H(M,N,t) \supset N$;

iii) $H(M,N,t) \subset [r,\infty) \cup K$ *if* $M \cup N \subset [r,\infty) \cup K$; *and*

iv) $H(M,N,t) \subset [r,s]$ *if* $M \cup N \subset [r,s]$ *and* $\pi([r,s]) \neq K$.

5.3. *Theorem. For* $X = [0,\infty) \cup K$ *as above, there exists a hyperspace retraction* $2^X \to C(X)$.

Proof. Let $F: 2^X \setminus 2^K \to C(X) \setminus C(K)$ denote the "smallest continuum" retraction, defined by

$$F(A) = \begin{cases} [\inf A, \sup A] & \text{if } A \subset [0,\infty), \\ [\inf(A \cap [0,\infty)),\infty) \cup K & \text{if } A \cap K \neq \emptyset. \end{cases}$$

Define a map $\Theta: 2^X \setminus 2^K \to [0,1]$ by the formula

$\Theta(A) = \min\{(2/\delta) \cdot \inf(A \cap [0,\infty)) \cdot \rho(\pi(A), \pi(F(A))),1\}$,

where $0 < \delta < 1$ is chosen such that $\{N \in C(K): \rho(N,K) < \delta\} \subset \mathcal{N}$, the neighborhood of K in C(K) given by (5.2). Note that $\Theta(M) = 0$ for all $M \in C(X)\setminus C(K)$.

Let $\mathcal{W} = \{A \in 2^X \setminus 2^K$: either $A \subset [0,\infty)$ or $\rho(\pi(A), K) < \delta\}$. Note that $\mathcal{W}$ is an open subset of 2^X, and $C(X)\setminus C(K) \subset \mathcal{W}$. Let $G: 2^X \to C(X)$ and $H: \mathcal{D} \times [0,1] \to C(X)$ be the maps given by (5.1) and (5.2). The desired retraction $R: 2^X \to C(X)$ is defined by

$$R(A) = \begin{cases} H(F(A), G(A), \Theta(A)) & \text{if } A \in \mathcal{W}, \\ G(A) & \text{if } A \in 2^X \setminus \mathcal{W}. \end{cases}$$

We first verify that for each $A \in \mathcal{W}$, $(F(A), G(A)) \in \mathcal{D}$, so that R is well-defined. There are two cases to be considered:

1) Suppose $A \in 2^X \setminus 2^K$ with $A \cap K \neq \emptyset$ and $\rho(\pi(A),K) < \delta$. Then $F(A) \supsetneq K \supset G(A) \supset \pi(A)$, therefore $\rho(G(A),K) < \delta$ and $G(A) \in \mathcal{N}$. Thus $(F(A), G(A)) \in \mathcal{D}$.

2) Suppose $A \subset [0,\infty)$. Then $F(A) \subset [0,\infty)$, and $(F(A) \cup K) \cap G(A) \supset (A \cup K) \cap G(A) \neq \emptyset$, so again $(F(A), G(A)) \in \mathcal{D}$.

We next verify that $R/C(X) = \text{id}$. Since $R/C(K) = G/C(K) = \text{id}$, we need only consider $M \in C(X)\setminus C(K)$. Then $\Theta(M) = 0$ and $M \in \mathcal{W}$, so $R(M) = H(F(M), G(M),0) = F(M) = M$.

It remains to show that R is continuous. Since $\mathcal{W}$ is open in 2^X, we have only to verify continuity of R at each $A \in \text{bd}\, \mathcal{W}$. Suppose to the contrary that R is *not* continuous at some such A. Then there exists a sequence $\{A_i\}$ in $\mathcal{W}$

converging to A, with no subsequence of $\{R(A_i)\}$ converging to $R(A) = G(A)$. In particular, $\Theta(A_i) \neq 1$ for almost all i. There are two cases to be considered.

1) Suppose $A \in 2^K$. Then $\inf(A_i \cap [0,\infty)) \to \infty$, which together with $\Theta(A_i) \neq 1$ implies that $\rho(\pi(A_i), \pi(F(A_i))) \to 0$. Thus $F(A_i) \to A \in C(K)$, and $G(A_i) \to G(A) = A$. If $A = K$, then $R(A_i) = H(F(A_i), G(A_i), \Theta(A_i)) \to K$ by the properties ii) and iii) of H, contrary to our choice of $\{A_i\}$. Thus $A \in C(K)\setminus\{K\}$, and $A_i \subset [0,\infty)$ for almost all i since $F(A_i) \to A$.

If $G(A_i) \cap K \neq \emptyset$ for infinitely many i, then $G(A_i) \supset \pi(A_i)$ by the property ii) of G, and since $F(A_i) \to A \neq K$ and $G(A_i) \to A$, it follows that $G(A_i) \supset \pi(F(A_i))$ for infinitely many i. By the property iv) of G, $G(A_i) \supset K$, contradicting the convergence of $\{G(A_i)\}$ to A.

On the other hand, if $G(A_i) \subset [0,\infty)$ for almost all i, then $F(A_i) \cap G(A_i) \supset A_i \cap G(A_i) \neq \emptyset$ by the property v) of G, so for almost all i, $F(A_i) \cup G(A_i) = [r_i,s_i]$, a subarc of $[0,\infty)$. Since both $\{F(A_i)\}$ and $\{G(A_i)\}$ converge to $A \neq K$, $\pi([r_i,s_i]) \neq K$ for almost all i. Then the properties ii) and iv) of H imply that $R(A_i) \to A = R(A)$, again contrary to our choice of $\{A_i\}$.

2) Suppose $A \in 2^X\setminus 2^K$, with $A \cap K \neq \emptyset$ and $\rho(\pi(A),K) \geq \delta$. Then for almost all i, $\pi(F(A_i)) = K$ and $\rho(\pi(A_i),K) \geq \delta/2$, yielding $\Theta(A_i) = 1$, which is impossible. This completes the verification of continuity for R.

Finally, we note that the retraction R is conservative if G is, since for each $A \in 2^X$, either $R(A) \supset F(A) \supset A$ or $R(A) \supset G(A)$. Thus, in the cases $n = 0,1$ where a conservative map G may be chosen, we obtain a conservative hyperspace retraction.

6. Admissible Expansions in K

As in the previous section, $X = [0,\infty) \cup K = X_n$ for some $n \geq 0$, with $\pi: X \to K$ the retraction defined by π_n. We call a map $e: K \times [0,\infty) \to C(K)$ an *expansion* if it satisfies the following conditions (for $A \in 2^X$, $e(A,t) = \cup\{e(a,t): a \in A\}$):

1) $e(x,t) \supset e(x,0) = \{x\}$ for all x and t;

2) for every $0 \leq s < t$, there exists $\delta > 0$ such that $e(e(x,s),\delta) \subset e(x,t)$ for all x;

3) for every $A \in 2^K$ and $\delta > 0$, $e(B,\delta) \supset A$ for all $B \in 2^K$ sufficiently close to A; and

4) for every $A \in 2^K$, $e(A,t) \in C(K)$ for some t.

An expansion e is *admissible* if it permits an extension to a map $\tilde{e}: X \times [0,\infty) \to C(X)$ satisfying the above condition 1) and such that, for all $x \in [1,\infty)$ and all t, $\tilde{e}(x,t) \subset [x - 1, x + 1]$ and $\pi(\tilde{e}(x,t)) = e(\pi(x),t)$. We refer to $\tilde{e}$ as a "lift" for e.

6.1. *Lemma. There exists an admissible expansion* $e: K \times [0,\infty) \to C(K)$.

Proof. With d the arc-length metric on K, we may obtain an expansion by simply setting $e(x,t) = \{y \in K: d(x,y) \leq t\}$. However, this "free" expansion is admissible only if $\pi/(0,\infty)$ is an open map, i.e., only for n = 0,1. Thus, for these cases the lemma is trivial, but for n > 1, some type of "partial" expansion is required.

Suppose then that $K = S$ and $n > 1$. Let $\omega: (-\infty,\infty) \to S$ be the covering projection defined by $\omega(r) = e^{2\pi ir}$, and let $\tilde{\pi}: [0,\infty) \to (-\infty,\infty)$ be a lift of the periodic surjection $\pi_n: [0,\infty) \to S$. Then $J = \text{im } \tilde{\pi}$ is a compact subinterval with length $n/2 \geq 1$. Let $p,q \in J$ be the points for which $J = [p - 1, q + 1]$. For each $z \in S$, let $z_p, z_q \in (0,1]$ be the unique values for which $\omega(p - z_p) = z = \omega(q + z_q)$.

Define maps $e_p, e_q: S \times [0,\infty) \to C(S)$ by the formulas

$$\begin{cases} e_p(z,t) = \omega([p - (1 + t)z_p, p - z_p] \cap J), \\ e_q(z,t) = \omega([q + z_q, q + (1 + t)z_q] \cap J). \end{cases}$$

Although the total image function $z \to e_p(z \times [0,\infty))$ is discontinuous at $z = \omega(p)$, the function e_p is continuous; similarly for e_q. These maps may be viewed quite simply. For $z \in S$, the restriction $e_p|z \times [0,\infty)$ is clockwise expansion around S from z to $\omega(p)$, where $\omega(p) = \pi(\{0,2,4,\cdots,\}) = (1,0)$ is the π-projection of those "turning points" in $[0,\infty)$ where the direction of travel (towards ∞) changes from clockwise rotation about S to counterclockwise rotation. Similarly, $e_q|z \times [0,\infty)$ is counterclockwise expansion from z to $\omega(q)$, where $\omega(q) = \pi(\{1,3,5,\cdots\})$ is the π-projection of those turning points where the direction of travel changes from counterclockwise to clockwise. For even n, $\omega(q) = (1,0)$, while for odd n, $\omega(q) = (-1,0)$.

We show that the map $e: S \times [0,\infty) \to C(S)$, defined by $e(z,t) = e_p(z,t) \cup e_q(z,t)$, is an admissible expansion. The admissibility of e should already be evident from the above discussion of the maps e_p and e_q. It remains to verify the expansion conditions 1) through 4).

Condition 1) is obvious. Condition 2) is satisfied with $\delta = t - s/(1 + s)$, since then $(1 + s)(1 + \delta) = (1 + t)$. The verification of condition 3) is more involved. The basic observation is that, for all $y,z \in S$ and $\delta > 0$,

$$\text{i)} \begin{cases} z_p/(1 + \delta) \leq y_p \leq z_p \text{ implies } z \in e_p(y,\delta); \\ z_q/(1 + \delta) \leq y_q \leq z_q \text{ implies } z \in e_q(y,\delta). \end{cases}$$

Let d be the metric on S defined by $d(y,z) = \min\{|u - v|: u,v \in (-\infty,\infty) \text{ with } \omega(u) = y \text{ and } \omega(v) = z\}$. The above observation i) implies that for all y,z,

$$\text{ii) if } d(y,z) \leq \min\{z_p,z_q\} \cdot \delta/(1 + \delta), \text{ then } z \in e(y,\delta).$$

Let $m = \min\{(\omega(p))_q, (\omega(q))_p\}$. Then i) also implies that for all y,

$$\text{iii)} \begin{cases} \text{if } y_q \leq m\delta/(1 + \delta), \text{ then } e_p(y,\delta) \supset \omega([q,q + y_q]); \\ \text{if } y_p \leq m\delta/(1 + \delta), \text{ then } e_q(y,\delta) \supset \omega([p - y_p,p]). \end{cases}$$

Assuming $\delta < 1$, iii) implies that for all y,z,

$$\text{iv)} \begin{cases} \text{if } d(y,z) \leq z_q/2 \leq m\delta/6, \text{ then} \\ \quad e_p(y,\delta) \supset \omega([q,q + z_q/2]); \\ \text{if } d(y,z) \leq z_p/2 \leq m\delta/6, \text{ then} \\ \quad e_q(y,\delta) \supset \omega([p - z_p/2,p]). \end{cases}$$

We can now verify condition 3). Given $A \in 2^S$ and $\delta > 0$, set $A_p = x_p/2$, for some $x \in A$ such that either $x_p \leq m\delta/3$ or $x_p = \min\{a_p: a \in A\}$; set $A_q = y_q/2$, for some $y \in A$ such that either $y_q \leq m\delta/3$ or $y_q = \min\{a_q: a \in A\}$. Let $\eta = \min\{A_p,A_q\} \cdot \delta/(1 + \delta)$. We claim that for every $B \in 2^S$ with $\rho(A,B) < \eta$, $e(B,\delta) \supset A$. There are three cases to be considered:

a) Consider $z \in A$ with $z_p \leq A_p$. Then $A_p = x_p/2 \leq m\delta/6$ for some $x \in A$. Choose $y \in B$ with $d(y,x) < \eta < A_p = x_p/2$.

By iv), $e_q(y,\delta) \supset \omega([p - x_p/2,p])$. Since $z_p \leq x_p/2$, we have $z = \omega(p - z_p) \in \omega([p - x_p/2,p])$. Thus $z \in e_q(y,\delta) \subset e(B,\delta)$.

b) An analogous argument shows that for $z \in A$ with $z_q \leq A_q$, $z \in e(B,\delta)$.

c) Consider $z \in A$ with $z_p \geq A_p$ and $z_q \geq A_q$. Choose $y \in B$ with $d(y,z) < \eta \leq \min\{z_p,z_q\} \cdot \delta/(1 + \delta)$. By (ii), $z \in e(y,\delta) \subset e(B,\delta)$.

We next verify condition 4). Note that for each $z \in S$, and sufficiently large t, $e_p(z,t) \supset \omega([p - 1, p - z_p])$, the arc (possibly degenerate) traversed in the clockwise direction from z to $\omega(p)$. Similarly, for large t, $e_q(z,t) \supset \omega([q + z_q,q + 1])$, the arc traversed in the counterclockwise direction from z to $\omega(q)$. If $\omega(p) = \omega(q)$, then for every $A \in 2^S$ with $A \neq \{\omega(p)\}$, $e(A,t) = S$ for large t. If $\omega(p) \neq \omega(q)$, let $\alpha \subset S$ be the subarc traversed in the clockwise direction from $\omega(q)$ to $\omega(p)$. Then for each $A \in 2^S$ with $A \setminus \alpha \neq \emptyset$, $e(A,t) = S$ for large t, and for $A \subset \alpha$, $e(A,t) = \alpha$ for large t. This completes the verification that e is an expansion. And as remarked earlier, e is by its construction admissible.

The above lemma will be used in section 8 for the construction of a map H with the properties specified in (5.2). At present, we apply (6.1) in the case $n > 1$ to obtain a result which will be essential for the construction in the next section of a map G with the properties specified in (5.1).

6.2. *Lemma.* *Let* $\pi = \pi_n\colon [0,\infty) \to S$, $n > 1$. *Then there exists a retraction* $E\colon 2^S \to C(S)$ *with the following properties:*

i) $E(A) \supset A$ *for each* $A \in 2^S$; *and*

ii) *for each* $A \in 2^S$ *and subinterval* $L \subset [0,\infty)$ *such that* $A \subset \pi(L) \subset E(A)$, *there exists a subinterval* $M \subset [0,\infty)$ *with* $L \subset M$ *and* $\pi(M) = E(A)$.

Proof. Let $e\colon S \times [0,\infty) \to C(S)$ be an admissible expansion given by (6.1). For each $A \in 2^S$, let $\tau(A)$ denote the smallest value of t for which $e(A,t) \in C(S)$, and define $E\colon 2^S \to C(S)$ by setting $E(A) = e(A,\tau(A))$. Then $E|C(S) = \mathrm{id}$, and $E(A) \supset A$.

We establish continuity for E by verifying continuity for the function $\tau\colon 2^S \to [0,\infty)$. The lower semi-continuity of τ is automatic, since $C(S)$ is closed in 2^S and e is continuous. Using the expansion properties 2) and 3) of e, we show that τ is upper semi-continuous. Given $A \in 2^S$ and $\varepsilon > 0$, there exists by property 2) a number $\delta > 0$ such that $e(e(B,\tau(A)),\delta) \subset e(B,\tau(A) + \varepsilon)$ for all $B \in 2^S$. By continuity of e and property 3), there exists a neighborhood $\mathcal{U}$ of A in 2^S such that $e(e(B,\tau(A)),\delta) \supset e(A,\tau(A))$ for every $B \in \mathcal{U}$. Thus, $e(B,\tau(A) + \varepsilon) \supset e(A,\tau(A))$. Also, by application of property 3) to each $\{a\}$, $a \in A$, we may assume the neighborhood $\mathcal{U}$ is small enough that for each $B \in \mathcal{U}$ and $b \in B$, $e(b,\tau(A) + \varepsilon)$ meets A. Thus, each component of $e(B,\tau(A) + \varepsilon)$ meets A, and since $A \subset e(A,\tau(A)) \subset e(B,\tau(A) + \varepsilon)$ and $e(A,\tau(A)) \in C(S)$, it follows that $e(B,\tau(A) + \varepsilon) \in C(S)$. Then $\tau(B) \leq \tau(A) + \varepsilon$ for every $B \in \mathcal{U}$, and τ is upper semi-continuous.

It remains to verify the property ii). Given $A \in 2^S$ and a subinterval $L \subset [0,\infty)$ such that $A \subset \pi(L) \subset E(A)$, we may assume that $E(A) \neq S$. Let $M \supset L$ be a *maximal* subinterval of $[0,\infty)$ for which $\pi(M) \subset E(A)$. We show that $\pi(M) = E(A)$. Let $\tilde{e}$: $X \times [0,\infty) \to C(X)$ be a lift for e. Since $A \subset \pi(L) \subset \pi(M)$, we may choose for each $a \in A$ an element $\tilde{a} \in M$ with $\pi(\tilde{a}) = a$. Set $N_a = \tilde{e}(\tilde{a},\tau(A))$. Then N_a is a subinterval of $[0,\infty)$ containing $\tilde{a}$, and $\pi(N_a) = \pi(\tilde{e}(\tilde{a},\tau(A))) = e(a,\tau(A)) \subset e(A,\tau(A)) = E(A)$. Since $\tilde{a} \in M \cap N_a$, $M \cup N_a$ is a subinterval, with $\pi(M \cup N_a) \subset E(A)$. By the maximal character of M, we must have $N_a \subset M$. Thus $E(A) = \cup\{e(a,\tau(A)): a \in A\} = \cup\{\pi(N_a): a \in A\} \subset \pi(M)$, and $\pi(M) = E(A)$.

7. Construction of the Map G

We consider first the case $n > 1$. Thus, $K = S$ and $\pi = \pi_n$: $[0,\infty) \to S$. As in the proof of (6.1), let ω: $(-\infty,\infty) \to S$ be the covering projection defined by $\omega(r) = e^{2\pi i r}$, and let $\tilde{\pi}$: $[0,\infty) \to (-\infty,\infty)$ be a lift of π. The desired map G: $2^X \to C(X)$ will be obtained as an extension of the retraction E: $2^S \to C(S)$ given by (6.2).

Let $\mathcal{U} \subset 2^X$ be the collection of those $A \in 2^X$ which satisfy the following conditions:

a) $A \subset [0,\infty)$;

b) $E(\pi(A)) \neq S$; and

c) $E(\pi(A)) \supset \omega([\inf \tilde{\pi}(A), \sup \tilde{\pi}(A)])$.

Although condition c) by itself defines a closed subspace of 2^X, $\mathcal{U}$ is an open subspace. This can be seen from the fact that, since $E(\pi(A)) \supset \pi(A) = \omega(\tilde{\pi}(A)) \supset \{\omega(\inf \tilde{\pi}(A)), \omega(\sup \tilde{\pi}(A))\}$ for each $A \in 2^X$, A satisfies conditions b) and

c) if and only if $E(\pi(A)) \cup \omega([\inf \tilde{\pi}(A), \sup \tilde{\pi}(A)]) \neq S$. Thus conditions b) and c) together define an open subspace of 2^X, as does condition a), and therefore $\mathcal{U}$ is open.

We claim that for each $A \in \mathcal{U}$ and $x \in A$, the continuum $E(\pi(A)) \subset S$ can be "lifted" through x, i.e., there exists a continuum $M \subset [0,\infty)$ with $x \in M$ and $\pi(M) = E(\pi(A))$. Suppose $x \in [i,i + 1]$, for some integer i; let $L \subset [i,i + 1]$ be the subinterval such that $\tilde{\pi}(L) = [\inf \tilde{\pi}(A), \sup \tilde{\pi}(A)]$ (note that $\tilde{\pi}|[i,i + 1]$ is a homeomorphism onto im $\tilde{\pi}$). Then $x \in L$, and $\pi(A) \subset \pi(L) = \omega(\tilde{\pi}(L)) \subset E(\pi(A))$ since $A \in \mathcal{U}$. The property ii) of the retraction E shows that L may be expanded to an interval $M \subset [i,i + 1]$ such that $\pi(M) = E(\pi(A))$.

In particular, if $A \in \mathcal{U}$ and $a = \sup A$ is the point of A nearest S, with $a \in [i,i + 1]$, then there exists a unique interval $M_i \subset [i,i + 1]$ with $a \in M_i$ and $\pi(M_i) = E(\pi(A))$. This permits the construction of a map $L: \mathcal{U} \to C(X)$ such that for each $A \in \mathcal{U}$, L(A) is an "approximate lift" of $E(\pi(A))$ through the point $a = \sup A$. We may construct L according to the following rules:

1) $L(A) = M_i$ if $\min\{a - i, i + 1 - a\} \geq 1/a$;

2) $L(A) = [i, \max M_i]$ if $a - i = 1/2a$, and $L(A) = [\min M_i, i + 1]$ if $i + 1 - a = 1/2a$;

3) $L(A) = M_{i-1} \cup M_i$ if $a = i > 0$, and $L(A) = M_i \cup M_{i+1}$ if $a = i + 1$.

For $1/2a < a - i < 1/a$ or $1/2a < i + 1 - a < 1/a$, L(A) is defined so that $M_i \subset L(A) \subset [i, \max M_i]$ or $M_i \subset L(A) \subset [\min M_i, i + 1]$, respectively, and for $0 < a - i < 1/2a$ or

$0 < i + 1 - a < 1/2a$, $[i, \max M_i] \subset L(A) \subset [\min M_{i-1}, \max M_i]$ or $[\min M_i, i + 1] \subset L(A) \subset [\min M_i, \max M_{i+1}]$, respectively.

The key properties of the map L are that $\sup A \in L(A) \subset [0,\infty)$ and $\pi(L(A)) \supset E(\pi(A))$ for each $A \in \mathcal{U}$, with $\inf L(A) \to \infty$ and $\rho(\pi(L(A)), E(\pi(A))) \to 0$ as $\sup A \to \infty$.

The desired map $G: 2^X \to C(X)$ is defined over $\mathcal{U}$ by modifying L as follows:

4) $G(A) = L(A)$ if $\rho(E(\pi(A)),S) \geq 1/\sup A$;

5) $G(A) = [\inf L(A),\infty) \cup S$ if $\rho(E(\pi(A)),S) = 1/(2 \sup A)$;

6) $G(A) = S$ if $\rho(E(\pi(A)),S) \leq 1/(4 \sup A)$.

For $1/(2 \sup A) < \rho(E(\pi(A)),S) < 1/\sup A$, $G(A)$ is defined so that $L(A) \subset G(A) \subset [\inf L(A),\infty)$, and for $1/(4 \sup A) < \rho(E(\pi(A)),S) < 1/(2 \sup A)$, $S \subset G(A) \subset [\inf L(A),\infty) \cup S$.

Note that for $A \in \mathcal{U}$, either $G(A) \cap S = \emptyset$ or $G(A) \supset S$, and $G(A) \cap (A \cup S) \neq \emptyset$.

Finally, G is defined over $2^X \setminus \mathcal{U}$ by the formula $G(A) = E(\pi(A))$. Since $\mathcal{U}$ is open, it suffices to verify continuity of G at each $B \in \mathrm{bd}\,\mathcal{U}$. Note that, since the condition c) in the definition of $\mathcal{U}$ is automatically satisfied by each $B \in \mathrm{bd}\,\mathcal{U}$, we must have either $E(\pi(B)) = S$ or $B \cap S \neq \emptyset$, otherwise $B \in \mathcal{U}$. If $G(B) = E(\pi(B)) = S$, then for any $A \in \mathcal{U}$ near B, either $G(A) = S$ by virtue of rule 6) above, or $1/(4 \sup A) < \rho(E(\pi(A)),S)$, in which case both $L(A)$ and $G(A)$ are near S. If $E(\pi(B)) \neq S$ and $B \cap S \neq \emptyset$, then for any $A \in \mathcal{U}$ near B, $L(A)$ is near $E(\pi(B))$ and $1/\sup A \leq \rho(E(\pi(A)),S)$, hence $G(A) = L(A)$ is near $G(B) = E(\pi(B))$. Thus G is a map.

We next verify that G has the required properties i) through v) of (5.1). Since G extends E, property i) is clear. Since either $G(A) \cap S = \emptyset$, $G(A) \supset S$, or

$G(A) = E(\pi(A)) \supset \pi(A)$, property ii) is satisfied. Property iii) is immediate from the definition of G over $2^X \setminus \mathcal{U}$. Property iv) is clear if $A \in \mathcal{U}$. On the other hand, if $A \subset [0,\infty)$ with $A \notin \mathcal{U}$ and $G(A) = E(\pi(A)) \neq S$, then $E(\pi(A)) \not\supset \omega([\inf \tilde{\pi}(A), \sup \tilde{\pi}(A)])$. However, this contradicts the hypothesis that $G(A) \supset \pi([\inf A, \sup A]) = \omega(\tilde{\pi}([\inf A, \sup A]))$, since $\tilde{\pi}([\inf A, \sup A]) \supset [\inf \tilde{\pi}(A), \sup \tilde{\pi}(A)]$. Finally, property v) has been previously noted for $A \in \mathcal{U}$, and is obvious for $A \in 2^X \setminus \mathcal{U}$. This completes the proof of (5.1) in the case $n > 1$.

In the cases $n = 0,1$, a streamlined version of the above construction yields a conservative map $G: 2^X \to C(X)$ with the required properties. For either $K = I$ or $K = S$, let $E: 2^K \to C(K)$ be any retraction such that $E(A) \supset A$ for each $A \in 2^K$. Let $\mathcal{V} = \{A \in 2^X: A \subset [0,\infty)\}$. As above, an approximate lifting map $L: \mathcal{V} \to C(X)$ may be constructed such that for each $A \in \mathcal{V}$, $\sup A \in L(A) \subset [0,\infty)$ and $\pi(L(A)) \supset E(\pi(A))$, with $\inf L(A) \to \infty$ and $\rho(\pi(L(A)), E(\pi(A))) \to 0$ as $\sup A \to \infty$. In fact, for $n = 0$, L is constructed in the same manner as above for $n > 1$. For $n = 1$, L is constructed such that $L(A) \subset [0,\infty)$ is the unique lift of $E(\pi(A))$ through $a = \sup A$ if $\rho(E(\pi(A)),S) \geq 1/a$; $a \in L(A) \subset [a - 2, a + 2]$ with $\pi(L(A)) \supset E(\pi(A))$ if $0 < \rho(E(\pi(A)),S) < 1/a$; and $L(A) = [a - 2, a + 2]$ if $E(\pi(A)) = S$.

In either case, L extends to a map $G: 2^X \to C(X)$ by the formula $G(A) = E(\pi(A))$ for $A \in 2^X \setminus \mathcal{V}$. Properties i) and iii) are immediate from the definition of G. Property ii) is a consequence of the fact that $E(\pi(A)) \supset \pi(A)$, and that $G(A) \subset [0,\infty)$ when $A \subset [0,\infty)$. Property iv) is satisfied vacuously. And finally, $G(A) \cap A \neq \emptyset$ for all $A \in 2^X$, since $G(A) = E(\pi(A)) \supset \pi(A)$ if $A \cap K \neq \emptyset$, and $G(A) = L(A) \ni \sup A$ if $A \cap K = \emptyset$.

8. Construction of the Map H

Let $e: K \times [0,\infty) \to C(K)$ be an admissible expansion given by (6.1). Set $\mathcal{N} = \{N \in C(K): e(N,t) = K \text{ for some } t\}$. By the expansion property 3), $\mathcal{N}$ is a neighborhood of K.

The domain $\mathcal{D} \subset C(X) \times C(X)$ of H can be partitioned into four subdomains as follows:

$$\mathcal{D}_1 = \{(M,N): M \supsetneq K \supset N \in \mathcal{N}\};$$
$$\mathcal{D}_2 = \{(M,N): M \cap K = \emptyset \text{ and } N \subset K\};$$
$$\mathcal{D}_3 = \{(M,N): M \cap K = \emptyset \text{ and } N \supsetneq K\}; \text{ and}$$
$$\mathcal{D}_4 = \{(M,N): M \cap K = \emptyset = N \cap K \text{ and } M \cap N \neq \emptyset\}.$$

We will define H separately over each $\mathcal{D}_i \times [0,1]$.

For $(M,N) \in \mathcal{D}_1$, set

$$\begin{cases} H(M,N,t) = M, & 0 \leq t \leq 1/4; \\ H(M,N,t) = K, & 1/2 \leq t \leq 3/4; \text{ and} \\ H(M,N,1) = N. & \end{cases}$$

Use the natural path in C(X) from M to K to define H(M,N,t) for $1/4 \leq t \leq 1/2$, and reverse the e-expansion $\{e(N,t): 0 \leq t < \infty\}$ of N to K to define H(M,N,t) for $3/4 \leq t \leq 1$.

For $(M,N) \in \mathcal{D}_2$, let $N^* = e(N, \sup M)$; then $N \subset N^* \in C(K)$. Set

$$\begin{cases} H(M,N,0) = M; \\ H(M,N,1/4) = [\inf M,\infty) \cup K: \\ H(M,N,1/2) = K; \\ H(M,N,3/4) = N^*; \text{ and} \\ H(M,N,1) = N. \end{cases}$$

Use the natural paths in C(X) to define H(M,N,t) for $0 \leq t \leq 1/4$ and $1/4 \leq t \leq 1/2$; reverse the free expansion (via an arc-length metric) in C(K) from N* to K to define H(M,N,t) for $1/2 \leq t \leq 3/4$; and reverse the e-expansion from N to N* to define H(M,N,t) for $3/4 \leq t \leq 1$.

For $(M,N) \in \mathcal{D}_3$, set

$$\begin{cases} H(M,N,0) = M: \\ H(M,N,1/4) = [\inf M,\infty) \cup K; \\ H(M,N,1/2) = [\max\{\inf M, \inf N\},\infty) \cup K; \text{ and} \\ H(M,N,t) = N,\ 5/8 \leq t \leq 1. \end{cases}$$

Use the natural paths in C(X) to define H(M,N,t) for all other t.

Define an index map $\tau\colon \mathcal{D}_4 \to [0,\infty)$ by the formula $\tau(M,N) = \max\{\inf N - \inf M - 2, 0\} \cdot \rho(\pi(N),K)$. For $(M,N) \in \mathcal{D}_4$, let $N^* = \tilde{e}(N,\tau(M,N))$, where $\tilde{e}$ is a lift for e. Then $N^* \in C(X)$, with $N \subset N^* \subset [\inf N - 1, \sup N + 1]$. Set

$$\begin{cases} H(M,N,0) = M; \\ H(M,N,1/4) = [\inf M, \max\{\sup M, \sup N^*\}]; \\ H(M,N,1/2) = [\max\{\inf M, \inf N^*\}, \max\{\sup M, \\ \qquad \sup N^*\}]; \\ H(M,N,5/8) = [\inf N^*, \max\{\sup M, \sup N^*\}]; \\ H(M,N,3/4) = N^*; \text{ and} \\ H(M,N,1) = N. \end{cases}$$

Use the natural paths in $C(X)$ to complete the definition of $H(M,N,t)$ for $0 \leq t \leq 3/4$, and reverse the $\tilde{e}$-expansion from N to N^* to define $H(M,N,t)$ for $3/4 \leq t \leq 1$.

We now verify that H is a map. For $i \neq j$, $\mathcal{D}_i \cap \overline{\mathcal{D}}_j \neq \emptyset$ only if $(i,j) = (1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, or $(3,4)$. Since each restriction $H/\mathcal{D}_i \times [0,1]$ is continuous, it suffices to check continuity of H at boundary points in the above cases. Considering first the case $(i,j) = (1,2)$, let (M_k,N_k) be a sequence in $\mathcal{D}_2$ converging to $(M,N) \in \mathcal{D}_1$. Then $\sup M_k \to \infty$, and since $N_k \to N \in \mathcal{N}$, we have $N_k^* = K$ for almost all k (use continuity of e, and the expansion properties 2) and 3)). It follows that $H(M_k,N_k,t_k) \to H(M,N,t)$ whenever $t_k \to t$. The cases $(i,j) = (1,3)$ or $(2,3)$ are routine. Consider a sequence (M_k,N_k) in $\mathcal{D}_4$ converging to $(M,N) \in \mathcal{D}_1$. Then if $N \neq K$, $\tau(M_k,N_k) \to \infty$ and $N_k^* \to K$; if $N = K$, obviously $N_k^* \to K$. This implies that $H(M_k,N_k,t_k) \to H(M,N,t)$ whenever $t_k \to t$. Finally, consider a sequence (M_k,N_k) in $\mathcal{D}_4$ converging to $(M,N) \in \mathcal{D}_3$. Then $\pi(N_k) = K$ for almost all k, hence $\tau(M_k,N_k) = 0$ and $N_k^* = N_k$, implying that $H(M_k,N_k,t_k) \to H(M,N,t)$ whenever $t_k \to t$. This completes the verification of continuity for $H: \mathcal{D} \times [0,1] \to C(X)$.

Clearly, H satisfies the required conditions i) and ii) of (5.2). Conditions iii) and iv) are also clear, except possibly for $(M,N) \in \mathcal{D}_4$ with $N^* \neq N$. However, $N^* \neq N$ implies $\tau(M,N) > 0$, which implies that $\inf N \geq \inf M + 2$. Then $\inf N^* \geq \inf N - 1 \geq \inf M$, and condition iii) is satisfied. And, $\operatorname{diam}(M \cup N) \geq 2$ implies that $\pi(M \cup N) = K$, so condition iv) is satisfied vacuously. This completes the proof of (5.2).

9. Means and Pseudo-Means

Let Y be a continuum. A map $\lambda\colon Y \times Y \to Y$ is called a *mean* if $\lambda(x,y) = \lambda(y,x)$ and $\lambda(y,y) = y$ for all $x,y \in Y$. A map $\lambda\colon Y \times Y \to C(Y)$ with the same properties is called a *pseudo-mean* for Y [7].

Every hyperspace 2^X admits a mean: define $\lambda(A,B) = A \cup B$. If there exists a retraction $2^X \to C(X)$, then C(X) also admits a mean, and X admits a pseudo-mean. Thus we have yet another necessary condition for the existence of a hyperspace retraction. In this section we describe examples from the class of regular half-line compactifications which show that the existence of a pseudo-mean neither implies nor is implied by the subcontinuum approximation property of section 2, and that both conditions together are still not sufficient for the existence of a hyperspace retraction. Recall that a regular compactification $X = [0,\infty) \cup K$ has the subcontinuum approximation property if and only if the remainder K is either an arc or a simple closed curve. We do not know in general which regular compactifications admit pseudo-means.

9.1. *Example.* Let $\pi\colon [0,\infty) \to I$ be the periodic surjection defined as follows:

i) $\pi(k) = 0$ if k is an odd integer;

ii) $\pi(k) = 1$ if $k \equiv 2,4 \pmod 6$;

iii) $\pi(k) = -1$ if $k \equiv 6 \pmod 6$; and

iv) π is linear over each interval $[k,k+1]$.

Then for $X = X(\pi)$, no retraction $2^X \to C(X)$ exists, since $X \not\approx X_0$; nonetheless, a pseudo-mean may be constructed for X, and in fact $C(X)$ admits a mean.

9.2. *Example.* Let $\pi: [0,\infty) \to I$ be the periodic surjection defined by:

i) $\pi(k) = 0$ if k is odd;

ii) $\pi(k) = 1$ if $k \equiv 2,4 \pmod 8$;

iii) $\pi(k) = -1$ if $k \equiv 6,8 \pmod 8$; and

iv) π is linear over each interval $[k,k+1]$.

Then $X = X(\pi)$ does not admit a pseudo-mean.

Proof. Suppose there exists a pseudo-mean $\lambda: X \times X \to C(X)$. Let k denote an integer of the form $8n + 2$. Then consideration of $\lambda(k - t, k + t)$, for $0 \leq t \leq 1$ and large n, shows that either $\lambda(k - 1, k + 1) \approx$ (approximates) $\{k - 1\}$ or $\lambda(k - 1, k + 1) \approx \{k + 1\}$. Similarly, either $\lambda(k + 1, k + 3) \approx \{k + 1\}$ or $\lambda(k + 1, k + 3) \approx \{k + 3\}$. If $\lambda(k - 1, k + 1) \approx \{k - 1\}$, then $\lambda(k, k + 2) \approx \{k\}$; if $\lambda(k + 1, k + 3) \approx \{k + 3\}$, then $\lambda(k, k + 2) \approx \{k + 2\}$. Thus, either $\lambda(k - 1, k + 1) \approx \{k + 1\}$ or $\lambda(k + 1, k + 3) \approx \{k + 1\}$. Letting $n \to \infty$, we see by continuity of λ that, for every $s \in I \subset X$ and the point $0 \in I$, either $\lambda(0,s) \subset [0,1]$ or $1 \in \lambda(0,s')$ for some s' between 0 and s. (Suppose that $\lambda(k - 1, k + 1) \approx \{k + 1\}$ for infinitely many k as above. Then for every $r \in [k - 2, k]$, either $\lambda(r, k + 1) \subset [k, k + 2]$ or $\lambda(r', k + 1) \cap \{k, k + 2\} \neq \emptyset$ for some r' between $k - 1$ and r. Note that $\pi(k - 2) = -1$, $\pi(k - 1) = \pi(k + 1) = 0$, and $\pi(k) = \pi(k + 2) = 1$). An analogous argument shows that either $\lambda(k + 3, k + 5) \approx \{k + 5\}$ or

$\lambda(k + 5, k + 7) \approx \{k + 5\}$, which implies that for every $s \in I$, either $\lambda(0,s) \subset [-1,0]$ or $-1 \in \lambda(0,s')$ for some s' between 0 and s. Consequently, $\lambda(0,s) = \{0\}$ for every $s \in I$. However, this implies that $\lambda(k - 1, k) \approx \{k - 1\} \approx \lambda(k - 1, k + 1)$ and also that $\lambda(k, k + 1) \approx \{k + 1\} \approx \lambda(k - 1, k + 1)$, a contradiction. Thus X does not admit a pseudo-mean.

9.3. *Example*. Let T be a triod, with branch point v and endpoints e_1, e_2, and e_3, and let $\pi: [0,\infty) \to T$ be the periodic surjection defined as follows:

i) $\pi(k) = v$ if k is odd;

ii) $\pi(k) = e_1$ if $k \equiv 4 \pmod 8$;

iii) $\pi(k) = e_2$ if $k \equiv 2,6 \pmod 8$;

iv) $\pi(k) = e_3$ if $k \equiv 8 \pmod 8$; and

v) π is linear over each interval $[k, k + 1]$.

Let $X = X(\pi)$. It can be shown that C(X) admits a mean.

9.4. *Example*. For T as above, let $\pi: [0,\infty) \to T$ be the periodic surjection defined by:

i) $\pi(k) = v$ if k is odd;

ii) $\pi(k) = e_1$ if $k \equiv 2 \pmod 6$;

iii) $\pi(k) = e_2$ if $k \equiv 4 \pmod 6$;

iv) $\pi(k) = e_3$ if $k \equiv 6 \pmod 6$; and

v) π is linear over each interval $[k, k + 1]$.

Then $X = X(\pi)$ does not admit a pseudo-mean.

Proof. Suppose there exists a pseudo-mean λ. Let k denote an integer of the form 6n + 1. Consideration of $\lambda(k, k + t)$ and $\lambda(k + 2, k + 2 - t)$, for $0 \leq t \leq 1$ and large n, shows that λ must have the following property with respect to e_1: for each $x \in [v,e_1]$, either $\lambda(v,x) \subset [v,e_1]$ or $e_1 \in \lambda(v,x')$ for some x' between v and x. Of course, λ has the analogous properties with respect to e_2 and e_3.

Now, consideration of $\lambda(k + 1 - t, k + 1 + t)$, for $0 \leq t \leq 1$ and $k = 6n + 1$ as above, shows that for large n, either $\lambda(k, k + 2) \approx \{k\}$ or $\lambda(k, k + 2) \approx \{k + 2\}$. We may suppose the former (for infinitely many n). Then consideration of $\lambda(k, k + 2 + t)$, for $0 \leq t \leq 1$, together with the above property of λ with respect to e_2, shows that $\lambda(v,x) = \{v\}$ for each $x \in [v,e_2]$. But this implies that $\lambda(k + 2, k + 3) \approx \{k + 2\} \approx \lambda(k + 2, k + 4)$ and also that $\lambda(k + 4, k + 3) \approx \{k + 4\} \approx \lambda(k + 4, k + 2)$, a contradiction. Thus X does not admit a pseudo-mean.

There also exist regular compactifications $X = [0,\infty) \cup S$ similar to the above examples. Let $\pi: [0,\infty) \to S$ be the periodic surjection defined by $\pi(t) = e^{i\pi t}$, $0 \leq t \leq 3 \pmod 4$, and $\pi(t) = e^{-i\pi t}$, $3 \leq t \leq 4 \pmod 4$. Then for $X = X(\pi)$, C(X) admits a mean. On the other hand, there exist periodic surjections $[0,\infty) \to S$ for which the corresponding compactifications do not admit pseudo-means. An example is the map π defined by $\pi(t) = e^{i2\pi t}$, $0 \leq t \leq 2 \pmod 3$, and $\pi(t) = e^{-i2\pi t}$, $2 \leq t \leq 3 \pmod 3$.

If there exists a *conservative* retraction $2^X \to C(X)$, then there exists a *conservative* pseudo-mean $\lambda: X \times X \to C(X)$, i.e., $\lambda(x,y) \cap \{x,y\} \neq \emptyset$ for all x,y. It can be shown that a regular compactification $X = [0,\infty) \cup K$ admits a conservative pseudo-mean only if X is homeomorphic to either X_0 or X_1. Thus, in the class of regular half-line compactifications, the existence of a conservative pseudo-mean is equivalent to the existence of a conservative hyperspace retraction. It seems unlikely that this would hold in general, but we do not have a counterexample.

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