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by

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TOPOLOGY OF THE SIEGEL SPACES OF DEGREE TWO AND THEIR COMPACTIFICATIONS

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1. Introduction

1.1. In this paper, we study the various compactifications of Siegel spaces of degree two. We discuss the Borel-Serre, the Satake, and the Igusa compactifications. This last one is of special interest because it gives a projective variety with at most finite quotient singularities, and can be treated in the framework of Mumford's toroidal compactification theory ([ARMT]).

We treat this smooth compactification in great detail. Apart from its intrinsic interest, it provides one of the first non-trivial examples of Mumford's theory, and the description here, for $G = Sp_4$, is a model for other rank two groups, e.g., $G = SO(2,q)$ and $G = SU(2,q)$.

Let $\mathfrak{S}_2$ denote the Siegel upper-half space of degree 2, consisting of 2-by-2 complex symmetric matrices with positive-definite imaginary part

$$\mathfrak{S} = \mathfrak{S}_2 = \{Z \in M_2(\mathbb{C}) \mid Z = {}^tZ, \operatorname{Im} Z > 0\}.$$

There is a natural action of the discrete group $G(\mathbb{Z}) = Sp_4(\mathbb{Z})$ on this space, and we consider as well the principal congruence subgroup of level $p \geq 3$

$$\Gamma = \{g \in Sp_4(\mathbb{Z}) \mid g \equiv I \bmod p\}.$$

Denote by $\mathfrak{S}/\Gamma^*$ the Igusa compactification of the quotient space $\mathfrak{S}/\Gamma$. We call $\mathfrak{S}/\Gamma^* - \mathfrak{S}/\Gamma$ the "boundary" of $\mathfrak{S}/\Gamma^*$. It is a union of boundary components, each of which is known as an elliptic modular surface. We study these surfaces in detail, determining their cohomology and Hodge structure, as well as the Chern classes of their normal bundles in $\mathfrak{S}_2/\Gamma^*$.

For Γ a normal subgroup of finite index in $G(\mathbb{Z})$, as in this example, these spaces admit a natural action of the factor group $G(\mathbb{Z})/\Gamma$. The above-mentioned data is essential in computing the holomorphic Lefschetz numbers ([AS]) for this action.

In the case of the Siegel modular spaces, this can be viewed as an extension of the work of Erich Hecke. In 1928, he discovered a relation between the class number $h(-p)$ of the imaginary quadratic field $Q(\sqrt{-p})$ and the multiplicities of certain cuspidal representations in "Über Ein Fundamental problem Aus Der Theorie Der Elliptichen Modulfunctionen" (see [H1]). Let $p \geq 7$ be a prime, $p \equiv 3 \pmod 4$, let $\mathbb{F}_p$ be the finite field of order p, and let $SL_2(\mathbb{F}_p)$ be the special linear group of order 2 over this finite field. There are two irreducible $SL_2(F_p)$-representations R_1, R_2 which are dual to each other, $R_1 \simeq R_2^*$, $\dim R_i = (p-1)/2$; in [H2] Hecke proved that if m_i is the multiplicity of R_i in the space of weight 2 cusp forms of level p, then the difference $m_1 - m_2$ is the same as the class number $h(-p)$ of $Q\sqrt{-p}$, $m_1 - m_2 = h(-p)$. For cusp forms of weight greater than 2, the same result is true, and these forms were investigated thoroughly by Hecke and Feldmann (see [F]).

In modern language, Hecke's work is about the rank one group $G = SL(2)(Sp_2)$ acting on the upper half plane $\mathfrak{S}_1$.

The space of cusp forms of weight k and level p can be identified with a cohomology group $H^0(\mathfrak{G}_l/\Gamma^*;\mathcal{L}_k)$ for an appropriate line bundle $\mathcal{L}_k$, and the representation computed by the holomorphic Lefschetz formula.

Let $S_k(\Gamma)$ be the space of cusp forms of weight k over $\mathfrak{G}/\Gamma$ (known as the space of cusp forms of weight k, level p, and degree 2). Since $Sp_4(\mathbf{F}_p)$ operates on $\mathfrak{G}/\Gamma$, this space $S_k(\Gamma)$ of cusp forms is a representation space of $Sp_4(\mathbf{F}_p)$. To generalize Hecke's result, it is natural to consider non-self-dual irreducible representations $\rho \neq \rho^*$ of $Sp_4(\mathbf{F}_p)$, and compute the difference in multiplicities of ρ and ρ^* appearing in $S_k(\Gamma)$. In [Y], Yamazaki proved that there is an isomorphism between this space $S_k(\Gamma)$ and the analytic cohomology $H^0(\mathfrak{G}/\Gamma^*;\mathcal{L}_k)$ with coefficients in a line bundle $\mathcal{L}_k$. It follows that the difference in multiplicities may be obtained by computing the holomorphic Lefschetz numbers. Some of these results were announced in [LW1], and a further treatment can be found in a forthcoming paper of Horikawa.

This paper is organized as follows:

We begin, in Section 2, by discussing the process of compactifying $\mathfrak{G}/\Gamma$. In fact, we discuss various compactifications of this quasi-projective variety, all of which are based on the combinatorial design known as the Tits building, considered in 2.2. In 2.3 we discuss the Borel-Serre compactification, a smooth manifold with boundary, in 2.4 the Satake compactification, a singular projective variety, and in 2.5 the Igusa compactification $\mathfrak{G}/\Gamma^*$, a non-singular variety which is a desingularization of the Satake compactification.

As mentioned above, each irreducible component in the boundary of $\mathfrak{S}/\Gamma^*$ is an elliptic modular surface of level p. This is a non-singular fibration over the modular curve of level p, i.e. it is a fibration except over a finite number of points whose inverse images are singular. These kinds of manifolds were first studied by Kodaira in [K], and the elliptic modular surface was further studied by Shioda in [So]. In Section 3, we study the algebraic topology and Hodge structure of the elliptic modular surface.

In Section 4, we determine the Chern classes of this surface and its normal bundle in $\mathfrak{S}/\Gamma^*$, as well as other Chern classes which enter into the computation of the holomorphic Lefschetz numbers.

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2. Compactifications

2.1. *Siegel space of degree* 2. Let $\mathfrak{S}_2$ denote the Siegel upper half space consisting of symmetric, complex, 2-by-2 matrices with positive definite imaginary component,

$$\mathfrak{S}_2 = \{Z \in M_2(C) \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}.$$

Let $Sp_4(\mathbb{Z})$ denote the integral symplectic group of degree 4. Every element g in this group can be written in the form of a 2-by-2 block matrix

$$(g) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where A,B,C,D, satisfy the following conditions:

$$A^tC - C^tA = 0, \quad B^tD - D^tB = 0, \quad A^tD - C^tB = I.$$

There is an action of $Sp_4(\mathbb{Z})$ on the Siegel upper half space $\mathfrak{S}_2$ defined by the formula:

$$Z \mapsto Z\cdot g = (ZB + D)^{-1}\cdot(ZA + C).$$

However, $Sp_4(Z)$ contains elements of finite order, and the quotient space is not a manifold. To avoid this difficulty, it is the usual practice to consider the principal congruence subgroup of level p

$$\Gamma = \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in Sp_4(\mathbb{Z}) \quad \begin{array}{c} A \equiv D \equiv I \bmod p \\ \\ B \equiv C \equiv 0 \bmod p \end{array} \right\}$$

Then the quotient space $\mathfrak{S}_2/\Gamma$ is a complex manifold of complex dimension 3, called the Siegel modular space. The only drawback of the Siegel modular space is that it is not a closed manifold, and there are various methods to compactify this manifold. In the next few sections, we will discuss the Borel-Serre compactification $\mathfrak{S}/\Gamma^c$, the Satake compactification $\overline{\mathfrak{S}/\Gamma}$, and the Igusa compactification $\mathfrak{S}/\Gamma^*$. From the topological point of view, the Borel-Serre compactification $\mathfrak{S}/\Gamma^c$ is the most natural because it gives the actual topological boundary of the manifold. From the point of view of history, the Satake compactification is the oldest. Finally, from the point of view of algebraic geometry, the Igusa compactification is most satisfactory because it results in a nonsingular projective variety. (See [BS], [St], [Ig], [AMRT].)

2.2. *Tits building.* All these compactifications are based on a combinatorial design called the *Tits building*.

Let V denote the free $\mathbb{Z}$-module of rank 4 with base e_1, e_2, f_1, f_2. Over this free module, there is a nonsingular, skew-symmetric, bilinear pairing $\lambda: V \times V \to \mathbb{Z}$ defined by the condition:

$$\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0, \qquad \lambda(e_i, f_j) = \delta_{ij}.$$

The subgroup of automorphisms of V which preserves this pairing λ is of course isomorphic to $Sp_4(\mathbb{Z})$ mentioned before. However, we will use the notation Sp(V) to denote this group. Whenever it is necessary to use a matrix presentation, we regard V as the space of integral row vectors (x_1, x_2, y_1, y_2) and regard $Sp(V) \cong Sp_4(\mathbb{Z})$ as a matrix group.

Associated to V, there are the vector spaces $V_{\mathbb{Q}} \cong V \otimes_{\mathbb{Z}} \mathbb{Q}$ defined over the rational field and the vector space $\bar{V} \cong V \otimes_{\mathbb{Z}} \mathbb{F}_p$ defined over the finite field $\mathbb{F}_p$ of p elements. Accordingly, there are the algebraic groups $Sp(V_{\mathbb{Q}})$ over the rationals, and $Sp(\bar{V})$ over the finite field.

The structure of the parabolic subgroups in $Sp(V_{\mathbb{Q}})$ is well known. There are two types of maximal parabolic subgroups, and one minimal parabolic. For example, as representatives for the maximal parabolic groups, we have the following subgroups of matrices:

$$(2.2.1) \qquad P_1 = \left\{ \left[\begin{array}{cc|cc} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{array}\right] \in Sp_4(\mathbb{Q}) \right\}$$

$$(2.2.2) \qquad P_2 = \left\{ \left[\begin{array}{cc|cc} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \in Sp_4(\mathbb{Q}) \right\}$$

and for the minimal parabolic subgroups, we have

$$(2.2.3) \qquad P_0 = P_1 \cap P_2 = \left\{ \left[\begin{array}{cc|cc} a_{11} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{array}\right] \in Sp_4(\mathbb{Q}) \right\}$$

Every parabolic subgroup in $Sp_4(\mathbb{Q})$ is conjugate to one of these parabolic subgroups P_i, $i = 0,1,2$.

To record the incidence relation among parabolic subgroups, we consider the set $\mathcal{P}$ of all rational parabolic subgroups partially ordered by inclusion. The geometric realization of this partially ordered set $\mathcal{P}$ is called the Tits building, and is denoted by the symbol $\mathcal{J}(V)$, $|\mathcal{P}| = (V)$.

In the present situation, $\mathcal{J}(V)$ is a graph (1-dimensional simplical complex) and can be explained in terms of the simplectic geometry of V. Let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_0$ denote respectively the set of lines $L_{\mathbb{Q}}$ in $V_{\mathbb{Q}}$, the set of 2-dimensional isotropic subspaces $H_{\mathbb{Q}}$ in $V_{\mathbb{Q}}$, and the set of flags $(L_{\mathbb{Q}}, H_{\mathbb{Q}})$ where $H_{\mathbb{Q}}$ is an isotropic subspace and $L_{\mathbb{Q}}$ is a line in $H_{\mathbb{Q}}$, $L_{\mathbb{Q}} \subset H_{\mathbb{Q}}$,

$$\mathcal{P}_1 = \{L_{\mathbb{Q}} | L_{\mathbb{Q}} \text{ is a line in } V_Q\}$$
$$\mathcal{P}_2 = \{H_{\mathbb{Q}} | H_{\mathbb{Q}} \text{ is a maximal isotropic subspace in } V_Q\}$$
$$\mathcal{P}_0 = \{(L_{\mathbb{Q}}, H_{\mathbb{Q}}) | H_{\mathbb{Q}} \in \mathcal{P}_2, L_{\mathbb{Q}} \in \mathcal{P}_1, L_{\mathbb{Q}} \subseteq H_{\mathbb{Q}}\}.$$

Using these sets $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ we now form a graph: for every line L_Q in $\mathcal{P}_1$ or every plane $H_{\mathbb{Q}}$ in $\mathcal{P}_2$ we provide a vertex, and for every flag $(L_{\mathbb{Q}}, H_{\mathbb{Q}})$ we provide an edge whose endpoints are the vertices $(L_{\mathbb{Q}})$, $(H_{\mathbb{Q}})$. The resulting graph is homeomorphic to $\mathcal{J}(V)$. For there is a one-to-one correspondence between the sets $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ and respectively the set of parabolic subgroups conjugate to P_0, P_1, P_2 respectively. Given a line $L_{\mathbb{Q}} \in \mathcal{P}_1$, a plane $H_{\mathbb{Q}} \in \mathcal{P}_2$, or a flag $(L_{\mathbb{Q}}, H_{\mathbb{Q}}) \in \mathcal{P}_0$, we have a parabolic subgroup defined by the stabilizers in $Sp(V_{\mathbb{Q}})$,

$$P(L_{\mathbb{Q}}) = \{g \in Sp(V_{\mathbb{Q}}) | L_{\mathbb{Q}} \cdot g = L_{\mathbb{Q}}\},$$
$$P(H_{\mathbb{Q}}) = \{g \in Sp(V_{\mathbb{Q}}) | H_{\mathbb{Q}} \cdot g = H_{\mathbb{Q}}\},$$
$$P(L_{\mathbb{Q}}, H_{\mathbb{Q}}) = \{g \in Sp(V_{\mathbb{Q}}) | L_{\mathbb{Q}} \cdot g = L_{\mathbb{Q}}, H_{\mathbb{Q}} \cdot g = H_{\mathbb{Q}}\}.$$

Example (2.2.4). Let e_1 denote the line generated by the vector (1,0,0,0) and let $e_1 \wedge f_1$ denote the plane generated by the vectors (1,0,0,0) and (0,1,0,0). Then $P(e_1)$, $P(e_1 \wedge f_1)$, $P(e_1, e_1 \wedge f_1)$ coincide with the group P_1, P_2, and P_0 defined in (2.2.1), (2.2.2), and (2.2.3).

The building $\mathcal{J}(V)$ in the above paragraph consists of infinitely many vertices and edges. To apply this to the theory of compactification, we form the quotient space $\mathcal{J}(V)/\Gamma$ of the building $\mathcal{J}(V)$ modulo the action of the congruence subgroup Γ. This is a finite graph whose vertices and edges are in one-to-one correspondence with Γ-conjugacy classes of parabolic subgroups in $\mathcal{P}_1/\Gamma \cup \mathcal{P}_2/\Gamma$ and $\mathcal{P}_0/\Gamma$

respectively. Once again, this can be explained in terms of the symplectic geometry of the vector space $\bar{V}$.

We consider nonzero vectors $\ell = (x_1, x_2, y_1, y_2)$ in $\bar{V}$, and identify two such vectors ℓ and ℓ' whenever they differ only by a sign ± 1,

$$\ell \sim \ell' \Leftrightarrow \ell = \pm \ell'.$$

The resulting set is called the set of based lines in $\bar{V}$, denoted by $\bar{\mathcal{P}}_1$. In a similar manner, we consider the nonzero exterior products $h = \ell_1 \wedge \ell_2$ in $\Lambda^2\bar{V}$ satisfying the condition:

$$\lambda(\ell_1, \ell_2) = 0 \bmod p.$$

Again, we say two such products $h = \ell_1 \wedge \ell_2$, $h' = \ell_1' \wedge \ell_2'$ are equivalent if they differ only by a sign ± 1, $h = \pm h'$. The set $\bar{\mathcal{P}}_2$ of all these equivalence classes is called the set of based isotropic planes in $\bar{V}$. Note that given any element h in $\bar{\mathcal{P}}_2$, there is an isotropic subspace H_h in $\bar{V}$ defined by the formula: $H_h = \{\xi \in \bar{V} | \xi \wedge h = 0\}$. However, two different elements h, h' in $\bar{\mathcal{P}}_2$ may define the same isotropic subspace, $H_h = H_{h'} \Leftrightarrow h = ah'$. For $a \neq 1$, h and h' represent different bases of the same isotropic subspace. Finally, we define $\bar{\mathcal{P}}_0$ to be the set of pairs (ℓ, h) such that $\ell \in \bar{\mathcal{P}}_1$, $h \in \bar{\mathcal{P}}_2$, $\ell \wedge h = 0$. This allows us to construct, as before, a 1-dimensional simplical complex $\mathcal{J}_{\pm}(\bar{V})$ which has $\bar{\mathcal{P}}_0$ as its set of edges and which has $\bar{\mathcal{P}}_1 \cup \bar{\mathcal{P}}_2$ as its set of vertices. We will refer to $\mathcal{J}_{\pm}(\bar{V})$ as the based Tits building of $\bar{V}$.

Lemma (2.2.5). *Let $\mathcal{T}_{\pm}(\bar{V})$ be the based Tits building defined as above. Then there is a simplical isomorphism of $\mathcal{T}_{\pm}(\bar{V})$ with the quotient space $\mathcal{T}(V)/\Gamma$.*

$$\mathcal{T}_{\pm}(\bar{V}) \cong \mathcal{T}(V)/\Gamma.$$

Proof. For the proof, it is enough to construct natural isomorphisms:

$$\mathcal{P}_0/\Gamma \to \bar{\mathcal{P}}_0, \quad \mathcal{P}_1/\Gamma \to \bar{\mathcal{P}}_1, \quad \mathcal{P}_2/\Gamma \to \bar{\mathcal{P}}_2.$$

Given a line L_Q in $\mathcal{P}_1$, we note that its intersection with the integral lattice V is a one dimensional subspace, and so it determines a pair of generators $\pm\ell$,

$$L_Q \cap V = L_{\mathbb{Z}} = \mathbb{Z}\langle \ell \rangle.$$

We define a map

$$\mathcal{P}_1 \to \bar{\mathcal{P}}_1$$

of $\mathcal{P}_1$ to $\bar{\mathcal{P}}_1$ by assigning to every line L_Q the vector $\pm\ell \pmod p$ obtained by reducing the element ℓ modulo p. An element g in Γ induces the identity map on the vector space $\bar{V}$, and so

$$\ell \cdot g \equiv \ell \bmod p.$$

As a result, the above map of $\mathcal{P}_1$ to $\bar{\mathcal{P}}_1$ can be factored through the quotient space

$$\begin{tikzcd} \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_1/\Gamma \arrow[d, dashed] \\ & \bar{\mathcal{P}}_1 \end{tikzcd}$$

It is not difficult to verify that this is a bijection.

In a similar manner, we define bijections

$$\mathcal{P}_0/\Gamma \to \bar{\mathcal{P}}_0, \quad \mathcal{P}_2/\Gamma \to \bar{\mathcal{P}}_2.$$

Given a maximal isotropic subspace H in $\mathcal{P}_2$, its intersection $H \cap V$ gives a two dimensional isotropic subspace in V. Let

ℓ_1, ℓ_2 be a base for this subspace. Then

$$\lambda(\ell_1, \ell_2) = 0$$

and the exterior product $\ell_1 \wedge \ell_2$ determines up to sign an elemnet in $\Lambda^2 V$. There are mappings

$$\mathcal{P}_0 \to \overline{\mathcal{P}}_0, \quad \mathcal{P}_2 \to \overline{\mathcal{P}}_2$$

defined by assigning to H the corresponding product $\ell_1 \wedge \ell_2$ mod p. It is easy to check that these factor through Γ and give rise to bijections.

Since $\mathcal{P}_0/\Gamma$, $\mathcal{P}_1/\Gamma$, $\mathcal{P}_2/\Gamma$ and $\overline{\mathcal{P}}_0$, $\overline{\mathcal{P}}_1$, $\overline{\mathcal{P}}_2$ are respectively the edges and vertices of the simplical complexes $\mathcal{J}(V)/\Gamma$, $\mathcal{J}_\pm(\overline{V})$, it follows that these complexes are the same. This proves the lemma.

2.3. *Borel-Serre compactification.* As mentioned before, the Borel-Serre compactification of $\mathfrak{G}_2/\Gamma$ is a compact manifold $\mathfrak{G}_2/\Gamma^c$ with boundary, or strictly speaking, a manifold with corners. Let $\partial\mathfrak{G}/\Gamma^c$ denote its boundary. Then its construction can be described by a procedure called "blowing up" the building $\mathcal{J}_\pm(\overline{V})$. The idea is to replace each of the simplices in $\mathcal{J}_\pm(\overline{V})$ by a $K(\pi,1)$-manifold, and then glue them together in a canonical manner according to the incidence relation in $\mathcal{J}_\pm(V)$.

To begin with, associated to each rational parabolic group $P \in \mathcal{P}$ there is a real parabolic group $P(\mathbb{R})$, and a discrete subgroup $\Gamma_P = \Gamma \cap P(\mathbb{R})$ in $P(\mathbb{R})$. We now construct a $K(\pi,1)$-manifold with Γ_P as its fundamental group. Let K be the unitary subgroup

$$K = \left\{ \left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array} \right) \in Sp_4(\mathbb{R}) \;\middle|\; \begin{array}{l} A^tA + B^tB = I \\ \\ A^tB = B^tA \end{array} \right\}$$

in $Sp_4(\mathbb{R})$. Then K is a maximal compact subgroup in $Sp_4(\mathbb{R})$, and there is a canonical isomorphism $\mathfrak{S}_2 \simeq K\backslash Sp_4(\mathbb{R})$ of $\mathfrak{S}_2$ onto the right-coset space $K\backslash Sp_4(\mathbb{R})$. In this way, the Siegel space $\mathfrak{S}_2/\Gamma$ can be identified with the double-coset space $K\backslash Sp_4(\mathbb{R})/\Gamma$. Associated to every rational parabolic subgroup $P \in \mathcal{P}$, we consider the subgroup $K_P = K \cap P(\mathbb{R})$ in the real parabolic group $P(\mathbb{R})$ obtained by taking the intersection of K and $P(\mathbb{R})$. In fact, K_P and Γ_P are subgroups of a smaller group. Let X(P) denote the character group of P, and

$$\mathring{P}(\mathbb{R}) = \cap_{\chi \in X(P)} \mathrm{Ker}\{\chi^2 : P(\mathbb{R}) \to \mathbb{R}^\times\}.$$

Then

$$\Gamma_P = \Gamma \cap P(\mathbb{R}) = \Gamma \cap \mathring{P}(\mathbb{R}),$$
$$K_P = K \cap P(\mathbb{R}) = K \cap \mathring{P}(\mathbb{R}).$$

The subgroup K_P is a maximal compact subgroup in $\mathring{P}(\mathbb{R})$, and so the symmetric space

$$e(P) = K_P\backslash\mathring{P}(\mathbb{R}), \quad P \in \mathcal{P}$$

is contractible. This allows us to form a $K(\pi,1)$-manifold by letting the discrete group Γ_P operate on this space, and taking the double coset space;

$$e'(P) = K_P\backslash\mathring{P}(\mathbb{R})/\Gamma_P = e(P)/\Gamma_P.$$

Note that if two parabolic subgroups P, Q are conjugate to each other $P = gQg^{-1}$, by an element g in $Sp_4(\mathbb{Q})$, then there is a natural induced mapping

$$g: e'(P) \to e'(Q)$$

between the corresponding $K(\pi,1)$-manifolds. Putting these mappings together, we have actions of $Sp_4(\mathbb{Q})$ on the disjoint unions

$$\coprod_{P\in\mathcal{P}_0} e'(P),\quad \coprod_{P\in\mathcal{P}_1} e'(P),\quad \coprod_{P\in\mathcal{P}_2} e'(P).$$

To "blow up" the building $\mathcal{J}_\pm(\bar{V})$, we form the quotient spaces

$$\coprod_{P\in\mathcal{P}_0} e'(P)/\Gamma,\quad \coprod_{P\in\mathcal{P}_1} e'(P)/\Gamma,\quad \coprod_{P\in\mathcal{P}_2} e'(P)/\Gamma.$$

Note that the connected components in the above spaces are in one-to-one correspondence with the set of simplices,

$$\mathcal{P}_0/\Gamma \cong \overline{\mathcal{P}}_0,\quad \mathcal{P}_1/\Gamma \cong \overline{\mathcal{P}}_1,\quad \mathcal{P}_2/\Gamma \cong \overline{\mathcal{P}}_2$$

in the building $\mathcal{J}_\pm(V)$. Thus for each based line $\ell \in \overline{\mathcal{P}}_1$, based plane $h \in \overline{\mathcal{P}}_2$, and based flag $(\ell,h) \in \overline{\mathcal{P}}_0$, we have $K(\pi,1)$-manifolds and we will denote these manifolds by the symbols

$$X(\ell),\ X(h),\ X(\ell,h).$$

Notation (2.3.1). We will also use the symbols L_1, L_2 and L_0 to denote the manifolds $X(e_1)$, $X(e_1\wedge f_1)$, and $X(e_1,e_1\wedge f_1)$, where e_1, $e_1 \wedge f_1$ are given as follows

$$L_1 = X(e_1), \qquad e_1 = \pm(1,0,0,0),$$
$$L_2 = X(e_1\wedge f_1), \qquad e_1\wedge f_1 = \pm(1,0,0,0) \quad (0,1,0,0),$$
$$L_0 = X(e_1,e_1\wedge f_1).$$

It remains to patch these manifolds together. For this, we need the Levi-decomposition of parabolic groups. Recall the choice of maximal compact subgroup K in $Sp_4(\mathbb{R})$. Associated to this, there is the Cartan involution

$$\theta\colon Sp_4(\mathbb{R}) \to Sp_4(\mathbb{R})$$

defined by $\theta(g) = {}^tg^{-1}$.

Let N be the nilradical of $P(\mathbb{R})$. Then the parabolic subgroups P(R) can be written as semi-direct products

$$P(\mathbb{R}) = M \cdot N = \mathring{M} \cdot A \cdot N$$

where M, N, A are invariant under θ,

$$\mathring{M} = \cap_{\chi \in X(M)} \mathrm{Ker}\{\chi^2 : M \to \mathbb{R}\}.$$

$\mathring{M}$ is semi-simple and A is in the centralizer of $\mathring{M}$.

The discrete group Γ_P has a similar decomposition

$$\Gamma_P = \Gamma_M \cdot \Gamma_N$$

where $\Gamma_M = \Gamma \cap M$, $\Gamma_N = \Gamma \cap N$. The maximal compact subgroup K_P in $\mathring{M}$ is obtained by taking the intersection of K with $\mathring{M}$,

$$K_P = K \cap P = K \cap \mathring{M}.$$

Let Z_M denote the symmetric space $K_P \backslash \mathring{M}$. Then this is a contractible space with an action of the discrete group Γ_M. Hence the quotient space

$$0 \to \Gamma_M \to Z_M \to Z_M/\Gamma_M \to 0$$

is a $K(\pi,1)$-manifold. Since

$$K_P \backslash P = K \backslash \mathring{M} \cdot N = (Z_M) \times N,$$

there is a diagram of fibrations:

$$(2.3.2) \qquad \begin{tikzcd} 0 \arrow[r] & N/\Gamma_N \arrow[r] & K_P \backslash \mathring{P}/\Gamma_P \arrow[r] & Z_M/\Gamma_M \arrow[r] & 0 \\ 0 \arrow[r] & \Gamma_N \arrow[r] \arrow[u] & K_P \; \mathring{P} \arrow[r] \arrow[u] & Z_M \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \Gamma_N \arrow[r] \arrow[u] & \Gamma_P \arrow[r] \arrow[u] & \Gamma_M \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The middle space on the top row is by definition the $K(\pi,1)$-manifold $e'(P)$. It follows from the top row of the above diagram that $e'(P)$ is the total space of a fiber bundle with Z_M/Γ_M as its base space, with nilmanifold N/Γ_N as its fiber, and with Γ_M as its structure group.

We now examine these fibrations in the case of the submanifolds L_1, L_2, L_0 in (2.3.1). Let $P_i(\mathbb{R})$, $i = 0,1,2$, be the parabolic subgroups defined in (2.2.1), (2.2.2) and (2.2.3), and let $\overset{\circ}{M}_i$, A_i, N_i be the corresponding subgroups appearing in the Levi decomposition of $P_i(\mathbb{R})$. Then, it is not difficult to see directly that for $i = 1,2$, the subgroup M_i contain $SL_2(\mathbb{R})$ as their identity components. Furthermore, the symmetric space $Z_{M_i} = K_{P_i} \backslash \overset{\circ}{M}_i$, $i = 1,2$, can be identified with the Siegel upper half plane $\mathfrak{S}_1$,

$$Z_{M_i} \simeq SO(2)\backslash SL_2(\mathbb{R}) \cong \mathfrak{S}_1,$$

and the double coset space $K_{P_i} \backslash \overset{\circ}{M}_i/\Gamma_{M_i}$ can be identified with the quotient space

$$\begin{aligned} Z_{M_i}/\Gamma_{M_i} &\cong SO(2)\backslash SL_2(R)/\Gamma(2,P) \\ &\cong \mathfrak{S}_1/\Gamma(2,P) \end{aligned}$$

where the fundamental group $\Gamma(2,P)$ is the full congruence subgroup in $SL_2(\mathbb{Z})$ of level p. In the following, we will refer to this quotient space as the open modular curve. Over this base space, there is a fibration

$$\begin{tikzcd} N_i/\Gamma_{N_i} \arrow[r] \arrow[d] & L_i \arrow[d] \\ * \arrow[r] & Z_{M_i}/\Gamma_{M_i} \end{tikzcd} \qquad i = 1,2.$$

where the fiber N_i/Γ_{N_i} is a 3-dimensional Heisenberg space for $i = 1$, and a 3-dimensional torus for $i = 2$.

The above manifold L_i can be compactified to a manifold $\overline{L}_i$ with boundary by compactifying the base space $\mathfrak{S}_1/\Gamma_1^c$ and then adding the torus, or nilmanifold, to each boundary component.

If we restrict our attention to the minimal parabolic subgroup P_0, and the manifold M_0, then the above fibration (2.3.2) is trivial. However, there are two different fibrations of L_0 over S^1 with nilmanifolds as fibers. These two fibrations f_i: $L_0 \to S^1$, $i = 1,2$ are induced by two natural homomorphisms:

$$f_i: N_0 \to P_{0i} \quad i = 1,2$$

where P_{0i} stands for the parabolic subgroup in M_i, $P_{0i} = P_0 \cap \overset{\circ}{M}_i$. The fiber of these maps are the nilmanifolds N_i/Γ_{N_i}

$$\begin{array}{ccc} N_i/\Gamma_{N_i} & \longrightarrow & L_0 \\ \downarrow & & \downarrow f_i \\ * & \longrightarrow & P_{0,i}/\Gamma_{P_{0,i}} \simeq S^1 \end{array}$$

Comparing this with the boundary components in the previous example, it follows that the base manifold $P_{0,i}/\Gamma_{P_{0,1}}$ can be identified in a natural manner with the boundary component of $Z_{M_i}/\Gamma^c_{M_i}$, and the manifold L_0 can be identified with one of the boundary components of $\bar{L}_i$, $i = 1,2$, ϕ_i: $L_0 \to \partial_0(\bar{L}_i)$. This allows us to patch these manifolds L_0, L_1, L_2, by gluing them together along the boundary component

$$\begin{array}{ccc} \bar{L}_1 & \cup & \bar{L}_2 \\ {\scriptstyle\phi_1}\nwarrow & & \nearrow{\scriptstyle\phi_2} \\ & L_0 & \end{array}$$

The above examples demonstrate the general pattern of gluing the components

$$\coprod_{\ell\in\mathcal{P}_1} X(\ell), \quad \coprod_{h\in\mathcal{P}_2} X(h), \quad \coprod_{(\ell,h)\in\mathcal{P}_0} X(\ell,h),$$

together. The manifolds $X(\ell)$, $X(h)$ are fibrations

$$X(\ell) \to \mathring{B}(\ell),\ X(h) \to \mathring{B}(h),$$

with base space the open modular curves $\mathring{B}(\ell)$, $\mathring{B}(h)$ and with fiber a torus or a nilmanifold. We compactify the base manifolds $\mathring{B}(\ell)$, $\mathring{B}(h)$ by adding a circle to each neighborhood of the cusp and then compactify $X(\ell)$, $X(h)$ by adding the corresponding fibration to these boundary circles. In this way, we obtain manifolds with boundaries $X(\ell)^c$, $X(h)^c$ which are fibrations over 2-dimensional manifolds $B(\ell)^c$, $B(h)^c$,

$$X(\ell)^c \to B(\ell)^c,$$
$$X(h)^c \to B(h)^c.$$

To obtain the Borel-Serre boundary, we identify these boundary components $\overline{\partial X(\ell)}$, $\overline{\partial X(h)}$ with the manifold $X(h,\ell)$ and glue all these manifolds together along their boundary components $\coprod_{(\ell,h)} X(\ell,h)$

$$\begin{tikzcd} \coprod_{\ell} X(\ell) & \cup & \coprod_{h} X(h) \\ & \coprod_{(\ell,h)} X(\ell,h) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We can describe the above gluing process more explicitly by means of an action of $Sp_4(\mathbb{F}_p)$. The fundamental group of L_i is a normal subgroup in $P_i(R) \cap Sp_4(R)$, and we define $\overline{P}_i$ to be the quotient group

$$\overline{P}_i = (P_i \cap Sp_4(\mathbb{Z}))/\Gamma_{P_i}.$$

In terms of matrices, these are matrix groups obtained by changing coefficients in formulas (2.2.1), (2.2.2), (2.2.3) to coefficients in the finite field $\mathbb{F}_p$. It is not difficult to see from our construction of L_i

$$L_i = K_{P_i} \smallsetminus \overset{\circ}{P}_i / \Gamma_{P_i},$$

that the group $\overline{P}_i$ operates in the manifold L_i. On the other hand, there is a natural action of $\overline{Sp}_4 = Sp_4(\overline{V})$ on the oriented building $\mathcal{J}_\pm(\overline{V})$, and the isotropy subgroups of the simplices (e_1), $(e_1 \wedge e_2)$, $(e_1, e_1 \wedge e_2)$ are precisely the groups $\overline{P}_1$, $\overline{P}_2$ and $\overline{P}_0$. This allows us to rewrite the union $\coprod_{\ell \in \mathcal{P}_1} X(\ell)$, $\coprod_{h \in \mathcal{P}_2} X(h)$, $\coprod_{(\ell,h) \in \mathcal{P}_0} X(\ell,h)$ in the following manner:

$$\coprod_{\ell \in \overline{\mathcal{P}}_1} X(\ell) \cong L_1 \underset{\overline{\mathcal{P}}_1}{\times} \overline{Sp}_4, \quad \coprod_{h \in \overline{\mathcal{P}}_2} X(h) \cong L_2 \underset{\overline{\mathcal{P}}_2}{\times} \overline{Sp}_4,$$

$$\coprod_{(\ell,h) \in \overline{\mathcal{P}}_0} X(\ell,h) \cong L_0 \underset{\overline{\mathcal{P}}_2}{\times} \overline{Sp}_4.$$

The action of $\overline{P}_i$, $i = 1,2$, on the manifold L_i can be extended to its compactification $\overline{L}_i$ so as to get compact manifolds with $\overline{Sp}_4$ actions

$$\coprod_{\ell \in \mathcal{P}_1} X(\ell)^c \cong \overline{L}_1 \underset{\overline{\mathcal{P}}_1}{\times} \overline{Sp}_4, \quad \coprod_{h \in \overline{\mathcal{P}}_2} X(h)^c \cong \overline{L}_2 \underset{\overline{\mathcal{P}}_2}{\times} \overline{Sp}_4.$$

In addition, the diffeomorphisms

$$\phi_1 : L_0 \to \partial_0 \overline{L}_1, \ \phi_2 : L_0 \to \partial_0 \overline{L}_2$$

can be chosen so that they are equivariant with respect to the action of $\overline{P}_0$. This allows us to extend ϕ_1, ϕ_2 to embeddings:

$$\Phi_1 : L_0 \underset{\overline{P}_0}{\times} \overline{Sp}_4 \to \partial \overline{L}_1 \underset{\overline{P}_1}{\times} \overline{Sp}_4 \to \overline{L}_1 \underset{\overline{P}_1}{\times} \overline{Sp}_4$$

$$\Phi_2 : L_0 \underset{\overline{P}_0}{\times} \overline{Sp}_4 \to \partial L_2 \underset{\overline{P}_2}{\times} \overline{Sp}_4 \to \overline{L}_2 \underset{\overline{P}_2}{\times} \overline{Sp}_4.$$

Finally, we can glue everything together by means of Φ_1, Φ_2:

$$\begin{tikzcd}[column sep=small]
(\overline{L}_1 \underset{\overline{P}_1}{\times} \overline{Sp}_4) & \coprod & (\overline{L}_2 \underset{\overline{P}_2}{\times} \overline{Sp}_4) \\
& L_0 \underset{\overline{P}_0}{\times} \overline{Sp}_4 \arrow[ul, "\Phi_1"] \arrow[ur, "\Phi_1"'] &
\end{tikzcd}$$

to get a closed five dimensional manifold with an action of $\overline{Sp}_4$.

Proposition (2.3.3). *There is a piecewise-smooth equivariant homeomorphism with respect to the action of* $\overline{Sp}_4$ *on the boundary* $\partial(\mathfrak{S}_2/\Gamma)^c$ *of Borel-Serre compactification to the manifold*

$$\begin{tikzcd}[column sep=small]
\overline{L}_1 \underset{\overline{P}_1}{\times} \overline{Sp}_4 & \cup & \overline{L}_2 \underset{\overline{P}_2}{\times} \overline{Sp}_4 \\
& L_0 \underset{\overline{P}_0}{\times} \overline{Sp}_4 \arrow[ul, "\Phi_1"] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}$$

defined as above.

2.3. *Satake compactification.* The oldest compactification of $\mathfrak{S}_2/\Gamma$ is due to I. Satake. The idea is a natural generalization of the classical SL_2-situation where the procedure is to add rational points p/q to the upper half plane $\mathfrak{S}_1$ and then form the quotient $\overline{\mathfrak{S}_1/\Gamma}_{(2,p)}$. Satake's approach was the same: adding rational boundary components to the Siegel half space $\mathfrak{S}_2$ and then forming the quotient space $\overline{\mathfrak{S}_2/\Gamma}$. With a suitable topology on $\overline{\mathfrak{S}}_2$, it was proven by Baily and Borel that $\overline{\mathfrak{S}_2/\Gamma}$ is a projective algebraic variety. Unlike the classical situation, the Satake compactification $\overline{\mathfrak{S}_2/\Gamma}$ is no longer a nonsingular variety.

Example (2.4.1). For $\Gamma = Sp_4(\mathbb{Z})$, the Satake compactification is the union of $\mathfrak{S}_2/Sp_4(\mathbb{Z})$ together with lower dimensional Siegel spaces

$$\overline{\mathfrak{S}_2/Sp_4(\mathbb{Z})} = \mathfrak{S}_2/Sp_4(\mathbb{Z}) \cup \mathfrak{S}_1/Sp_2(\mathbb{Z}) \cup \mathfrak{S}_0/Sp_0(\mathbb{Z}).$$

Here $\mathfrak{S}_0/Sp_0(\mathbb{Z})$ is understood to be an isolated point, and it is added to $\mathfrak{S}_1/Sp_2(\mathbb{Z})$ to compactify this manifold.

For the principal congruence subgroup Γ of level p the Satake compactification $\overline{\mathfrak{S}_2/\Gamma}$ can be described in terms of the lower dimensional Satake compactification $\overline{\mathfrak{S}_1/\Gamma}(2,p)$, and the building $\mathcal{J}_{\pm}(V)$. In [Z], S. Zucker described a map

$$\mathfrak{S}_2/\Gamma^c \to \overline{\mathfrak{S}_2/\Gamma}$$

of the Borel-Serre compactification onto the Satake compactification which extends the identity map in the interior. This map was exploited by R. Lee and R. Charney in their paper [CL].

Let $\partial\overline{\mathfrak{S}_2/\Gamma}$ be the complement of $\mathfrak{S}_2/\Gamma$ in the Satake compactification

$$\partial\overline{\mathfrak{S}_2/\Gamma} = \overline{\mathfrak{S}_2/\Gamma} - \mathfrak{S}_2/\Gamma.$$

This is called the singular set of $\mathfrak{S}_2/\Gamma$. We now construct this singular set $\partial\overline{\mathfrak{S}_2/\Gamma}$ by means of the building $\mathcal{J}_{\pm}(\overline{V})$. Recall that the manifolds $\overline{X}(\ell)$, $\ell \in \overline{\mathcal{P}}_1$ are singular fibrations over the base manifolds $\overline{B}(\ell)$. These base manifolds are compact 2-dimensional manifolds, and their boundary components are in one-to-one correspondence with the isotropic planes h, $h \supseteq \ell$. In other words, each boundary circle corresponds to an edge $(\ell,h) \in \mathcal{J}_{\pm}(\overline{V})$ attached to the vertex (ℓ). In fact, we can think of $\overset{\circ}{B}(\ell)$ as sitting above the vertex (ℓ), and on the edges coming out from this vertex there are the collar neighborhoods of the boundary component. We can compactify $\overset{\circ}{B}(\ell)$ by adding a point $b(\ell,h)$ to the end of each of these collar neighborhoods

$$B(\ell) = \cup_{h\in\overline{\mathcal{P}}_2} \overset{\circ}{B}(\ell) \cup b(\ell,h)$$

and we can think of these added points as sitting above the second type of vertices (h), $h \in \mathcal{P}_2$. In this way, the manifolds $B(\ell)$ are connected up to each other, and the resulting manifold $\cup_{\ell\in\overline{\mathcal{P}}_1} B(\ell)$ is homeomorphic to the singular set $\partial\overline{\mathfrak{S}_2/\Gamma}$. The manifold $B(\ell)$ is a 1-dimensional projective variety, and is referred to in the literature as the modular curve.

This attaching process can be described more explicitly in terms of the manifold L_1. As mentioned before, there is a fibration

$$\begin{array}{ccc} N_1/\Gamma_{M_1} & \longrightarrow & L_1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z_{M_1}/\Gamma_{M_1} \end{array}$$

with base space the open modular curves $B(\ell_1) = SO(2)\diagdown SL(2)/\Gamma(2,p)$. Instead of adding a circular boundary component, we add one point to Z_{M_1}/Γ_{M_1} for each cusp. The group $\overline{P}_1$ acts transitively on these boundary points of $B(\ell_1)$, with isotropy subgroup isomorphic to $\overline{P}_0$. This gives rise to equivariant embeddings

$$\Phi_1\colon \overline{P}_0\backslash\overline{Sp} \to B(\ell_1) \underset{\overline{P}_1}{\times} \overline{Sp},$$

$$\Phi_2\colon \overline{P}_0\backslash\overline{Sp} \to \overline{P}_2\backslash\overline{Sp}.$$

By means of Φ_i, $i = 1,2$, we glue the spaces $B(\ell_i)$ and $\overline{P}_2\backslash\overline{Sp}$ together to get a space

$$W = B(\ell_1) \underset{\overline{P}_1}{\times} \overline{Sp} \underset{\overline{P}_0\backslash\overline{Sp}}{\cup} \overline{P}_2\backslash\overline{Sp}$$

with an action of $\overline{Sp}$.

Proposition (2.4.2). *Let* $\partial\overline{\mathfrak{S}_2/\Gamma}$ *denote the singular set of the Satake compactification. Then there is an equivariant homeomorphism with respect to the action of* $\overline{Sp}$ *from the singular set* $\overline{\mathfrak{S}_2/\Gamma}$ *on to the space* W *defined as above.*

The proof of the above proposition follows immediately from the same argument as in [CL], and we will not repeat it here.

2.5. *The Theorem of torus embeddings.* Let $\mathfrak{S}_2/\Gamma$ be the Satake compactification described in the previous section. It was proven by Borel and Baily that the space $\overline{\mathfrak{S}_2/\Gamma}$ is a projective variety. As pointed out in the introudction, the singularities of $\overline{\mathfrak{S}_2/\Gamma}$ are extremely complicated and they present a major obstacle to studying it by algebraic-geometric methods.

Igusa was the first to find the cure for this problem by constructing a desingularization of this variety. We will refer to the resulting nonsingular projective variety $\mathfrak{S}_2/\Gamma^*$ as the Igusa compactification. The object of the next two sections is to explain the Igusa compactification in the modern language of "toroidal compactification" as developed by D. Mumford (see [AMRT]).

First, by an algebraic torus over $\mathbb{C}$, $T = T(\mathbb{C})$, we mean an algebraic group isomorphic to a finite cartesian product $\mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ where $\mathbb{C}^* = GL_1(\mathbb{C})$ denotes the multiplicative group of non-zero elements of $\mathbb{C}$. Associated to this is the ring of algebraic functions $\Gamma(\mathcal{O}_T)$ which is isomorphic to a polynomial ring, $\Gamma(\mathcal{O}_T) \cong \mathbb{C}[T_1,T_1^{-1},\cdots,T_n,T_n^{-1}]$, where the T_i are indeterminates. This can be explained in terms of the group of characters of T, $M = \mathrm{Hom}_{alg\cdot gr}(T,\mathbb{C}^*)$. For every element $\underline{r} \in M$, we have the corresponding character function $\chi^{\underline{r}}$ in $\Gamma(\mathcal{O}_T)$, and they form a base of $\Gamma(\mathcal{O}_T)$ as a complex vector space,

$$\Gamma(\mathcal{O}_T) \cong \mathbb{C}[\cdots,\chi^{\underline{r}},\cdots] \cong \mathbb{C}[M].$$

There is a dual object associated to M, namely the fundamental group $N = \mathrm{Hom}_{alg\cdot gp.}(\mathbb{C}^*,T)$. Every element in N can be expressed in the form

$$\lambda_\alpha(t) = (t^{a_1},t^{a_2},\cdots,t^{a_n}), \quad a_1 \in \mathbb{Z}.$$

There is a natural nonsingular pairing

$$M \times N \to \mathrm{Hom}(\mathbb{C}^*,\mathbb{C}^*) \cong \mathbb{Z}, \ (r,\alpha) \mapsto \langle r,\alpha\rangle$$

defined by the formula $\chi^{\underline{r}}(\lambda_\alpha(t)) = t^{\langle r,\alpha\rangle}$.

By a *torus embedding*, we mean an algebraic variety V which contains T as an open set, $T \subset V$, and which has a torus action T, $T \times V \to V$, extending the natural torus action of T on itself. A morphism of two torus embeddings $T \subset V_1$, $T \subset V_2$ is a map $f\colon V_1 \to V_2$ such that its restriction to T is an epimorphism $g\colon T \to T$ and the diagram

$$\begin{tikzcd} T \times V_1 \arrow[r] \arrow[d, "(g,f)"] & V_1 \arrow[d, "f"] \\ T \times V_2 \arrow[r] & V_2 \end{tikzcd}$$

is commutative.

The simplest form of a torus embedding occurs when $V = A$ is isomorphic to an affine space $A \simeq \mathbb{C}^n$, and this is called an *affine torus embedding*.

Example (2.5.1). Every algebraic torus $(\mathbb{C}^*)^n$ has an affine torus embedding. For this, we only have to consider the affine space $\mathbb{A}^n(\mathbb{C}) \cong \mathbb{C}^n$, the inclusion $(\mathbb{C}^*)^n \subseteq \mathbb{C}^n$, and the torus action $(\mathbb{C}^*)^n \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $(a_1,\cdots,a_n)\cdot(z_1,\cdots,z_n) = (a_1z_1,\cdots,a_nz_n)$.

Note that in the above example the ring of algebraic functions $\Gamma(\mathcal{O}_{A^n})$ is the subring of polynomials $\mathbb{C}[T_1,T_2,\cdots,T_n]$ embedded in $\Gamma(\mathcal{O}_T) \cong \mathbb{C}[T_1,T_1^{-1},\cdots,T_n,T_n^{-1}]$. In terms of the character group M of T, this amounts to taking the semi-group M_- generated by $T_1,T_2,\cdots,T_n$, $M_- \subset M$, and forming the associated group ring $\mathbb{C}[M_-]$ of this semi-group M_-. The general theory of affine torus embedding can be studied in the same way by considering the character group M and semi-groups lying inside M. However, for our purpose, it is convenient to describe the theory in terms of the dual objects: $N_{\mathbb{R}}$, the Lie algebra of the torus, $N_{\mathbb{R}} = \mathrm{Lie}(T)$, and *convex rational polyhedron cones* (c.r.p.) in $N_{\mathbb{R}}$. By the last term, we mean a convex set σ

$$\sigma = \{x \in N_{\mathbb{R}} \mid \ell_i(X) \geq 0,\ i = 1,\cdots,m\}$$

defined by rational linear functionals ℓ_i.

Theorem (2.5.2). *There is a one-to-one correspondence*

$$(\sigma,N_{\mathbb{R}}) \leftrightarrow (T \subset \mathrm{Temb}(\sigma))$$

between the set of c.r.p. *cones* σ *in* $N_{\mathbb{R}}$ *which do not contain any linear subspace and the set of normal affine torus* *embeddings of* T, $T \subset \mathrm{Temb}(\sigma)$. *In addition, morphisms of such torus embeddings correspond bijectively to linear maps* $(\sigma, N_{\mathbb{R}}) \to (\sigma', N_{\mathbb{R}})$ *of* $N_{\mathbb{R}}$ *which has finite cokernels and sends* σ *into* σ'.

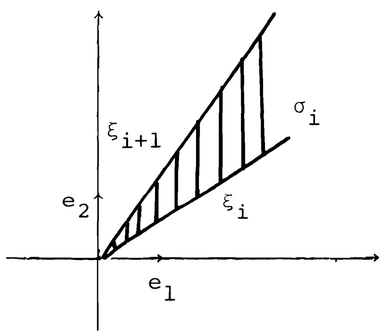
For the proof of this theorem, we refer the readers to [AMRT], [O].

Example (2.5.3). If we consider the affine torus embedding in Example (2.5.1), $(\mathbb{C}^*)^n \subseteq (\mathbb{C})^n$, then the polyhedral cone in $N_{\mathbb{R}}$ is the positive cone defined by the coordinate planes,

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}.$$

Example (2.5.4). Let $T = (\mathbb{C}^*)^2$ be the 2-dimensional torus, and let σ_i be the convex cone in $N_{\mathbb{R}} = \mathbb{R}^2$ generated by the elements $\xi_i = (1,i)$, $\xi_{i+1} = (1,i+1)$,

$$\sigma_i = \{x\xi_i + y\xi_{i+1} \mid x \geqq 0,\ y \geqq 0\}$$



Since the matrix $\begin{pmatrix} 1 & i \\ 1 & i+1 \end{pmatrix}$ is unimodular, it follows that there is an isomorphism of N which takes ξ_i into (1,0) and ξ_{i+1} to (0,1), and so σ_i to the standard positive cone $\mathbb{R}^2_+$ in $\mathbb{R}^2$, f: $(\sigma_i, N_{\mathbb{R}}) \to (\mathbb{R}^2_+, N_{\mathbb{R}})$. This isomorphism induces an isomorphism of the torus $f_*: T \to T$. The affine torus embedding $\mathrm{Temb}(\sigma_i)$ associated to σ_i is obtained by taking the composition $T \xrightarrow{f_*} T \subset \mathbb{C}^2$. Here the affine torus embedding, $T \subset \mathbb{C}^2$, in the right hand side is defined as in Example (2.5.1).

Let σ be a convex rational polyhedral cone (c.r.p. cone) in $N_{\mathbb{R}}$, and let $\mathrm{Temb}(\sigma)$ be the affine torus embedding associated to σ. Then the torus action on $\mathrm{Temb}(\sigma)$ can be analyzed by the following theorem.

Theorem (2.5.5). *There is a one-to-one correspondence*

$$\tau \leftrightarrow \mathrm{orb}(\tau)$$

between the set of simplices in σ *and the set of* T*-orbits* $\mathrm{orb}(\tau)$ *in* $\mathrm{Temb}(\sigma)$ *such that*

(i) $\mathrm{Orb}(0) = T$,

(ii) $\dim \tau + \dim \mathrm{orb}(\tau) = \dim T$,

(iii) $\tau_1 \subset \tau_2$ *if and only if the closure of* $\mathrm{Orb}(\tau_1)$ *contains* $\mathrm{Orb}(\tau_2)$, $\overline{\mathrm{Orb}(\tau_1)} \supset \mathrm{Orb}(\tau_2)$.

Example (2.5.6). Let σ_i be the convex rational polyhedral cones defined as in Example (2.5.4). Then there are two faces, τ_i, τ_{i+1}, corresponding to the lines generated by ξ_i and ξ_{i+1}. Accordingly, there are two codimension-one T-orbits defined by the coordinate axes in $\mathrm{Temb}(\sigma_i) \simeq \mathbb{C}^2$.

Definition (2.5.7). A *rational partial polyhedron* (r.p.p.) *decomposition* of $N_{\mathbb{R}}$ is a collection $\Delta = \{\sigma\}$ of convex rational polyhedral cones σ in $N_{\mathbb{R}}$ such that

(i) if $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$,

(ii) if $\sigma, \sigma' \in \Delta$, then their intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

Suppose we are given an r.p.p. decomposition $(\Delta, N_{\mathbb{R}})$. By patching affine torus embeddings $\mathrm{Temb}(\sigma)$, $\sigma \in \Delta$, together, we obtain an algebraic variety $\mathrm{Temb}(\Delta)$ containing T as a Zariski open set. Furthermore the patching process is so canonical that any morphism $f\colon (\Delta, N) \to (\Delta', N')$ gives rise to a morphism $f_*\colon \mathrm{Temb}(\Delta) \to \mathrm{Temb}(\Delta')$ of the corresponding torus embeddings. Hence the general theory of torus embeddings can be summed up as follows.

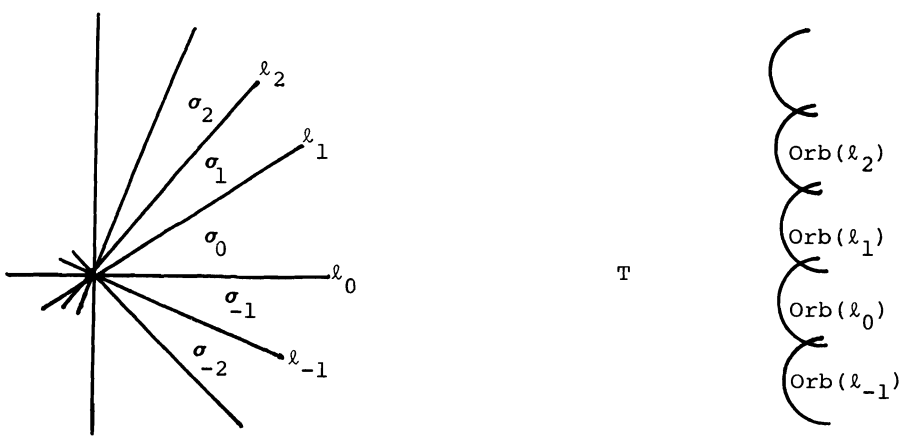
Theorem (2.5.8). *There is an equivalence*

$$(\Delta, N) \leftrightarrow T \subset \mathrm{Temb}(\Delta)$$

between the category of r.p.p. *decompositions* (Δ, N) *and the category of torus embeddings.*

Here are two examples of r.p.p. decompositions.

Example (2.5.9). Let σ_i be the c.r.p. cones defined in Example (2.5.4) and let ℓ_i be the positive ray generated by $\xi_i = (1, i)$. Then the collection $\Delta = \{\sigma_i, \ell_i, (0,0)\}$ forms a r.p.p. decomposition of $\mathbb{R}^2$. The torus embedding $\mathrm{Temb}(\Delta)$ can be schematically represented as follows:



Recall that the affine torus embedding $\mathrm{Temb}(\sigma_i)$ is obtained by inserting two affine lines $\overline{\mathrm{Orb}(\ell_i)}$, $\overline{\mathrm{Orb}(\ell_{i-1})}$ in T (see (2.5.6)). These two affine lines $\overline{\mathrm{Orb}(\ell_i)}$ in $\mathrm{Temb}(\sigma_i)$ intersect each other transversally at a point $\mathrm{Orb}(\sigma_i)$. To obtain the torus embedding $\mathrm{Temb}(\Delta)$, we form the union $\cup\, \mathrm{Temb}(\sigma_i)$, and glue the tori $T \subset \mathrm{Temb}(\sigma_i)$ and the affine lines $\mathrm{Orb}(\tau_i)$ in the consecutive affine spaces $\mathrm{Temb}(\sigma_i)$, $\mathrm{Temb}(\sigma_{i-1})$ together. Hence we have an infinite family of projective lines, $\overline{\mathrm{Orb}(\ell_i)} \simeq \mathbb{P}^1(\mathbb{C})$, which has empty intersection $\overline{\mathrm{Orb}(\ell_i)} \cap \overline{\mathrm{Orb}(\ell_j)} = \phi$ for $|i-j| > 1$, and has one intersection point for $|i-j| = 1$, $\overline{\mathrm{Orb}(\ell_i)} \cap \overline{\mathrm{Orb}(\ell_{i+1})} = \mathrm{Orb}(\sigma_{i+1})$.

Let us describe $\mathrm{Temb}(\Delta)$ more explicitly. The dual cone to σ_i is the convex cone generated by $(i+1,-1)$ and $(-i,1)$, yielding a torus embedding $\mathrm{Temb}(\sigma_i)$, whose image we denote by $\mathbb{C}_i^2$, given by the map $f_i\colon (\mathbb{C}^*)^2 \to \mathbb{C}^2$ by $f_i(z,w) = (z^{i+1}w^{-1}, z^{-i}w)$. Then $\mathrm{Temb}(\Delta) = \coprod_i \mathbb{C}_i^2/\sim$ where $\sim$ is the following identification: First we observe that $\coprod_i (\mathbb{C}_i^*)^2$ are all identified to $(\mathbb{C}^*)^2$, (and we see T is open and dense in $\mathrm{Temb}(\Delta)$) so that $(z_i,w_i) \in (\mathbb{C}_i^*)^2 \sim (z_j,w_j) \in (\mathbb{C}_j^*)^2$ if for some $(z,w) \in (C^*)^2$, $(z_i,w_i) = f_i(z,w)$ and $(z_j,w_j) = f_j(z,w)$. In particular, $(z_{i+1},w_{i+1}) \sim (z_i,w_i)$ if $z_{i+1} = z_i^2 w_i$, $w_{i+1} = z_i^{-1}$.

Second we note that $\mathrm{Orb}(\ell_i) \subset \mathrm{Temb}(\sigma_i)$ is identified with $\mathrm{Orb}(\ell_i) \subset \mathrm{Temb}(\sigma_{i+1})$. This is the identification of $(z_i,0)$ with $(0,w_{i+1})$ which we see is $\mathbb{C}^* \cup_i \mathbb{C}^*$ where $i(u) = u^{-1}$. Thus $\overline{\mathrm{Orb}(\ell_i)} = \mathbb{C} \cup_i \mathbb{C} = \mathbb{P}^1(\mathbb{C})$, the projective line, and $\mathrm{Orb}(\sigma_i) = \overline{\mathrm{Orb}(\ell_i)} \cap \overline{\mathrm{Orb}(\ell_{i+1})}$ is the point ∞ in $\mathrm{Orb}(\ell_i)$, which is identified with the point 0 in $\overline{\mathrm{Orb}(\ell_{i+1})}$.

Example (2.5.10). Let $P_2(\mathbb{R})$, $N_2(\mathbb{R})$ denote the maximal parabolic subgroup and its unipotent radical defined in (2.3.3). We denote by $N_2(\mathbb{Z})$ the integral lattice in $N_2(\mathbb{R})$,

$$N_2(\mathbb{Z}) = \left\{ \left[\begin{array}{c|c} I & B \\ \hline 0 & I \end{array} \right] \;\middle|\; B \text{ is a 2-by-2 symmetric integer matrix} \right\}.$$

Clearly, $N_2(\mathbb{Z})$ is a free abelian group of rank 3, and associated to this abelian group there is a 3-dimensional algebraic torus T_{P_2}. The advantage of considering the algebraic torus in this manner is that there is an action of $GL_2(\mathbb{Z}) \simeq M_2(\mathbb{Z}) \cdot A_2(\mathbb{Z})$ on the subgroup $N_2(\mathbb{Z})$, and so an induced action of $GL_2(\mathbb{Z})$ on the algebraic torus T_{P_2}. We now describe a torus embedding of T_{P_2} which is equivariant with respect to this action of $GL_2(\mathbb{Z})$.

We note that the vector space $N_{P_2}(\mathbb{R})$ can be identified with the space of 2-by-2, real symmetric matrices. Inside this space, there is the open convex cone Ω of positive definite symmetric matrices. On the boundary of this cone Ω, there are three semi-definite symmetric integral matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and they span a r.p. cone

$$\tau_3 = \left\{ \begin{pmatrix} \lambda_1+\lambda_3 & -\lambda_3 \\ -\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} \;\middle|\; \lambda_1, \lambda_2, \lambda_3 \geqq 0 \right\}.$$

If we consider the translates $\tau_3 \cdot g$ of τ_3 by elements g in the group $GL_2(\mathbb{Z})$, then $\tau_3 \cdot g$ and τ_3 either do not intersect or their intersection $\tau_3 \cdot g \cap \tau_3$ is a lower dimensional face. Hence there is a r.p.p. decomposition Δ of $N_2(\mathbb{R})$ defined by $\Delta = \{\tau_i \cdot g \mid 0 \leq i \leq 3,\ g \in GL_2(\mathbb{Z})\}$, where

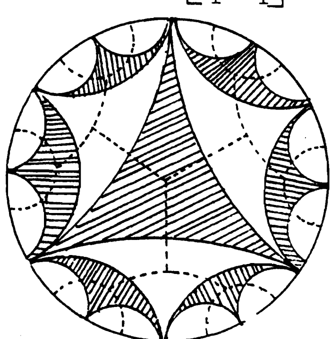
$$\tau_0 = \{0\},\quad \tau_1 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda \geqq 0\right\},$$

$$\tau_2 = \left\{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \begin{matrix} \lambda_1 \geqq 0, \\ \lambda_2 \geqq 0 \end{matrix}\right\}, \text{ and}$$

$$\tau_3 = \left\{\begin{pmatrix} \lambda_1+\lambda_3 & -\lambda_3 \\ -\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} \middle| \lambda_1, \lambda_2, \lambda_3 \geqq 0\right\}.$$

This decomposition is well-known from the reduction theory of quadratic forms. Under the equivalence relation of homothety, the space of positive semidefinite symmetric matrices has another model: namely the Poincare disk $\mathfrak{S}_1^* \cong \{z \mid |z| \leq 1\}$. The cone τ_3 corresponds to an equilateral geodesic triangle in the disk, and the group of tesselations is the same as $GL_2(\mathbb{Z})$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Poincaré disk with a triangulation by equilateral triangles and the dual triangulation represented by dotted lines

For our application in the next section, we have to consider the subspace

$$\mathcal{O} = \cup_{\tau\in\Delta\cap\Omega}\mathrm{Orb}(\tau)$$

in Temb(Δ) where τ runs through all the c.r.p. cones lying in the interior of Ω. In other words, we delete all the orbits Orb(τ) in Temb(Δ) associated to $\tau_0\cdot g$, $\tau_1\cdot g$. Since the remaining cones are of the form $\tau_2\cdot g$, $\tau_3\cdot g$, the corresponding orbits Orb($\tau_2\cdot g$), Orb($\tau_3\cdot g$) are of dimension one and dimension zero by the Theorem (2.5.8 ii). As in Example (2.5.9) it is not difficult to verify that the closure $\overline{\mathrm{Orb}(\tau_2\cdot g)}$ is isomorphic to a projective line, $\overline{\mathrm{Orb}(\tau_2\cdot g)} \cong P^1(\mathbb{C})$. Hence the above space $\mathcal{O}$ consists of an infinite number of projective lines $P^1(\mathbb{C})$, and two of them either do not intersect or intersect transversely at one point. To give a precise description, we need the "dual" triangulation obtained by taking the center of each of the above triangles as a vertex, and the edge connecting the centers of two adjacent triangles as a vertex, and the edge connecting the centers of two adjacent triangles as a 1-simplex (see figure). Every edge of this triangle corresponds to a projective line $P^1(\mathbb{C})$ in $\mathcal{O}$, and two projective spaces intersect if they share a common vertex.

Let $\Gamma(2,p)$ be the full congruence subgroup of level p in GL($2,\mathbb{Z}$). Since the r.p.p. decomposition Δ is equivariant with respect to the action of $GL_2(\mathbb{Z})$, there is an induced action of $\Gamma(2,p)$ on the torus embedding Temb(Δ) and its subspace $\mathcal{O}$. The quotient space $\mathfrak{S}_1/\Gamma(2,p)^*$ of the Poincaré space $\mathfrak{S}_1$ under the action of $\Gamma(2,p)$ is known as the modular curve of level p, and the triangulation Δ gives us a triangulation of $\mathfrak{S}_1/\Gamma(2,p)^*$. For $p \geqq 3$, the group $\Gamma(2,p)$ is torsion free, and its action on the triangulation does not fix any c.r.p. cones in the interior of Ω. Hence the action of the congruence subgroup $\Gamma(2,p)$ on $\mathcal{O}$ does not fix any of the curves $\overline{\mathrm{Orb}(\tau_2 \cdot g)} \simeq \mathbb{P}^1(\mathbb{C})$ and so the quotient space $\mathcal{O}/\Gamma(2,p)$ is a union of projective lines $P^1(\mathbb{C})$. Since $\mathfrak{S}/\Gamma(2,p)^*$ is compact, there are a finite number of cells in $\mathfrak{S}_1/\Gamma(2,p)^*$ and so there are a finite number of projective spaces in the corresponding space $\mathcal{O}/\Gamma(2,p)$. In particular, $\mathcal{O}/\Gamma(2,p)$ is compact.

It is worthwhile to point out that the dual triangulation gives rise to a triangulation of $\mathfrak{S}_1/\Gamma(2,p)^*$ by regular polygons with p sides. For $p = 3,4,5$ the modular curve $\mathfrak{S}_1/\Gamma(2,p)^*$ is the Riemann sphere and the corresponding triangulations are the tetrahedron, the cube and the icosahedron. All these are, of course, known since antiquity. For higher values of p, we obtain a tesselation of the modular curve of genus $1 + (p-6)(p^2-1)/24$ by $(p^2-1)/2$ regular p-gons.

2.6. *The Igusa Compactification.* We are now in a position to give a precise description of the Igusa compactification $\mathfrak{S}_2/\Gamma^*$.

Let $\pi\colon \mathfrak{S}_2/\Gamma^* \to \overline{\mathfrak{S}_2/\Gamma}$ be the projection of the Igusa space onto the Satake space. Since this is a desingularization, π is an isomorphism in the interior $\mathfrak{S}_2/\Gamma$. Recall the description in (2.4) that the singular set of the Satake space, $\partial\overline{\mathfrak{S}}_2/\Gamma = \overline{\mathfrak{S}}_2/\Gamma - \mathfrak{S}_2/\Gamma$, consists of modular curves:

$$\overline{\partial \mathfrak{S}_2/\Gamma} = \bigcup_{\ell \in \overline{\mathcal{P}}_1} B(\ell), \quad B(\ell) = \overset{\circ}{B}(\ell) \cup \partial B(\ell),$$

$$\partial B(\ell) = \bigcup_{\substack{\ell \subset h \\ h \in \mathcal{P}_2}} b(\ell,h),$$

Let $\pi^{-1}(\overset{\circ}{B}(\ell))$, $\pi^{-1}(b(\ell,h))$, $\pi^{-1}(B(\ell))$ denote respectively the inverse image in $\mathfrak{S}_2/\Gamma^*$ of the open modular curve $\overset{\circ}{B}(\ell)$, the boundary cusp $b(\ell,h)$, and the modular curve $B(\ell)$. The finite symplectic group $Sp_4(\mathbb{F}_p)$ operates on both the spaces $\mathfrak{S}_2/\Gamma^*$, $\overline{\mathfrak{S}_2/\Gamma}$, and the map π is equivariant with respect to these actions. It is clear that the action of $Sp_4(\mathbb{F}_p)$ on the set $\mathcal{P}_1$ is transitive, and so the induced action on the irreducible components $\{B(\ell)\}_{\ell \in \overline{\mathcal{P}}_1}$ is also transitive. It follows immediately that the same is true for the action on the set of inverse images $\pi^{-1}(\overset{\circ}{B}(\ell))$, $\pi^{-1}(b(\ell,h))$, $\pi^{-1}(B(\ell))$. Thus the subspaces $\pi^{-1}(\overset{\circ}{B}(\ell')) \cong \pi^{-1}(\overset{\circ}{B}(\ell'))$, $\pi^{-1}(b(\ell,h)) \cong \pi^{-1}(b(\ell',h'))$, $\pi^{-1}(B(\ell)) \cong \pi^{-1}(B(\ell'))$ are isomorphic and they do not depend on the choice of ℓ or h. To describe $\mathfrak{S}_2/\Gamma^*$, we will concentrate on the following spaces:

(2.6.1) $\pi^{-1}(\overset{\circ}{B}(\ell_1))$,

(2.6.2) $\pi^{-1}(b(\ell_1,h_1))$,

(2.6.3) $\pi^{-1}(B(\ell_1))$,

where $\ell_1 = \pm(1,0,0,0)$, $h_1 = \pm(1,0,0,0) \wedge (0,1,0,0)$, and we will describe how they are glued together.

In (2.2.4), we associated to the line ℓ_1 and the isotropic plane h_1, parabolic subgroups $P_1 = P(\ell_1)$, $P_2 = P(h_1)$. Let N_i be the unipotent radical in P_i, let Z_i be the center of N_i, let $Z_i(\mathbb{C})$ be the complexification of Z_i, and let Γ_{Z_i} be the intersection of Γ with Z_i, $\Gamma_{Z_i} = Z_i \cap \Gamma$. Then

Γ_{Z_i} is a free abelian discrete group, and the quotient $Z_i(\mathbb{C})/\Gamma_{Z_i}$ is an algebraic torus $T_{P_i} \cong Z_i(\mathbb{C})/\Gamma_{Z_i}$ with Γ_{Z_i} as its fundamental group. According to the results of Borel and Harish-Chandra, there is an embedding of $\mathfrak{S}_2$ as an open subspace in its compact dual symmetric space $\check{\mathfrak{S}}_2 = Sp_4(\mathbb{C})/P_0(\mathbb{C})$, $\mathfrak{S}_2 \subset \check{\mathfrak{S}}_2$. Over the last space the group $Z_i(\mathbb{C})$ operates, and so by translation there is an open subspace $\mathfrak{S}_2 \cdot Z_i(\mathbb{C})$ containing $\mathfrak{S}_2$ and invariant under the group action of $Z_i(\mathbb{C})$, $\mathfrak{S}_2 \subset \mathfrak{S}_2 \cdot Z_i(\mathbb{C}) \subset \check{\mathfrak{S}}_2$. From the theory of Siegel domains (see [Y]), there is a decomposition of $\mathfrak{S}_2 \cdot Z_i(\mathbb{C})$ into a product

(2.6.4) $\quad \mathfrak{S}_2 \cdot Z_i(\mathbb{C}) \cong \mathfrak{S}_{2-i} \times (Z_i \backslash N_i) \times Z_i(\mathbb{C})$

where the first factor $\mathfrak{S}_{2-i}$ is the upper half space of degree 2-i, and the second factor $Z_i \backslash N_i$ is the quotient of N_i modulo its center. Factoring down by the action of Γ_{Z_i}, we obtain an embedding of the quotient manifold $\mathfrak{S}_2/\Gamma_{Z_i}$ as an open subspace in $\mathfrak{S}_2 \cdot Z_i(\mathbb{C})/\Gamma_{Z_i} \cong \mathfrak{S}_{2-i} \times (Z_i \backslash N_i) \times T_{P_i}$,

(2.6.5) $\quad \mathfrak{S}_2/\Gamma_{Z_i} \to \mathfrak{S}_{2-i} \times (Z_i \backslash N_i) \times T_{P_i}$.

Before proceeding, let us consider some examples.

Example (2.6.6). In the case i = 2, the unipotent radical N_2 is an abelian and so it coincides with its center Z_2,

$$N_2 = Z_2 = \left\{ \left(\begin{array}{c|c} I & B \\ \hline 0 & I \end{array} \right) \middle| \; B = B^t \right\}.$$

Both the spaces $_0$ and $Z_2 \backslash N_2$ consist of a single point. The group Γ_{Z_2} is the group of integral matrices:

$$\left[\begin{array}{c|c} I & B \\ \hline 0 & I \end{array}\right], \quad B = {}^tB$$

with B congruent to zero modulo p. Since Γ_{Z_2} is naturally isomorphic to $N_2(Z)$ in (2.5.10), the algebraic torus T_{P_2} can be identified with the space of complex, symmetric 2-by-2 matrices

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}, \quad z_{ij} \neq 0$$

described there. As for the embedding

$$\mathfrak{S}_2/\Gamma_{Z_2} \to \mathfrak{S}_0 \times Z_2\backslash N_2 \times T_{P_2} \cong T_{P_2},$$

it is defined by sending an element $\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$ in $\mathfrak{S}_2/\Gamma_{Z_2}$ to the symmetric matrix

$$\begin{pmatrix} e(\tau_{11}/p) & e(\tau_{12}/p) \\ e(\tau_{12}/p) & e(\tau_{22}/p) \end{pmatrix}, \quad e(\cdot) = \exp(2\pi\sqrt{-1}\ \cdot)$$

in T_{P_2}. The image of this embedding consists of matrices (z_{ij}) in T_{P_2} such that $(-\log|z_{ij}|)$ is positive definite.

Example (2.6.7). In the case i = 1, the unipotent radical N_1 is the Heisenberg group

$$N_1 = \left\{ \left[\begin{array}{cc|cc} 1 & & & \\ a_{12} & 1 & & \\ \hline & & 1 & -a_{12} \\ & & & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline a_{31} & a_{32} & 1 & \\ a_{32} & 0 & & 1 \end{array}\right] \middle| \; a_{21}, a_{31}, a_{32} \in \mathbb{R} \right\}.$$

The center Z_1 is the one-parameter subgroup

$$Z_1 = \left\{ \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline a_{31} & 0 & 1 & \\ 0 & 0 & & 1 \end{array} \right] \middle| a_{31} \in \mathbb{R} \right\} .$$

The discrete group Γ_{Z_1} is infinite cyclic, $a_{31} \in \mathbb{Z}$ with $a_{31} \equiv 0 \bmod p$, and the algebraic torus $T_{P_1} = Z_1(\mathbb{C})/\Gamma_{Z_1}$ is the one dimensional algebraic torus, $T_{P_1} \cong \mathbb{C}^*$. As for the embedding,

$$\mathfrak{S}_2/\Gamma_{Z_2} \to \mathfrak{S}_1 \times Z_1/N_1 \times T_{P_1}$$

it is given by the formula:

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \longmapsto (\tau_{11}, \tau_{12}, e(\tau_{22}/p))$$

where τ_{11} with $\mathrm{Im}(\tau_{11}) > 0$ lies in the upper half space $\mathfrak{S}_1$, τ_{12} lies in a one-dimensional complex vector space identified with $Z_1 \setminus N_1$, and $e(\tau_{22}/p) \neq 0$ lies in the algebraic torus $T_{P_1} \cong \mathbb{C}^*$.

We now return to the general theory. The Lie algebra of the algebraic torus T_{P_1} is naturally isomorphic to the space Z_i, $\Gamma_{Z_i} \otimes \mathbb{R} \cong Z_i$. There is a composite mapping

$$\mathfrak{S}_2 \to \mathfrak{S}_{2-i} \times Z \setminus N \times Z_i(\mathbb{C}) \xrightarrow{pr_3} Z_i(\mathbb{C}) \xrightarrow{\mathrm{Im}} Z_i$$

of the upper half space into this vector space Z_i. It can be shown that the image of this mapping is an open convex cone Ω_i in Z_i. The key step in the theory of toroidal compactification is to choose a r.p.p. decomposition Δ_i of Z_i with the following properties:

(2.6.8) the union $\bigcup_{\sigma\in\Delta_i} \sigma$ is the rational closure of Ω_i,

(2.6.9) the decomposition Δ_i is invariant under the induced action of Γ_{P_i} on Z_i,

(2.6.10) the number of equivalent cones modulo Γ_{P_i} is finite,

(2.6.11) if Z_i is a subgroup in Z_j, then Δ_i is the same as the set of cones $\bar{\sigma} \in \Delta_i$, $\sigma \in \Delta_j$.

In the situation of (2.6.11), $Z_i \subset Z_j$, it can be shown that the cone Ω_i is the intersection of Z_i with the rational closure $\bar{\Omega}_j$ of Ω_j.

Once such a system of r.p.p. decompositions Δ_i is chosen, our theory in (2.5) gives us a torus embedding $\mathrm{Temb}(\Delta_i)$ of T_{P_i}. From this, we have the space

$$\mathcal{O}_i = \bigcup_{\tau\in\Delta_i\cap\Omega_i} \mathfrak{S}_{2-i} \times Z_i\backslash N_i \times \mathrm{Orb}(\tau)$$

defined as a subspace in $\mathfrak{S}_{2-i} \times Z_i\backslash N_i \times \mathrm{Temb}(\Delta_i)$. Because of condition (2.6.10), there is an action of Γ_{P_i} on this space. The quotient spaces $\mathcal{O}_1/\Gamma_{P_1}$, $\mathcal{O}_2/\Gamma_{P_2}$ are respectively the spaces $\pi^{-1}(\mathring{B}(\ell_1))$, $\pi^{-1}(b(\ell_1,h_1))$ required in our compactification. The projection $\mathcal{O}_i/\Gamma_{P_i}$ to the Satake space is induced by the projection of $\mathcal{O}_i$ onto the first factor $\mathfrak{S}_{2-i}$.

Example (2.6.12). As mentioned before, in the case $i = 2$, the algebraic torus T_{P_2} can be identified with the algebriac torus described in (2.5.10). From the definition, it is easy to check that the convex cone Ω_2 coincides with the cone of positive definite symmetric matrices discussed there. The r.p.p. decomposition $\Delta_2 = \{\tau_i \cdot g \mid 0 \le i \le 3, g \in GL_2(\mathbb{Z})\}$ satisfies all the above conditions (2.6.9)-(2.6.11), and so it can be used to construct our torus embedding $\mathrm{Temb}(\Delta_2)$. Since the factors $\mathfrak{G}_0$, $Z_2 \setminus N_2$ are trivial, we have

$$\begin{aligned} \mathcal{O}_2 &= \bigcup_{\tau \in \Delta_2 \cup \Omega_2} \mathfrak{G}_0 \times Z_2 \setminus N_2 \times \mathrm{Orb}(\tau) \\ &= \bigcup_{\tau \in \Delta_2 \cup \Omega_2} \mathrm{Orb}(\tau) \\ &= \mathcal{O}. \end{aligned}$$

The subgroup Γ_{N_2} operates trivially on this space, and so the quotient under the action of $\Gamma_{P_2} = \Gamma(2,p) \cdot \Gamma_{N_2}$ is the same as the quotient of $\mathcal{O}$ under the action of $\Gamma(2,p)$, $\mathcal{O}_2/\Gamma_{P_2} \cong \mathcal{O}/\Gamma(2,p)$. The structure of this space was studied thoroughly in (2.5.10), and this is our space $\pi^{-1}(b(\ell_1,h_1))$ in $\mathfrak{G}_2/\Gamma^*$.

Example (2.6.13). The situation for i = 1 is simpler. This is because in this case we have a one-dimensional algebraic torus $T_{P_1} \cong \mathbb{C}^*$. There is a single cone τ_1, $\tau_1 \neq 0$, in Δ_1, and the corresponding torus embedding $\mathrm{Temb}(\Delta_1) \cong \mathbb{C}$. Clearly this satisfies all the conditions (2.6.8)-(2.6.11).

Since $\mathrm{Orb}(\tau_1)$ consists of a single point, we have an isomorphism

$$\begin{aligned} \mathcal{O}_1 &\cong \mathfrak{G}_1 \times Z_1 \setminus N_1 \times \mathrm{Orb}(\tau_1) \\ &\cong \mathfrak{G}_1 \times Z_1 \setminus N_1 \\ &\cong \mathfrak{G}_1 \times \mathbb{C} \end{aligned}$$

of $\mathcal{O}_1$ with the product of upper half space $\mathfrak{S}_1$ and the complex affine space. As for the action of Γ_{P_1}, we observe that the subgroup Γ_{Z_1} operates trivially and so there is an induced action of $\Gamma_{P_1}/\Gamma_{Z_1}$ on $\mathcal{O}_1$. Using our description of Γ_{P_1} as a semi-direct product in (2.3.3), $\Gamma_{P_1} \cong \Gamma_{M_1} \cdot \Gamma_{N_1}$, it is easy to see that there is a semi-direct product decomposition of $\Gamma_{P_1}/\Gamma_{Z_1}$

$$\Gamma_{P_1}/\Gamma_{Z_1} \cong \Gamma_{M_1} \cdot (\Gamma_{N_1}/\Gamma_{Z_1}) \cong \Gamma(2,p) \cdot Z^2$$

with the congruence subgroup $\Gamma(2,p) \cong \Gamma_{M_1}$ as the quotient and with the free abelian group of rank 2, $\mathbb{Z}^2 \cong \Gamma_{N_1}/\Gamma_{Z_1}$, as the kernel. In fact, it is more convenient to identify $\Gamma_{P_1}/\Gamma_{Z_1}$ with the group of matrices:

$$\Gamma_{P_1}/\Gamma_{Z_1} = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix} \middle| \begin{array}{l} a_{11}a_{22} - a_{12}a_{21} = 1 \\ a_{11} \equiv a_{22} \equiv 1 \bmod p \\ a_{12} \equiv a_{21} \equiv a_{31} \equiv a_{32} \equiv 0 \bmod p \end{array} \right\}$$

The upper 2-by-2 block $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ constitutes the congruence subgroup $\Gamma(2,p)$, and the last row (a_{31},a_{32}) gives the abelian kernel $\mathbb{Z}^2$. An element (z_1,z_2) in the product $\mathfrak{S}_1 \times \mathbb{C}$ is sent under the action of $\Gamma_{P_1}/\Gamma_{Z_1}$ to the element

$$\left(\frac{a_{11}z_1 + a_{21}}{a_{12}z_1 + a_{22}} \quad \frac{z_2 + a_{13}z_1 + a_{32}}{a_{12}z_1 + a_{22}}\right).$$

It follows that the quotient space $\mathcal{O}_1/\Gamma_{P_1}$ is a fibration with the open modular curve $\overset{\circ}{B}(\ell_1) = \mathfrak{S}_1/\Gamma(2,p)$ as its base and with the elliptic curve $\mathbb{C}/\mathbb{Z}^2$ as its fiber. Throughout the rest of the paper, this total space is known as the open elliptic modular surface, and is denoted by $\mathring{D}(\ell_1)$, $\mathring{D}(\ell_1) = \mathcal{O}/\Gamma_{P_1} \cong \mathfrak{G}_1 \times \mathbb{C}/\Gamma(2,p) \times \mathbb{Z}^2$,

$$\begin{tikzcd} \mathbb{C}/\mathbb{Z}^2 \arrow[r] \arrow[d] & \mathring{D}(\ell_1) \arrow[d, "\pi"] \\ * \arrow[r] & \mathring{B}(\ell_1) \end{tikzcd}$$

From the previous discussion, this total space is the portion of the Igusa compactification sitting above $\mathring{B}(\ell_1)$, $\mathring{D}(\ell_1) = \pi^{-1}(\mathring{B}(\ell_1))$, and π is the projection of the Igusa compactification onto the Satake compactification.

It remains to explain how $\mathcal{O}_i/\Gamma_{P_i}$ are glued together.

For this, we return to the general theory of toroidal compactification. If $Z_i \subset Z_j$, then there is a commutative diagram:

$$\begin{tikzcd} \mathfrak{G}_2/\Gamma_{Z_i} \arrow[r] \arrow[d] & \mathfrak{G}_2/\Gamma_{Z_j} \arrow[d] \\ \mathfrak{G}_2 \cdot Z_i(\mathbb{C})/\Gamma_{Z_i} \arrow[r] \arrow[d, "\cong"] & \mathfrak{G}_2 \cdot Z_j(\mathbb{C})/\Gamma_{Z_j} \arrow[d, "\cong"] \\ \mathfrak{G}_{2-i} \times Z_i \smallsetminus N_i \times T_{P_i} \arrow[r] & \mathfrak{G}_{2-j} \times Z_j \smallsetminus N_j \times T_{P_j} \end{tikzcd}$$

where the vertical maps are given as in (2.6.5), and the horizontal maps are induced by inclusions $\Gamma_{Z_i} \subset \Gamma_{Z_j}$, $\mathfrak{G}_2 \cdot Z_i(\mathbb{C}) \subset \mathfrak{G}_2 \cdot Z_j(\mathbb{C})$. Because of condition (2.6.11), the bottom horizontal map can be further extended to a map e_{ij}:

$$\begin{tikzcd} \mathfrak{G}_{2-i} \times Z_i \smallsetminus N_i \times \mathrm{Temb}(\Delta_i) \arrow[r, "e_{ij}"] & \mathfrak{G}_{2-j} \times Z_j \smallsetminus N_j \times \mathrm{Temb}(\Delta_j) \\ \mathfrak{G}_{2-i} \times Z_i \smallsetminus N_i \times T_{P_i} \arrow[r] \arrow[u] & \mathfrak{G}_{2-j} \times Z_j \smallsetminus N_j \times T_{P_j} \arrow[u] \end{tikzcd}$$

The spaces $\mathcal{O}_i$, $\mathcal{O}_j$ are subspaces in $\mathfrak{G}_{2-i} \times Z_i \smallsetminus N_i \times \mathrm{Temb}(\Delta_i)$, $\mathfrak{G}_{2-j} \times Z_j \smallsetminus N_j \times \mathrm{Temb}(\Delta_j)$, and we can glue them together by considering the union:

(2.6.14) $$\mathcal{O}_i \xleftarrow{\ \ } \mathcal{O}_i \xrightarrow{e_{ij}} \bigcup \{\mathfrak{G}_{2-j} \times Z_j \setminus N_j \times \mathrm{Temb}(\Delta_j)\}\ .$$

Notice that the image $e_{ij}(\mathcal{O}_i)$ of $\mathcal{O}_i$ in the product $\mathfrak{G}_{2-j} \times Z_j \setminus N_j \times \mathrm{Temb}(\Delta_j)$ consists of the following

$$\mathfrak{G}_{2-j} \times Z_j \setminus N_j \times \bigcup_{\tau \subset \Delta_j \cup Z_i} \mathrm{Orb}(\tau)$$

and so it is disjoint from $\mathcal{O}_j$. However, the closure $\overline{e_{ij}(\mathcal{O}_i)}$ of this space contains the subspace

$$\mathfrak{G}_{2-j} \times Z_j \setminus N_j \times \bigcup_{\substack{\tau \subset \Delta_j \cup \Omega_j \\ \overline{\tau} \cap \Omega_i \neq 0}} \mathrm{Orb}(\tau)$$

which lies also in $\mathcal{O}_j$.

In our situation $Z_i = Z_1$, $Z_j = Z_2$, we can describe this gluing procedure more explicitly. The map

$$e_{12}\colon \mathfrak{G}_1 \times Z_1 \setminus N_1 \times T_{P_1} \to T_{P_2}$$

is described by the formula

$$(\tau_{11}, \tau_{12}, z) \longmapsto \begin{pmatrix} e(\tau_{11}/p) & e(\tau_{12}/p) \\ e(\tau_{12}/p) & z \end{pmatrix} .$$

Under the inclusion $Z_1 \subset Z_2$, the convex cone Ω_1 is mapped to the positive ray τ_1, $\tau_1 \in \Delta_2$, generated by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $\overline{\Omega}_2$. If the above map e_{12} is extended to their torus embeddings

$$e_{12}\colon \mathfrak{G}_1 \times Z_1 \setminus N_1 \times \mathrm{Temb}(\Delta_1) \to \mathrm{Temb}(\Delta_2)$$

we obtain a covering mapping of $\mathfrak{G}_1 \times Z_1 \setminus N_1 \times \mathrm{Temb}(\Delta_1)$ onto its image $\mathrm{Orb}(\tau_0) \cup \mathrm{Orb}(\tau_1)$, and the abelian group $\Gamma_{Z_1}/\Gamma_{Z_2}$ is the covering transformation group. If we restrict this to the subspace $\mathcal{O}_1 \cong \mathfrak{G}_1 \times \mathbb{C}$, then its image is a two-dimensional algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$ with $\Gamma_{Z_1}/\Gamma_{Z_2}$ as its fundamental group.

Define the star of τ_1, star(τ_1), to be the $\underline{r}$ cones σ in Δ_2 which contain τ_1 as a face. Then, from (2.5.10), it is easy to see that the union of all the orbits $\mathrm{Orb}(\sigma)$ associated to σ, $\sigma \in \mathrm{star}(\tau_1)$, is an infinite chain of projective lines $\cup_i \mathbf{P}^1(\mathbb{C})_i$, $\overline{\mathrm{Orb}(\sigma)} \cong \mathbf{P}^1(\mathbb{C})$, dim $\tau = 2$, with two consecutive members intersecting transversely at one point, $\overline{\mathrm{Orb}(\sigma)} \cap \overline{\mathrm{Orb}(\sigma')} = \{pt\}$ $\sigma = \partial\nu$, $\sigma' = \partial\nu$. From our previous discussion, this is the subspace in $\mathcal{O}_2$ which lies in the closure of $e_{12}(\mathcal{O}_1)$.

On the other hand, the star star(τ_1) induces a r.p.p. decomposition Δ_0 on the Lie algebra $\Gamma_{Z_1}/\Gamma_{Z_2} \otimes \mathbb{R}$ of the algebraic torus $e_{12}(\mathcal{O}) \cong \mathbb{C}^* \times \mathbb{C}^*$. Such a r.p.p. decomposition coincides with the triangulation Δ_0 defined in (2.5.9) above. The algebraic closure $e_{12}(\mathcal{O}_1)$ is the same as the torus embedding $\mathrm{Temb}(\Delta_0)$ associated to Δ_0, $\bar{e}_{12}(\mathcal{O}_1) = \mathrm{Temb}(\Delta_0)$, and so we have an infinite chain of projective lines as explained in (2.5.9).

Recall that Γ_{P_0} is the isotropy subgroup in Γ which keeps both the line ℓ_1 and the plane h_1 invariant, $\Gamma_{P_0} = \Gamma \cap P(\ell_1, h_1)$ (see (2.3.3)). This group Γ_{P_0} operates on the spaces $\mathcal{O}_1$, $\mathrm{Temb}(\Delta_2)$, and $\mathrm{Temb}(\Delta_0)$, and the map e_{12} is equivariant under this action. In particular, this operates on the closure $\overline{e_{12}(\mathcal{O}_1)}$ and the infinite union of projective lines $\cup\mathbf{P}(\mathbb{C})_i$. The quotient of the last space under this action is a p-gon $\cup_{i\in\mathbb{Z}/p}(\mathbf{P}^1(\mathbb{C}))_i$ as explained in (2.5.10). The spaces $\mathcal{O}_1/\Gamma_{P_1}$, $\mathcal{O}/\Gamma_{P_2}$ are glued together in the union $\mathcal{O}_1/\Gamma_{P_1} \underset{\mathcal{O}_1}{\cup} \mathrm{Temb}(\Delta_2)/\Gamma_{P_2}$, and this p-gon is precisely the subspace in $\mathcal{O}_2/\Gamma_{P_2} = \pi^{-1}(b(\ell_1,h_1))$, which lies in the closure of $\mathcal{O}_1/\Gamma_{P_1} = \pi^{-1}(\mathring{B}(\ell_1)) = \mathring{D}(\ell_1)$.

The procedure of attaching p-gons to the open elliptic modular surface $\mathring{D}(\ell_1) = \mathfrak{S} \times C/\Gamma(2,p)\cdot\mathbb{Z}^2$ is well known. It was discovered by Kodaira (see [K]), and was explained by D. Mumford in great detail as an example of toroidal compactification in [AMRT]. In the above notation, this is achieved by forming the union

$$\mathcal{O}_1/\Gamma_{P_1} \nwarrow_{\mathcal{O}_1} \nearrow \mathrm{Temb}(\Delta_0)/\Gamma_{P_0}$$

$$= \mathfrak{S}_1 \times \mathbb{C}/\Gamma(2,p)\cdot\mathbb{Z}^2 \nwarrow \underset{\mathcal{O}_1\times\mathbb{C}}{\cup} \nearrow \mathrm{Temb}(\Delta_0)/\Gamma_{P_0} \, .$$

There is a projection of this union onto the modular curve $B(\ell_1)$ which sends the above p-gon to the infinite cusp $i\infty$. Thus the p-gon is attached to a boundary neighborhood of $\mathring{D}(\ell_1)$ near the infinite cusp. Since the group $\overline{P}_1$ permutes transitively all the boundary neighborhoods, we can attach other p-gons to other cusps by means of this action. The result is a projective variety called the elliptic modular surface $D(\ell_1)$.

As mentioned at the beginning of this section, the inverse image $\pi^{-1}(B(\ell))$ over other components $B(\ell)$ is isomorphic to this elliptic modular surface. We will denote $\pi^{-1}(B(\ell))$ by $D(\ell)$ and refer to this as the elliptic modular surface associated to ℓ. It contains the subspace $\pi^{-1}(\mathring{B}(\ell))$ as a Zariski open set.

3. The Elliptic Modular Surface

In this section, we study the algebraic topology of the modular surface. First we describe its cohomology, and then describe its Hodge structure.

Our results here are valid for any $N \geq 3$.

As we are concentrating on a single elliptic modular surface $D(\ell_1)$, we shall, in order to simplify the notation, denote it by D_1. As before, we have the natural fibration $\pi\colon \mathring{D}_1 \to \mathring{B}_1$, which extends to a map of the compactifications $\pi\colon D_1 \to B_1$, $B_1 = B(\ell_1)$, (see 2.5). Let $i\infty$ denote the infinite cusp in B_1. Then sitting above this infinite cusp there is a rational N-gon,

$$\pi^{-1}(i\infty) = \cup_{i=1}^{N} (\mathbb{P}^1(\mathbb{C}))_i .$$

We denote by $V_{i\infty}$ a disc neighborhood of the infinite cusp $i\infty$ in B_1 and denote by $U_{i\infty}$ its inverse image in D_1, $U_{i\infty} = \pi^{-1}(V_{i\infty})$. The special linear group $\overline{SL} = SL_2(\mathbb{Z}_N)$ over the ring of integers mod N operates on D_1 and B_1. This action is transitive on the boundary cusps in B_1 with the subgroup

$$\overline{P} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in \overline{SL} \;\middle|\; a_{11} = \pm 1 \right\}$$

as the isotropy subgroup of $i\infty$. Wihtout loss of generality, we may assume that $V_{i\infty}$, $U_{i\infty}$ are invariant under the action of this subgroup $\overline{P}$. Translation by $\overline{SL}$ yields an equivariant neighborhood

$$U = U_{i\infty} \underset{\overline{P}}{\times} \overline{SL}, \; V = V_{i\infty} \underset{\overline{P}}{\times} \overline{SL}$$

covering all the boundary components.

The cohomology of the space U can be expressed as an induced representation

$$H^*(U) = \mathrm{Ind}_{\overline{P}}^{\overline{SL}}(H^*(U_{i\infty})),$$

as U is a disjoint union of copies of $U_{i\infty}$. Since the space $U_{i\infty}$ has the homotopy type of an N-gon, it is easy to verify that the cohomology $H^*(U_i)$ is given by the formulas:

$$(3.1.1)\qquad \begin{aligned} H^0(U_{i\infty}) &= \mathbb{Z} \\ H^1(U_{i\infty}) &= \mathbb{Z} \\ H^2(U_{i\infty}) &= \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \text{ (N copies)} \\ H^i(*_{i\infty}) &= 0, \quad i > 2. \end{aligned}$$

Here in the third formula, there is a natural basis for the vector space $H^2(U_{i\infty}) \cong \oplus_{i=1}^N \mathbb{Z}$, given by the Poincare dual of the rational curves $(\mathbb{P}^1(\mathbb{C}))_i$, $1 \leq i \leq N$, in $\pi^{-1}(i\infty)$.

As for the boundary manifold $\partial U_{i\infty}$, it follows from our description of Γ_{P_i} that this is a torus bundle over S^1 with monodromy $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$. (This sort of manifold is known as a Heisenberg manifold.) Hence, it has the following cohomology:

$$(3.1.2)\qquad \begin{aligned} H^0(\partial U_{i\infty}) &= \mathbb{Z} \\ H^1(\partial U_{i\infty}) &= \mathbb{Z}a \oplus \mathbb{Z}b \\ H^2(\partial U_{i\infty}) &= \mathbb{Z}c \oplus \mathbb{Z}d \oplus \mathbb{Z}/N\mathbb{Z} \\ H^3(\partial U_{i\infty}) &= \mathbb{Z} \end{aligned}$$

where a and b denote the two generators of H^1, and c and d the two generators of H^2. Let $i^*: H^*(U_{i\infty}) \to H^*(\partial U_{i\infty})$ be the natural homomorphism induced by inclusion. Then from the description of classes in (3.1.1) the image of i^* in $H^1(\partial U_{i\infty})$, $H^2(\partial U_{i\infty})$ consists of the one dimensional subspaces $\mathbb{Z}a$, $\mathbb{Z}c$ respectively. The other generator b of H^1 lies in the image of

$$\pi^*: H^1(\partial V_{i\infty}) \to H^1(\partial U_{i\infty}).$$

Finally, we denote by the symbol $H^*_!(U_{i\infty})$ the kernel of i^*,

(3.1.3) $$H^*_!(U_{i\infty}) = \ker(i^*) = \mathrm{Image}(H^*(U_{i\infty}, \partial U_{i\infty}) \to H^*(U_{i\infty})).$$

Then the space $H^1_!(U_{i\infty})$ is trivial, and the space $H^2_!(U_{i\infty})$ is a free abelian group of rank N-1.

3.2. We now consider the fibration $\pi: \mathring{D}_1 \to \mathring{B}_1$ with torus T as its fiber, and with $\mathring{B}_1$ as its base space. Since $\mathring{B}_1$ is the quotient of the upper half plane under the action of $\Gamma(2,N)$, it is a $K(\pi,1)$-manifold. Associated to this fibration, there is a spectral sequence converging to the cohomology $H^*(\mathring{D}_1)$ with its $E_2^{r,s}$-terms given by:

(3.2.1) $$E_2^{r,s} = H^r(\mathring{B}_1; H^s(T)) = H^r(\Gamma(2,N); H^s(T)).$$

The cohomology $H^s(T)$ of the torus has only three nonzero terms:

(3.2.2) $$H^0(T) = \mathbb{Z}$$ $$H^1(T) = \mathbb{Z} \oplus \mathbb{Z} \simeq E$$ $$H^2(T) = \mathbb{Z}.$$

We use the notation E to denote the standard representation of $\Gamma(2,N)$ on the free abelian group of rank 2 (the restriction of the standard representation of $SL_2(\mathbb{Z})$ to $\Gamma(2,N)$. This follows from checking the fundamental group of D_1^0). At any rate the nonzero terms of the spectral sequence are concentrated in the lower corner as in the figure:

$$\begin{array}{|lll}
0 & 0 & 0 \\
\mathbb{Z} & H^1(\Gamma(2,N)) & 0 \\
0 & H^1(\Gamma(2,N);E) & 0 \\
\mathbb{Z} & H^1(\Gamma(2,N)) & 0 \\
\hline
\end{array}$$

Since the differentials are zero, the spectral sequence collapses, and the cohomology of $\mathring{D}_1$ is given by:

$$\begin{aligned}
&H^0(\mathring{D}_1) \cong \mathbb{Z} \\
&H^1(\mathring{D}_1) \cong H^1(\mathring{B}_1) \cong H^1(\Gamma(2,N)) \\
(3.2.3)\qquad &H^2(\mathring{D}_1) \cong H^1(\mathring{B}_1;E) \oplus \mathbb{Z} \\
&\qquad\quad\ \cong H^1(\Gamma(2,N);E) \oplus \mathbb{Z} \\
&H^3(\mathring{D}_1) \cong H^1(\mathring{B}_1)
\end{aligned}$$

On the other hand, there are long exact sequences:

$$(3.2.4)\qquad 0 \longrightarrow H^1_{cusp}(\mathring{B}_1) \longrightarrow H^1(\mathring{B}_1) \xrightarrow{j^*_{\mathbb{Z}}} H^1(\partial\mathring{B}_1) \xrightarrow{\delta} H^2(\mathring{B}_1,\partial\mathring{B}_1) \longrightarrow \cdots$$

$$(3.2.5)\qquad 0 \longrightarrow H^1(\mathring{B}_1,\partial\mathring{B}_1;E) \longrightarrow H^1(\mathring{B}_1;E) \xrightarrow{j^*_E} H^1(\partial\mathring{B}_1;E) \longrightarrow H^2(\mathring{B}_1,\partial\mathring{B}_1;E) \longrightarrow \cdots$$

where $H^1_{cusp}(\mathring{B}_1) = \operatorname{coker}(H^0(\partial\mathring{B}_1) \to H^1(\mathring{B}_1,\partial\mathring{B}_1))$, and j^* and j^*_E are induced by inclusions. We are abusing our language by writing $\partial\mathring{B}_1$ for $\mathring{B}_1 \cap V_\infty$. As $\mathring{B}_1$ is an open manifold and $\mathring{B}_1 \cap V$ is a collar of each end, this is harmless.

In the first exact sequence (3.2.4), the cohomology group $H^2(\mathring{B}_1,\partial\mathring{B}_1)$ is of rank 1 generated by the orientation class, and the coboundary map δ is surjective, so

$$(3.2.6)\qquad \operatorname{Coker} j^*_{\mathbb{Z}} \cong \mathbb{Z}.$$

Also, it is easy to see that

$$(3.2.7)\qquad H^1_{cusp}(\mathring{B}_1) \cong H^1(B_1).$$

(3.3) *Theorem* (3.3.1). *The cohomology of* D_1 *as a representation space of* $\Gamma(2,N)$ *is given by the following formulas:*

$$(3.3.2)\quad \begin{cases} \text{(i)}\ H^0(D_1) = \mathbb{Z} \\ \text{(ii)}\ H^1(D_1) = H^1(B_1) \\ \text{(iii)}\ H^2(D_1) = H^1(\mathring{B}_1, \partial\mathring{B}_1; E) \oplus \mathrm{Ind}_{\overline{P}}^{\overline{SL}}(H^2_!(U_{i\infty})) \oplus \mathbb{Z} \oplus \mathbb{Z} \\ \text{(iv)}\ H^3(D_1) = H^1(B_1) \\ \text{(v)}\ H^4(D_1) = \mathbb{Z} \end{cases}$$

where $H^2_!(U_{i\infty})$ *is defined in* (3.1.3).

Proof. Consider the Mayer-Vietoris sequence:

$$\begin{aligned} 0 &\to H^3(D_1) \to H^3(\mathring{D}_1) \oplus H^3(U) \xrightarrow{\gamma} H^3(\partial U) \\ &\to H^2(D_1) \to H^2(\mathring{D}_1) \oplus H^2(U) \xrightarrow{\beta} H^2(\partial U) \\ &\to H^1(D_1) \to H^1(\mathring{D}_1) \oplus H^1(U) \xrightarrow{\alpha} H^1(\partial U) \to 0. \end{aligned}$$

It is enough to determine the groups coker γ, ker β, coker β, ker α because we have group extensions:

$$(3.3.3)\quad \begin{aligned} &0 \to \operatorname{coker}\gamma \to H^2(D_1) \to \ker\beta \to 0 \\ &0 \to \operatorname{coker}\beta \to H^1(D_1) \to \ker\alpha \to 0 \end{aligned}$$

and the group $H^3(D_1)$ is dual to $H^1(D_1)$ by Poincaré duality. From the discussion in the previous paragraphs, we can decompose $H^*(\partial U)$, $H^*(\mathring{D}_1) \oplus H^*(U)$ into direct sums as follows:

$$(3.3.4)\quad \begin{cases} H^1(\partial U) \cong H^1(\partial\mathring{B}_1) \oplus \mathrm{Im}[i^1\colon H^1(U) \to H^1(\partial U)], \\ H^2(\partial U) \cong H^2(\partial\mathring{B}_1; E) \oplus \mathrm{Im}[i^2\colon H^2(U) \to H^2(\partial U)], \\ H^3(\partial U) \cong H^1(\partial\mathring{B}_1; H^2 T) \oplus H^1(\partial\mathring{B}_1), \end{cases}$$

$$\begin{cases} H^1(\mathring{D}_1) \oplus H^1(U) \cong H^1(\mathring{B}_1) \oplus H^1(U) \\ H^2(\mathring{D}_1) \oplus H^2(U) \cong H^2(T) \oplus H^2(\mathring{B}_1; E) \oplus H^2(U) \\ H^3(\mathring{D}_1) \oplus H^3(U) \cong H^1(\mathring{B}_1; H^2 T) \oplus H^1(\mathring{B}_1) \end{cases}$$

Accordingly the mappings α, β, γ can be written in the form of block matrices

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{pmatrix}, \quad \gamma = (\gamma_{11}).$$

For the coefficients of these matrices, we have

$$\begin{aligned} &\alpha_{11} = j^*, \quad \alpha_{22} = i^1, \quad \alpha_{12} = \alpha_{21} = 0. \\ (3.3.5) \quad &\beta_{21} = j_E^*, \quad \beta_{32} = i^2, \quad \beta_{31} = \beta_{11} = 0, \quad \beta_{22} = 0 \\ &\beta_{12} = \text{the composite map } H^2(T) \to H^2(U) \to H^2(\partial U), \\ &\gamma_{11} = j^*. \end{aligned}$$

A straightforward computation shows that

$$\begin{aligned} \ker \alpha &= \ker j^* \oplus \ker i^1 = H^1_{cusp}(\mathring{B}_1) \\ \operatorname{coker} \beta &= 0 \\ \ker \beta &= \ker j_E^* \oplus \ker i^2 \oplus \mathbb{Z} = H_1(\mathring{B}_1, \partial\mathring{B}_1; E) \\ &\qquad \oplus \operatorname{Ind}_{\overline{P}}^{\overline{SL}}(H^2_!(U_{i\infty})) \\ &\qquad \oplus \mathbb{Z}, \\ \operatorname{coker} \gamma &= \operatorname{coker} j^* = \mathbb{Z}. \end{aligned}$$

From (3.2.7), $H^1_{cusp}(\mathring{B}_1) = H^1(B_1)$ which is self-dual. Putting these results into (3.3.3), we obtain the required formulas immediately.

From (2.6.13) we can see that there is a natural action of the group $\overline{SL}^A$ on the elliptic modular surface D_1, where $\overline{SL}^A$ is given by an extension

$$1 \to \mathbb{Z}_N \oplus \mathbb{Z}_N \to \overline{SL}^A \overset{\rho}{\to} \overline{SL} \to 1$$

and we can describe $H^*(D_1)$ as a representation space of $\overline{SL}^A$. As we have described the action of $\overline{SL}$ in (3.3.1), it suffices to describe the action of $\mathbb{Z}_N \oplus \mathbb{Z}_N = \operatorname{Ker}(\rho)$. The action of $\operatorname{Ker}(\rho)$ covers the trivial action on B_1, and while $\ker(\rho)$ acts non-trivially on fibers, it acts trivially on the homology of the general torus fiber T. Thus $\ker(\rho)$

acts trivially on every term in (3.3.1) except for the subspace $\mathrm{Ind}_{\overline{P}}^{\overline{SL}}(H^2_!(U_{i\infty}))$. Since it does not permute the cusps, it suffices to determine its action on $H^2_!(U_{i\infty}) = \ker(i^2: H^2(U_{i\infty}) \to H^2(\partial U_{i\infty}))$, a free abelian group of rank N-1. Using 2.6.13 and 3.1.2, it is not hard to check that one factor of $\mathbb{Z}_N$ acts trivially, and that the action of the other factor of $\mathbb{Z}_N$ is its natural action on the kernel of the augmentation map from the group ring $\mathbb{Z}[\mathbb{Z}_N]$ to $\mathbb{Z}$.

We now determine the ranks of the cohomology groups of D_1. First, we recall the following well-known facts (see [Sm]). Let t denote the number of cusps of B_1, and g the genus of B_1, then

(3.3.6) $t = (N^2/2)\Pi(1-p^{-2})$ where the product is taken over all primes p dividing N

(3.3.7) $g = 1 + (N-6)t/12$.

Next we determine the Euler characteristic of D_1. Since $\mathring{D}_1$ is a torus bundle over $\mathring{B}_1$ it contributes 0 to the Euler characteristic χ. This leaves t cusps, each with Euler characteristic N (from 3.1.1), so

(3.3.8) $\chi = Nt$.

Corollary (3.3.9). *The ranks of the groups* $H^i(D_1)$ *are as follows:*

(i) $\mathrm{rank}(H^0(D_1)) = \mathrm{rank}(H^4(D_1)) = 1$

(ii) $\mathrm{rank}(H^1(D_1)) = \mathrm{rank}(H^3(D_1)) = 2g$

(iii) $\mathrm{rank}(H^2(D_1)) = \chi + 4g - 2$

where g *and* χ *are as above.*

Corollary (3.3.10). $\mathrm{Rank}(H^1(\mathring{B}_1,\partial\mathring{B}_1;E)) = (N-3)t/3$.

3.4. In this section we further identify $H^*(D_1)$ and compute its Hodge structure. Here we have taken cohomology with coefficients in $\mathbb{C}$.

Proposition (3.4.1). *The chern number* $c_1^2(D_1) = 0$.

Proof. By a theorem of Kodaira ([K], Theorem 12.1], the canonical bundle $K(D_1)$ is the pullback of a bundle M_1 over B_1, $K(D_1) = \pi^*(M_1)$. Recall that the canonical bundle is the highest exterior power of the cotangent bundle, and hence (see [Hi]) we have the following fact, which we shall use repeatedly below: $c_1(D_1) = -c_1(K(D_1))$. In particular, here $c_1^2(D_1) = 0$.

Corollary (3.4.2). (i) *The Todd genus* $\tau(d_1) = \chi/12$

(ii) *The signature* $\sigma(D_1) = -2\chi/3$.

Proof. Hirzebruch ([Hi]) has shown that $\tau(D_1) = (c_1^2(D_1) + c_2(D_1))/12$ and $\sigma(D_1) = (c_1^2(D_1) - 2c_2(D_1))/3$. Here $c_1^2 = 0$ and of course $c_2 = \chi$.

Corollary (3.4.3). *The Hodge numbers of* D_1 *are as follows:*

(i) $h^{0,0} = h^{2,2} = 1$

(ii) $h^{1,0} = h^{0,1} = h^{2,1} = g$

(iii) $h^{2,0} = h^{0,2} = (N-3)t/6$

(iv) $h^{1,1} = 2 + (N-1)t$.

Proof. The Todd genus $= 1 - h^{1,0} + h^{2,0}$.

Corollary (3.4.4). *Let* $S_i(\Gamma(2,N))$ *denote the space of cusp forms of weight* i *for the group* $\Gamma(2,N)$, *and* $\overline{S_i(\Gamma(2,N))}$ *its dual.*

(i) $H^{1,0} = H^{2,1} = S_2(\Gamma(2,N))$, $H^{0,1} = H^{1,2} = \overline{S_2(\Gamma(2,N))}$

(ii) $H^{2,0} = S_3(\Gamma(2,N))$, $H^{0,2} = \overline{S_3(\Gamma(2,N))}$

(iii) $H^{1,1} = \mathrm{Ind}_{\overline{P}}^{\overline{SL}}(H^2_!(U_{i\infty}: \mathbb{C})) \oplus \mathbb{C} \oplus \mathbb{C}$.

Proof. By the results of Shimura [Sh], $H^1_{cusp}(\mathring{B})$ can be identified with $S_2(\Gamma(2,N)) \oplus \overline{S_2(\Gamma(2,N))}$, and similarly, $H^1(\mathring{B}_1, \partial\mathring{B}_1; E)$ can be identified with $S_3(\Gamma(2,N)) \oplus \overline{S_3(\Gamma(2,N))}$. Clearly these summands transform holomorphically and anti-holomorphically respectively. Thus to prove the corollary it suffices to show that all classes in $\mathrm{Ind}_{\overline{P}}^{\overline{SL}}(H^2_!(U_{i\infty};\mathbb{C})) \oplus \mathbb{C} \oplus \mathbb{C}$ are of type (1,1), since the dimension of this space is equal to $h^{1,1}$. This we do by showing that they are all represented by algebraic cycles. This is clear for the first summand, as by 3.1.1 all classes in $H^2(U_{i\infty};\mathbb{C})$ are represented by algebraic cycles. There remain two summands $\mathbb{C} \oplus \mathbb{C}$. One is represented by the general elliptic curve fiber. The other is represented by the section of $\pi: D_1 \to B_1$ which is given by the identity in the group law in each elliptic curve, which extends over each cusp as a nonsingular section (as is verified in [So]--compare 4.2.3 and 4.2.6).

4. The Chern Classes of Certain Bundles

4.1. In this section we calculate the chern classes of D_1, of its normal bundle in the Igusa compactification $\mathfrak{S}/\Gamma^*$, and various other chern classes and numbers that we need for our work in [LW1,2].

While our method here works for an arbitrary level $N \geq 3$, in order to simplify computations we restrict ourselves to the case $N = p$, p an odd prime, which is the situation of interest in [LW1,2]. We rely on the results of Yamazaki [Y], and we follow his notation with the following exception:

If P is a complex codimension 1 submanifold of the complex manifold Q, we denote by $[P] \in H^2(Q)$ the cohomology class dual to P. This cohomology class determines a complex line bundle (the line bundle associated to the divisor P) which we denote by NP. Then $c_1(NP) = [P]$, and the restriction of the bundle NP to P is indeed the normal bundle of P in Q. Here Yamazaki uses $c_1(P)$, but we follow the topologists' convention in writing $c_1(P)$ for $c_1(TP)$, where TP is the tangent bundle of P. (We may sometimes write NP as N_QP, if it is important to emphasize Q.)

We will follow Yamazaki in identifying a top-dimensional cohomology class of a complex manifold with its evaluation on the fundamental class of the manifold.

We have that $D = \mathfrak{S}_2/\Gamma^* - \mathfrak{S}_2/\Gamma$ is the union of irreducible components $D(\ell)$. Following Yamazaki, we re-index these as D_i, $i = 1,\cdots,(p^4-1)/2$, and denote by $\pi\colon D_i \to B_i$ the projection of each one of these singular fibrations onto its base B_i. Thus $B_i = \mathfrak{S}_1/\Gamma(2,p)^*$ and if x is a generic point of B_i, $x \in \mathfrak{S}_1/\Gamma(2,p)$, then its fiber $T = \pi^{-1}(x)$ is a complex torus in D_i. Note that in our notation $[x]$ is the fundamental cohomology class of B_i, and $\pi^{-1}([x]) = [T]$.

First we deal with a single boundary component, an elliptic modular surface D_1.

Theorem (4.1.1). (i) $c_2(D_1) = p(p^2-1)/2$

(ii) $c_1(D_1) = -2^{-3}(p^2-1)(p-4)[T]$

(iii) $c_1^2(D_1) = 0.$

Proof. We have already shown (i) and (iii)--see (3.3.8), (3.4.1), (3.4.2). As for (ii), let us quote Kodaira's theorem ([K], Theorem 12.1) more precise than in (3.4.1). Kodaira showed that

$$K(D_1) = \pi^*(K(B_1) - f)$$

where $c_1(f) = -(p_a+1)$, with p_a the arithmetic genus of D_1. But $p_a+1 = \tau(D_1)$, the Todd genus, and $c_1(D(B_1)) = -(2-2g)$. From (3.3.7) and (3.4.2) we have that

$$(4.1.2) \quad -c_1(D_1) = c_1(K(D_1)) = \pi^*(2^{-3}(p^2-1)(p-4)[x])$$
$$= +2^{-3}(p^2-1)(p-4)[T] \text{ as required.}$$

Proposition (4.1.3). *Let* S_1 *be a projective line in an exceptional fiber of* D_1. *Then* $c_1(N_{D_1}S_1) = -2$.

Proof. The general fiber T is homologous to the sum $S_1 + \cdots + S_p$ of the projective lines in an exceptional fiber, and T has self-intersection number 0.

Now $c = c_1(N_{D_1}S_1)$ is the self-intersection number of S_1, so the self-intersection matrix of the span of the classes $S_1,\cdots,S_p$ is

$$\begin{pmatrix} c & 1 & & & & 1 \\ 1 & c & 1 & & & \\ & 1 & c & & & \\ & & & \ddots & & \\ & & & & c & 1 \\ 1 & & & & 1 & c \end{pmatrix}$$

From this matrix, $S_1 + \cdots + S_p$ has self-intersection number $p(c+2)$, so $c = -2$.

4.2. Now we turn to the normal bundle of D_1.

Consider $c_1(ND_1) \in H^2(D_1)$. If we let P denote the stabilizer of D_1 under the action of $Sp_4(\mathbb{F}_p)$ on $\mathfrak{S}_2/\Gamma^*$, then $c_1(ND_1)$ is in the invariant cohomology $H^2(D_1)^P$. We see from Section 3.3 that $H^2(D_1: \mathbb{C})^P = \mathbb{C} \oplus \mathbb{C}$. The cohomology class [T] is invariant, and $[T]\cdot[T] = 0$ as NT is trivial. We let $[S] \in H^2(D_1: \mathbb{Q})^P$ be the dual of [T], so $[T]\cdot[S] = 1$, $[S]\cdot[S] = 0$. ([T] is a primitive integral cohomology class, and it turns out that [S] is a half-integral class, i.e. $2[S] \in H^2(D_1: \mathbb{Z})$.)

Theorem (4.2.1). $c_1(ND_1) = \frac{-(p^2-1)}{24}[T] - 2p[S]$.

Proof. We have $c_1(ND_1) = a[T] + b[S]$ for some a,b. We first determine b, and then determine a with the aid of a couple of lemmas.

Yamazaki ([Y], proof of Theorem 5) shows that

(4.2.2) $\quad c_1(K(D_1))c_1(ND_1) = -2^{-2}p(p^2-1)(p-4)$

where $K(D_1)$ is the canonical bundle of D_1.

Applying (4.1.2), we have

$$-2^{-2}p(p^2-1)(p-4) = (2^{-3}(p^2-1)(p-4)[T])(a[T] + b[S]),$$

yielding $b = -2p$.

Let Δ be the closure of the image of the diagonal matrices of $\mathfrak{S}_2$ in $\mathcal{M} = \mathfrak{S}_2/\Gamma^*$, and let E be the sub-variety of $\mathcal{M}$ consisting of the union of the translates of Δ under the action of the group $Sp(4,\mathbb{F}_p)$ on $\mathcal{M}$. (Thus the divisor [E] is an invariant class in $H^2(\mathcal{M})$.) E is a union of $p^2(p^2+1)/2$ disjoint irreducible components E_α, and for any component D_i of D, $E \cap D_i$ is the union of the points of order p on D_i ([Y], Lemma 3).

The structure of E is easy to describe. All the components are identical, so we concentrate on the one, E_δ, which is the image of Δ itself. It is the quotient of Δ by its stabilizer in Γ, which is isomorphic to $\Gamma(2,p) \times \Gamma(2,p)$. Then $E_\delta = \mathfrak{G}_1/\Gamma(2,p)^* \times \mathfrak{G}_1/\Gamma(2,p)^* = B_1^1 \times B_1^2$, $B_1^i = B_1$ as in Section 3.

Lemma (4.2.3). $c_1^2(NE)c_1(ND_1) = -p^3(p^2-1)/24$.

Proof. We have
$$c_1^2(NE)c_1(ND) = c_1^2(NE)\Sigma_i c_1(ND_i) = \Sigma_i c_1^2(NE)c_1(ND_i) = 2^{-1}(p^4-1)c_1^2(NE)c_1(ND_1).$$
However, by [Y], Theorem 2,

$$(4.2.4) \qquad c_1^2(NE)c_1(ND) = -2^{-4}3^{-1}p^3(p^2-1)(p^4-1),$$

and the lemma follows.

Lemma (4.2.5). $c_1(NE|D_1) = -\frac{p(p^2-1)}{48}[T] + p^2[S]$.

(Here, as below, $c_1(NF|G) = i^*c_1(NF)$, $i\colon G \to \mathfrak{M}$).

Proof. As [E] is invariant, $c_1(NE|D_1) = y[T] + z[S]$ for some y,z.

Now $E \cap D_1$ is the union of the section of π consisting of the points of order p. There are p^2 such sections, each of which has intersection number 1 with T, so $c_1(NE|D_1)\cdot[T] = p^2$, so $z = p^2$.

Hence $c_1^2(NE|D_1) = 2p^2y = c_1^2(NE)c_1(ND_1) = -2^{-3}3^{-1}p^3(p^2-1)$ so y is as claimed.

Proof of (4.2.1) *(continued)*. For any component E_α of E, the intersection $E_\alpha \cap D_1$ is either empty, or is $\{pt\} \times B_1^2$ or $B_1^1 \times \{pt\}$ in E_α, which has self-intersection number zero. Thus $c_1(NE|D_1)c_1(ND_1) = 0$. But

$$c_1(NE|D_1)c_1(ND_1) = \left(\frac{-p(p^2-1)}{48}[T] + p^2[S]\right)(a[T] - 2p[S])$$

yielding $a = -(p^2-1)/24$.

Corollary (4.2.6). (i) $c_1^2(ND_1) = p(p^2-1)/6$

(ii) $c_1(ND_1|T) = -2p$

(iii) $c_1(ND_1)c_1(D_1) = p(p^2-1)(p-4)/4$.

Let E_0^1 be the section of $\pi\colon D_1 \to B_1$ consisting of the origin for the group law in each general fiber (each of which is an elliptic curve), extended to the singular fibers as well. Then, for some α, $E_0^1 = E_\alpha \cap D_1$.

Lemma (4.2.7). $c_1(N_{D_1}|E_0^1)^2 = -p(p^2-1)/24$.

Proof. As E_0^1 is one of p^2 components of $E^1 = E \cap D_1$ (and $E_\alpha \cap E_\beta \cap D_1 = \phi$ for $\alpha \neq \beta$),

$$\begin{aligned} c_1(N_{D_1}|E_0^1)^2 &= p^{-2}c_1(N_{D_1}|E^1)^2 \\ &= p^{-2}c_1(NE^1)^2 c_1(ND_1) \\ &= p^{-2}c_1(NE)^2 c_1(ND_1) \\ &= -p(p^2-1)/24 \text{ by } (4.2.3). \end{aligned}$$

Remark (4.2.8). It follows that $c_1(NE) = \frac{-p(p^2-1)}{24}([B_1^1] + [B_1^2])$, though we do not need this fact.

4.3. For our work in [LW1,2] we also need to consider the following:

Let C_2 be the sections of $\pi\colon D_1 \to B_1$ consisting of the points of order one or two. We observe first that C_2 contains E_0^1 as one component. Also, over a general fiber C_2 has four points, but in a singular fiber two of these points become identified. Lastly, $C_2 \cap E^1 = E_0^1$ as 2 is prime to p. We shall write $C_2 = E_0^1 \cup \tilde{C}_2$, and denote the

the fundamental cohomology classes of each term in the union C_2 by $[x]$ and $[\tilde{x}]$ respectively.

Proposition (4.3.1). (i) $c_1(C_2) = -(p^2-1)(2p-9)/6$

(ii) $c_1(N_{D_1}|C_2) = -(p^2-1)(p-3)/6$

(iii) $c_1(ND_1|C_2) = -(p^2-1)/4$.

Proof. (i) $\pi\colon C_2 \to B_1$ is four-to-one everywhere except over the $(p^2-1)/2$ cusps, where it is three-to-one, so $c_1(C_2) = \chi(C_2) = 4\chi(B_1) - (p^2-1)/2$. But B_1 is a Riemann surface of genus $g = 1 + \frac{(p-6)(p^2-1)}{24}$, and the result follows.

(ii) Since $N_{D_1}|C_2 \oplus TC_2 = TD_1|C_2$, $c_1(N_{D_1}|C_2) + c_1(C_2) = c_1(TD_1|C_2)$. Now $c_1(C_2) = \chi(B)[x] + (3\chi(B) - (p^2-1)/2)[\tilde{x}]$ as in (i), and $c_1(TD_1|C_2) = i^*(-2^{-3}(p^2-1)(p-4)[T])$ by (4.1.1), where $i\colon C_2 \to D_1$ is the inclusion. Now E_0^1 intersects T in one point, and $\tilde{C}_2$ intersects T in three points. Hence $i^*([T]) = [x] + 2[\tilde{x}]$ and the result follows.

(iii) We have
$$
\begin{aligned}
c_1(ND_1|C_2) &= i^*(c_1(ND_1)) \\
&= i^*(-\frac{(p^2-1)}{24}[T] - 2p[S]) \text{ by } (4.2.1).
\end{aligned}
$$

By (4.2.5), $[S] = p^{-2}(\frac{p(p^2-1)}{48}[T] + c_1(NE|D_1))$, so $c_1(ND_1|C_2) = i^*(-\frac{(p^2-1)}{12}[T] - \frac{2}{p}c_1(NE|D_1))$. Since $E \cap C_2 = E_0^1$ (and $E \cap \tilde{C}_2 = \phi$),
$$
\begin{aligned}
i^*(c_1(NE|D_1)) &= i_1^*(c_1(NE_0^1|D_1)) \\
&= (E_0^1 \cdot E_0^1 \cdot D_1) \\
&= c_1(N_{D_1}E_0^1)^2 = -p(p^2-1)/24 \text{ by } (4.2.7).
\end{aligned}
$$
where $i_1\colon E_0^1 \to D_1$ is the natural projection, $(\cdot\cdot)$ is the intersection number. Thus $c_1(ND_1|C_2) = (-(p^2-1)/12)(4) - (2/p)(-p(p^2-1)/24) = -(p^2-1)/4$.

4.4. Finally, we consider the line bundle L corresponding to modular forms of weight one.

Proposition (4.4.1). $c_1(L|D_1) = \frac{p(p^2-1)}{24}[T]$.

Proof. By [Y], $L|D_1 = \pi^*(L_1')$, L_1' a bundle over B_1, so, as above, $c_1(L|D_1) = z\cdot[T]$. To determine z, note that [Y] also shows that

$$c_1(L|D_1)c_1(N(D_1)) = -2^{-2}3^{-1}p^2(p^2-1).$$

But $c_1(L|D_1)c_1(N(D_1)) = -2pz$ by (4.2.1), and the proposition follows.

References

[AMRT] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactification of locally symmetric varieties*, Math Sci. Press (1975).

[AS] M. F. Atiyah and I. M. Singer, *The index of elliptic operators III*, Ann. Math. 87 (1968), 546-604.

[BB] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. 84 (1966), 442-528.

[BH] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. 75 (1962), 485-535.

[BS] A. Borel and J. P. Serre, *Corners and arithmetic groups*, Comm. Math. Helv. 48 (1973), 436-491.

[CL] R. Charney and R. Lee, *Cohomology of the Satake compactification*, Topology 22 (1983), 387-423.

[F] H. Feldmann, *Uber das Verhalten der Modulfunktienen von Primzahlstufe bei beliekigen Modulsubstitutionen*, Abh. Math. Semin. Hamburg Univ. 8 (1931), 323-347.

[G] G. van der Geer, *On the geometry of a Siegel modular threefold*, Math. Ann. 260 (1982), 317-350.

[Hi] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer (1966).

[H1] E. Hecke, *Uber ein Fundamental problem aus der Theorie der eilliptischen Modulfunktionen*, Abh. Math. Semin. Hamburg Univ. 6 (1928), 235-237.

[H2] ________, *Uber das Verhalten der Integrale 1. Gattung bei Abbildungen, insfesondere in der Theorie der elliptischen Modulfunktionen*, Abh. Math. Semin. Hamburg Univ. 8 (1931), 271-281.

[Ig] J.-I. Igusa, *On the desingularization of Satake compactifications*. In: Proc. Symp. Pure Math. vol. IX, pp. 113-121, Providence (1966).

[K] K. Kodaira, *On compact analytic surfaces III*, Ann. Math. 78 (1963), 1-40.

[LW1] R. Lee and S. H. Weintraub, *On a generalization of a theorem of Erich Hecke*, Proc. Natl. Acad. Sci. USA, 79 (1982), 7955-7957.

[LW2] ________, *A generalization of a theorem of Hecke to the Siegel space of degree two* (to appear).

[N] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Math. No. 812, Springer-Verlag, Berlin, Heidelberg, New York (1980).

[O] T. Oda, *Torus embeddings and applications*, Lectures on Math. and Phy. No. 58, Tata Institute of Fundamental Research, Bombay (1978).

[St] I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc. 20 (1956), 259-281.

[So] T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan 24 (1972), 20-59.

[Sm] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publ. Math. Soc. Japan 11, Iwanami, Tokyo and Princeton Univ. Press (1971).

[Y] T. Yamazaki, *On Siegel modular forms of degree two*, Amer. J. Math. 98 (1976), 39-53.

[Z] S. Zucker, *Satake compactifications*, manuscript, Indiana University (1982).

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