**TOPOLOGY PROCEEDINGS**

**Volume 11, 1986**

Pages 177–208

**http://topology.auburn.edu/tp/**

# COMPLETIONS OF METRIC SIMPLICIAL COMPLEXES BY USING $\ell_p$-NORMS

by

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**Topology Proceedings**

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**ISSN:** 0146-4124

# COMPLETIONS OF METRIC SIMPLICIAL COMPLEXES BY USING $\ell_p$-NORMS

**Katsuro Sakai**

## 0. Introduction

Let K be a simplicial complex. Here we consider K as an abstract one, that is, a collection of non-empty finite subsets of the set $V_K$ of its vertices such that $\{v\} \in K$ for all $v \in V_K$ and if $\emptyset \neq A \subset B \in K$ then $A \in K$. Then a simplex of K is a non-empty finite set of vertices. The realization $|K|$ of K is the set of all functions $x: V_K \to I$ such that $C_x = \{v \in V_K | x(v) \neq 0\} \in K$ and $\Sigma_{v \in V_K} x(v) = 1$. There is a metric $d_1$ on $|K|$ defined by

$$d_1(x,y) = \sum_{v \in V_K} |x(v) - y(v)|.$$

Then the metric space $(|K|, d_1)$ is a metric subspace the Banach space $\ell_1(V_K)$ which consists all real-valued functions $x: V_K \to \mathbb{R}$ such that $\Sigma_{v \in V_K} |x(v)| < \infty$, where $\|x\|_1 = \Sigma_{v \in V_K} |x(v)|$ is the norm of $x \in \ell_1(V_K)$. The topology induced by the metric $d_1$ is the *metric topology* of $|K|$ and the space $|K|$ with this topology is denoted by $|K|_m$. The completion of the metric space $(|K|, d_1)$ is the closure $c\ell_{\ell_1(V_K)}|K|$ of $|K|$ in $\ell_1(V_K)$. We will call this the $\ell_1$-*completion* of $|K|_m$ and denoted by $\overline{|K|}^{\ell_1}$. It is well known that $|K|_m$ is an ANR (e.g., see [Hu]). In Section 1, we prove that the $\ell_1$-completion preserves this property, that is, 0.1. *Theorem. For any simplicial complex* K, *the* $\ell_1$-*completion* $\overline{|K|}^{\ell_1}$ *is an* ANR *and the inclusion* $|K|_m \subset \overline{|K|}^{\ell_1}$ *is a fine homotopy equivalence.*

Here a map $f: X \to Y$ is a *fine homotopy equivalence* if for each open cover $\mathcal{U}$ of Y there is a map $g: Y \to X$ called a $\mathcal{U}$-*inverse* of f such that fg is $\mathcal{U}$-homotopic to $id_Y$ and gf is $f^{-1}(\mathcal{U})$-homotopic to $id_X$.

By F(V), we denote the collection of all non-empty finite subsets of V. Then F(V) is a simplicial complex with V the set of vertices. Such a simplicial complex is called a *full simplicial complex*. From the following known result, our theorem makes sense in case K contains an infinite full simplicial complex.

0.2. *Proposition. For a simplicial complex* K, *the following are equivalent:*

(i) $|K|_m$ *is completely metrizable;*

(ii) K *contains no infinite full simplicial complex;*

(iii) $(|K|, d_1)$ *is complete (i.e.,* $|K| = \overline{|K|}^{\ell_1}$*).*

For the proof, refer to [Hu, Ch. III, Lemma 11.5], where only the equivalence between (i) and (ii) are mentioned but the implications (i) ⇒ (ii) ⇒ (iii) are proved (the implication (iii) ⇒ (i) is trivial).

We can also consider $|K|_m$ as a topological subspace of the Banach space $\ell_p(V_K)$ for any $p > 1$, where

$$\ell_p(V_K) = \{x \in \mathbb{R}^{V_K} \mid \sum_{v \in V_K} |x(v)|^p < \infty\}$$

and the norm of $x \in \ell_p(V_K)$ is

$$\|x\|_p = (\sum_{v\in V_K} |x(v)|^p)^{1/p}.$$

Let $d_p$ be the metric defined by the norm $\|\cdot\|_p$. Then the completion of the metric space $(|K|,d_p)$ is $c\ell_{\ell_p(V_K)}|K|$ and denoted by $\overline{|K|}^{\ell_p}$. We will call $\overline{|K|}^{\ell_p}$ the $\ell_p$-*completion* of $|K|_m$. And also $|K|_m$ can be considered as a topological subspace of the Banach space $m(V_K)$ which consists all bounded real-valued functions $x\colon V_K \to \mathbb{R}$ with the norm $\|x\|_\infty = \sup\{|x(v)| \mid v \in V_K\}$. Let $c_0(V_K)$ be the closed linear subspace of all those $x$ in $m(V_K)$ such that for each $\varepsilon > 0$, $\{v \in V_K \mid |x(v)| > \varepsilon\}$ is finite. Then $|K|_m \subset c_0(V_K)$. Let $d_\infty$ be the metric defined by the norm $\|\cdot\|_\infty$. The completion of the metric space $(|K|,d_\infty)$ is $c\ell_{m(V_K)}|K| = c\ell_{c_0(V_K)}|K|$ and denoted by $\overline{|K|}^{c_0}$. We will call $\overline{|K|}^{c_0}$ the $c_0$-*completion* if $|K|_m$. However *the metrics* $d_2,d_3,\cdots,d_\infty$ *on* $|K|$ *are uniformly equivalent*. In fact, for each $x,y \in |K|$,

$$\begin{aligned} d_2(x,y) &= \|x - y\|_2 = (\sum_{v\in V_K}(x(v) - y(v))^2)^{1/2} \\ &\le (\sup_{v\in V_K} |x(v) - y(v)| \cdot \sum_{v\in V_K}|x(v) - y(v)|)^{1/2} \\ &\le (\|x - y\|_\infty \cdot (\sum_{v\in V_K}x(v) + \Sigma_{v\in V_K}y(v)))^{1/2} \\ &= (2 \cdot d_\infty(x,y))^{1/2} \end{aligned}$$

and since $\|\cdot\|_2 \ge \|\cdot\|_3 \ge \cdots \ge \|\cdot\|_\infty$,

$$d_2(x,y) \ge d_3(x,y) \ge \cdots \ge d_\infty(x,y).$$

Therefore *the* $\ell_p$-*completions of* $|K|_m$, $p > 1$, *are the same as the* $c_0$-*completion, that is,* $\overline{|K|}^{\ell_p} = \overline{|K|}^{c_0}$ *for* $p > 1$.

For the $c_0$-completion, Section 2 is devoted. In relation to Proposition 0.2, the following is shown.

0.3. *Proposition. For a simplicial complex* K, *the metric space* $(|K|, d_\infty)$ *is complete if and only if* K *is finite-dimensional.*

From Propositions 0.2 and 0.3, it follows that $\overline{|K|}^{\ell_1} \neq \overline{|K|}^{c_0}$ for an infinite-dimensional simplicial complex K which contains no infinite full simplicial complex. And it is also seen that in general, $\overline{|K|}^{c_0}$ is not an ANR, actually not locally connected (2.8). This is related to the existence of arbitrarily high dimensional principal simplexes and the fact that $\overline{|K|}^{c_0}$ contains $0 \in c_0(K_V)$. In Section 2, we have the following

0.4. *Theorem. Let* K *be a simplicial complex. If* K *has no prinicpal simplex than* $\overline{|K|}^{c_0}$ *is an* AR, *in particular, contractible. And if all principal simplexes of* K *have bounded dimension then* $\overline{|K|}^{c_0}$ *is an* ANR.

0.5. *Theorem. For any simplicial complex* K, $\overline{|K|}^{c_0} \setminus \{0\}$ *is an* ANR *and the inclusion* $|K| \subset \overline{|K|}^{c_0} \setminus \{0\}$ *is a homotopy equivalence.*

By Sd K, we denote the barycentric subdivision of a simplicial complex K. Let $\theta$: $|\mathrm{Sd}\ K| \to |K|$ be the natural bijection. As well known, $\theta$: $|\mathrm{Sd}\ K|_m \to |K|_m$ is a homeomorphism. For the $\ell_1$- and $c_0$-completions of the barycentric subdivision, we have the following result in Section 3.

0.6. *Theorem. For any infinite-dimensional simplicial complex* K, *the natural homeomorphism* $\theta\colon |\mathrm{Sd}\ K|_m \to |K|_m$ *extends to a homeomorphism* $\bar{\theta}\colon \overline{|\mathrm{Sd}\ K|}^{\ell_1} \to \overline{|K|}^{\ell_1}$ *but cannot extend to any homeomorphism* $h\colon \overline{|\mathrm{Sd}\ K|}^{c_0} \to \overline{|K|}^{c_0}$.

Let $\ell_2^f$ be the dense linear subspace of the Hilbert space $\ell_2 = \ell_2(\mathbb{N})$ consisting of $\{x \in \ell_2 | x(i) = 0$ except for finitely many $i \in \mathbb{N}\}$. A *Hilbert (space) manifold* is a separable manifold modeled on the Hilbert space $\ell_2$ and simply called an $\ell_2$-*manifold*. A separable manifold modeled on the space $\ell_2^f$ is called an $\ell_2^f$-*manifold*. An $\ell_2^f$-manifold M is characterized as a dense subset of some $\ell_2$-manifold $\tilde{M}$ with the finite-dimensional compact absorption property, so-called an f-d cap set for $\tilde{M}$ (see [$\mathrm{Ch}_2$]). In [$\mathrm{Sa}_{3,4}$], the author has proved that a simplicial complex K is a combinatorial $\infty$-manifold if and only if $|K|_m$ is an $\ell_2^f$-manifold. Here a *combinatorial* $\infty$-*manifold* is a countable simplicial complex such that the star of each vertex is combinatorially equivalent to the countably infinite full simplicial complex $\Delta^\infty = F(\mathbb{N})$, that is, they have simplicially isomorphic subdivisions [$\mathrm{Sa}_2$]. In Section 4, using the result of [CDM], we see

0.7. *Proposition. The pair* $(\overline{|\Delta^\infty|}^{\ell_1}, |\Delta^\infty|_m)$ *is homeomorphic to the pair* $(\ell_2, \ell_2^f)$.

Thus we conjecture as follows:

0.8. *Conjecture. For a combinatorial* $\infty$*-manifold* K, *the* $\ell_1$*-completion* $\overline{|K|}^{\ell_1}$ *is an* $\ell_2$*-manifold and* $|K|_m$ *is an* f-d *cap set for* $\overline{|K|}^{\ell_1}$.

Similarly as the $\ell_1$-completion of $|\Delta^\infty|_m$, we can prove that $(\overline{|\Delta^\infty|}^{c_0}, |\Delta^\infty|_m)$ is homeomorphic to the pair $(\ell_2, \ell_2^f)$ but the same conjecture as 0.8 does not hold for the $c_0$-completion. In fact, let K be a non-contractible combinatorial $\infty$-manifold. Then $\overline{|K|}^{c_0} \setminus \{0\}$ is not homotopically equivalent to $\overline{|K|}^{c_0}$ by Theorems 0.4 and 0.5, hence the one-point set $\{0\}$ is not a Z-set in $\overline{|K|}^{c_0}$. Therefore $\overline{|K|}^{c_0}$ is not an $\ell_2$-manifold (cf. [$Ch_1$]).

The second half of Conjecture 0.8 is proved in Section 4 as a corollary of the second half of Theorem 0.1.

0.9. *Corollary. For a combinatorial* $\infty$*-manifold* K, $|K|_m$ *is an* f-d *cap set for the* $\ell_1$*-completion* $\overline{|K|}^{\ell_1}$.

## 1. The $\ell_1$-Completion of a Metric Complex

Recall F(V) is the all of non-empty finite subsets of V, namely, the full simplicial complex with V the set of vertices. For each real-valued function $x: V \to \mathbb{R}$, we denote

$$C_x = \{v \in V \mid x(v) \neq 0\}.$$

If $x \in c_0(V)$ then $C_x$ is countable. The set of vertices of a simplicial complex K is always denoted by $V_K$.

1.1. *Lemma. Let* K *be a simplicial complex and* $x \in \ell_1(V_K)$. *Then* $x \in \overline{|K|}^{\ell_1}$ *if and only if* $x(v) \geq 0$ *for all* $v \in V_K$, $\|x\|_1 = \Sigma_{v\in C_x} x(v) = 1$ *and* $F(C_x) \subset K$.

*Proof.* First we see the "only if" part. For each $v \in V_K$, let $v^*\colon \ell_1(V_K) \to \mathbb{R}$ be defined by $v^*(x) = x(v)$. Then clearly $v^*$ is continuous, so $x \in \overline{|K|}^{\ell_1}$ implies $x(v) = v^*(x) \geq 0$. And $\|x\|_1 = 1$ follows from the continuity of the norm $\|\cdot\|_1$. Let $A \in F(C_x)$ and choose $\varepsilon > 0$ so that $x(v) > \varepsilon$ for all $v \in A$. Since $x \in \overline{|K|}^{\ell_1}$, we have $y \in |K|$ with $\|x - y\|_1 < \varepsilon$. Then $y(v) \geq x(v) - |x(v) - y(v)| > x(v) - \varepsilon > 0$ for all $v \in A$, that is, $A \subset C_y$. This implies $A \in K$ because $C_y \in K$.

Next we see the "if" part. In case $C_x$ is finite obviously $x \in |K|$. In case $C_x$ is infinite, for any $\varepsilon > 0$ choose $A \in F(C_x)$ so that

$$\textstyle\sum_{v\in V_K\setminus A} x(v) = \|x\|_1 - \sum_{v\in A} x(v) < \frac{\varepsilon}{2} .$$

Let $v_0 \in A$ and put $\alpha = \Sigma_{v\in V_K\setminus A} x(v)$. Then $x(v_0) + \alpha \in I$. We define $y \in |K|$ as follows:

$$y(v) = \begin{cases} x(v_0) + \alpha & \text{if } v = v_0, \\ x(v) & \text{if } v \in A\setminus\{v_0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\|x - y\|_1 = 2\alpha < \varepsilon$. Therefore $x \in \overline{|K|}^{\ell_1}$.

To prove the first half of Theorem 0.1, we use a local equi-connecting map. A space X is *locally equi-connected* (LEC) provided there are a neighborhood U of the diagonal $\Delta X$ in $X^2$ and a map $\lambda\colon U \times I \to X$ called a *(local)* *equi-connecting map* such that

$$\lambda(x,y,0) = x,\ \lambda(x,y,1) = y \text{ for all } (x,y) \in U,$$

$$\lambda(x,x,t) = x \text{ for all } x \in X,\ t \in I.$$

Then a subset A of X is *λ-convex* if $A^2 \subset U$ and $\lambda(A^2 \times I) \subset A$. The following is well known.

1.2. *Lemma* [Du]. *If a metrizable space* X *has a local equi-connecting map* λ *such that each point of* X *has arbitrarily small* λ*-convex neighborhoods then* X *is an* ANR. *Moreover if* λ *is defined on* $X^2 \times I$ *then* X *is an* AR.

Now we prove the first half of Theorem 0.1.

1.3. *Theorem. For a simplicial complex* K, *the* $\ell_1$*-completion* $\overline{|K|}^{\ell_1}$ *is an* ANR.

*Proof.* Let $\mu\colon \ell_1(V_K)^2 \to \ell_1(V_K)$ be defined by

$$\mu(x,y)(v) = \min\{|x(v)|,|y(v)|\}.$$

Then μ is continuous. In fact, for each $(x,y),(x',y') \in \ell_1(V_K)^2$ and for each $v \in V_K$,

$$|\min\{|x(v)|,|y(v)|\} - \min\{|x'(v)|,|y'(v)|\}|$$
$$\leq \max\{||x(v)| - |x'(v)||,||y(v)| - |y'(v)||\}$$
$$\leq \max\{|x(v) - x'(v)|,|y(v) - y'(v)|\}$$
$$\leq |x(v) - x'(v)| + |y(v) - y'(v)|,$$

hence we have

$$\|\mu(x,y) - \mu(x',y')\|_1 \leq \|x - x'\|_1 + \|y - y'\|_1.$$

And note that $\mu(x,y) = 0$ if and only if $x(v) = 0$ or $y(v) = 0$ for each $v \in V_K$, which implies $\|x - y\|_1 = \|x\|_1 + \|y\|_1$. Then $\|x - y\|_1 < \|x\|_1 + \|y\|_1$ implies $\mu(x,y) \neq 0$. And observe $C_{\mu(x,y)} = C_x \cap C_y$ for each $(x,y) \in \ell_1(V_K)^2$. Let

$$U = \{(x,y) \in \overline{|K|}^{\ell_1} \mid \|x - y\|_1 < 2\}.$$

Then U is an open neighborhood of the diagonal $\Delta\overline{|K|}^{\ell_1}$ in $(\overline{|K|}^{\ell_1})^2$. For each $(x,y) \in U$, $\mu(x,y) \neq 0$ by the preceding observation. And it is easily seen that

$$x, \frac{\mu(x,y)}{\|\mu(x,y)\|_1} \in \overline{|F(C_x)|}^{\ell_1} \subset \overline{|K|}^{\ell_1} \text{ and}$$

$$y, \frac{\mu(x,y)}{\|\mu(x,y)\|_1} \in \overline{|F(C_y)|}^{\ell_1} \subset \overline{|K|}^{\ell_1}.$$

Since $\overline{|F(C_x)|}^{\ell_1}$ and $\overline{|F(C_y)|}^{\ell_1}$ are convex sets in $\ell_1(V_K)$, we have

$$(1-t)x + \frac{t\cdot\mu(x,y)}{\|\mu(x,y)\|_1},\ (1-t)y + \frac{t\cdot\mu(x,y)}{\|\mu(x,y)\|_1} \in \overline{|K|}^{\ell_1}$$

for any $t \in I$.

Thus we can define a local equi-connecting map $\lambda\colon U \times I \to \overline{|K|}^{\ell_1}$ as follows

$$\lambda(x,y,t) = \begin{cases} (1-2t)x + \dfrac{2t\mu(x,y)}{\|\mu(x,y)\|_1} & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (2t-1)y + \dfrac{(2-2t)\mu(x,y)}{\|\mu(x,y)\|_1} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now we show that each point of $\overline{|K|}^{\ell_1}$ has arbitrarily small $\lambda$-convex neighborhoods. Let $z \in \overline{|K|}^{\ell_1}$ and $\varepsilon > 0$. Choose an $A \in F(C_z)$ so that $\Sigma_{v\in A} z(v) > 1 - 2^{-1}\varepsilon$ and select $0 < \alpha(v) < z(v)$ for all $v \in A$ so that $\Sigma_{v\in A}\alpha(v) > 1 - 2^{-1}\varepsilon$. Let

$$W = \{x \in \overline{|K|}^{\ell_1} \mid x(v) > \alpha(v) \text{ for all } v \in A\}.$$

Then W is an open neighborhood of z in $\overline{|K|}^{\ell_1}$. For each $x,y \in W$,

$$\begin{aligned}\|x - y\|_1 &\le \sum_{v\in A}|x(v) - y(v)| + \sum_{v\in V_K\setminus A} x(v) \\ &\quad + \sum_{v\in V_K\setminus A} y(v) \\ &\le \sum_{v\in A}(x(v) - \alpha(v)) + \sum_{v\in A}(y(v) - \alpha(v)) \\ &\quad + 1 - \sum_{v\in A} x(v) + 1 - \sum_{v\in A} y(v) \\ &= 2 - 2\sum_{v\in A}\alpha(v) < \varepsilon.\end{aligned}$$

Therefore diam $W \le \varepsilon$. To see that W is $\lambda$-convex, let $(x,y,t) \in W^2 \times I$ and $v \in A$. Note $\|\mu(x,y)\|_1 \le 1$. If $t \le 1/2$,

$$\begin{aligned}\lambda(x,y,t)(v) &= (1-2t)x(v) + \frac{2t\cdot\min\{x(v),y(v)\}}{\|\mu(x,y)\|_1} \\ &\ge (1-2t)\cdot\min\{x(v),y(v)\} \\ &\quad + 2t\cdot\min\{x(v),y(v)\} \\ &= \min\{x(v),y(v)\} > \alpha(v).\end{aligned}$$

If $t \ge 1/2$, similarly $\lambda(x,y,t)(v) > \alpha(v)$. Then $\lambda(x,y,t) \in W$. Therefore W is $\lambda$-convex. The result follows from Lemma 1.2.

To prove the second half of Theorem 0.1, we use a SAP-family introduced in [$Sa_1$]. Let $\mathcal{F}$ be a family of closed sets in a space X. We call $\mathcal{F}$ a SAP-*family* for X if $\mathcal{F}$ is directed, that is, for each $F_1, F_2 \in \mathcal{F}$ there is an $F \in \mathcal{F}$ with $F_1 \cap F_2 \subset F$, and $\mathcal{F}$ has the *simplex absorption property*, that is, for each map $f: |\Delta^n| \to X$ of any n-simplex such that $f(\partial|\Delta^n|) \subset F$ for some $F \in \mathcal{F}$ and for each open cover $\mathcal{U}$ of X there exists a map $g: |\Delta^n| \to X$ such that $g(|\Delta^n|) \subset F$ for some $F \in \mathcal{F}$, $g|\partial|\Delta^n| = f|\partial|\Delta^n|$ and g is $\mathcal{U}$-near to f. Let L be a subcomplex of a simplicial complex K. We say that L is *full in* K if any simplex of K with vertices of L belongs to L. For a subcomplex L of K, we always consider $|L| \subset |K|$, that is, $x \in |L|$ is a function $x: V_L \to I$ but is considered a function $x: V_K \to I$ with $x(V_K\setminus V_L) = 0$.

1.4. *Lemma* (cf. [$Sa_1$, Lemma 3]). *Let* K *be a simplicial complex. Then the family*

$$\mathcal{F}(K) = \{|L| \; | L \text{ is a finite subcomplex of } K \text{ which is full in } K\}$$

*is a* SAP-*family for* $|K|^{\ell_1}$.

*Proof.* It is clear that $\mathcal{F}(K)$ is a direct family of closed (compact) set in $|K|^{\ell_1}$. Let $|L| \in \mathcal{F}(K)$ and define a map $\phi_L: |K|^{\ell_1} \to I$ by

$$\phi_L(x) = \textstyle\sum_{v \in V_L} x(v).$$

Then $\phi_L^{-1}(1) = |L|$. In fact, if $x \in |L|$ then $\phi_L(x) = \|x\|_1 = 1$. Conversely if $\phi_L(x) = 1$ then $C_x \subset V_L$ and $C_x \in K$ by Lemma 1.1. Since L is full in K, $C_x \in L$, which implies $x \in |L|$. Let $N(|L|,2)$ be the 2-neighborhood of $|L|$ in $|K|^{\ell_1}$, that is,

$$N(|L|,2) = \{x \in |K|^{\ell_1} \mid d_1(x,|L|) < 2\}.$$

Then $\phi_L(x) \neq 0$ for all $x \in N(|L|,2)$ because if $\phi_L(x) = 0$ then $x(v) = 0$ for all $v \in V_L$, hence for any $y \in |L|$,

$$\begin{aligned}\|x - y\|_1 &= \textstyle\sum_{v \in V_K} |x(v) - y(v)| \\ &= \textstyle\sum_{v \in V_K} x(v) + \textstyle\sum_{v \in V_K} y(v) = 2.\end{aligned}$$

We define a retraction $r_L: N(C|L|,2) \to |L|$ $(\subset |K|)$ by

$$r_L(x)(v) = \begin{cases} \dfrac{x(v)}{\phi_L(x)} & \text{if } v \in V_L, \\ 0 & \text{otherwise.} \end{cases}$$

Then for each $x \in N(|L|,2)$,

$$\begin{aligned}\|r_L(x) - x\|_1 &= \textstyle\sum_{v \in V_L} \left|\dfrac{x(v)}{\phi_L(x)} - x(v)\right| + \textstyle\sum_{v \in V_K \setminus V_L} x(v) \\ &= \left(\dfrac{1}{\phi_L(x)} - 1\right) \textstyle\sum_{v \in V_L} x(v) + 1 - \phi_L(x)\end{aligned}$$

$$= (\frac{1}{\phi_L(x)} - 1)\phi_L(x) + 1 - \phi_L(x)$$

$$= 2 - 2\phi_L(x).$$

On the other hand $1 - \phi_L(x) \leq d_1(x, |L|)$ since for any $y \in |L|$,

$$\|x - y\|_1 = \sum_{v \in V_K} |x(v) - y(v)|$$

$$= \sum_{v \in V_K \setminus V_L} x(v) + \sum_{v \in V_L} |x(v) - y(v)|$$

$$\geq 1 - \sum_{v \in V_L} x(v)$$

$$= 1 - \phi_L(x).$$

Therefore we have

$$d_1(r_L(x), x) \leq 2 \cdot d_1(x, |L|) \text{ for each } x \in N(|L|, 2).$$

By Lemma 2 in [$Sa_1$], $\mathcal{J}(K)$ is a SAP-family in $\lceil K \rceil^{\ell_1}$.

Now we prove the second half of Theorem 0.1.

1.5. *Theorem. For a simplicial complex* K, *the inclusion* $i: |K|_m \subset \lceil K \rceil^{\ell_1}$ *is a fine homotopy equivalence.*

*Proof.* By $|K|_w$, we denote the space $|K|$ with the weak (or Whitehead) topology. Then the identity of $|K|$ induces a fine homotopy equivalence $j: |K|_w \to |K|_m$ [$Sa_1$, Theorem 1]. By the same arguments in the proof of [$Sa_1$, Theorem 1] using the above lemma instead of [$Sa_1$, Lemma 3], $ij: |K|_w \to \lceil K \rceil^{\ell_1}$ is also a fine homotopy equivalence. Then the result follows from the following lemma.

1.6. *Lemma. Let* $f: X \to Y$ *and* $g: Y \to Z$ *be maps. If* f *and* gf *are fine homotopy equivalences then so is* g.

*Proof.* Let $\mathcal{U}$ be an open cover of Z. Then gf has a $\mathcal{U}$-inverse h: $Z \to X$. Let $\mathcal{V}$ be an open cover of Y which refines both $g^{-1}(\mathcal{U})$ and $g^{-1}h^{-1}f^{-1}g^{-1}(\mathcal{U})$. Then f has a $\mathcal{V}$-inverse k: $Y \to X$. Since hgf is $f^{-1}g^{-1}(\mathcal{U})$-homotopic to $id_X$, fhgfk is $g^{-1}(\mathcal{U})$-homotopic to fk which is $g^{-1}(\mathcal{U})$-homotopic to $id_Y$. Since fk is $g^{-1}h^{-1}f^{-1}g^{-1}(\mathcal{U})$-homotopic to $id_Y$, fhgfk is $g^{-1}(\mathcal{U})$-homotopic to fhg. Hence fhg is st $g^{-1}(\mathcal{U})$-homotopic to $id_Y$. Recall gfh is $\mathcal{U}$-homotopic to $id_Z$. Therefore g is a fine homotopy equivalence.

## 2. The $c_0$-Completion of a Metric Complex

As seen in Introduction, for any $p > 1$, the $\ell_p$-completion of a metric simplicial complex is the same as the $c_0$-completion. In this section, we clarify the difference between the $\ell_1$-completion and the $c_0$-completion. The "only if" part of Proposition 0.3 is contained in the following

2.1. *Proposition. Let* K *be a simplicial complex. Then* K *is infinite-dimensional if and only if* $0 \in \overline{|K|}^{c_0}$.

*Proof.* To see the "if" part, let $n \in \mathbb{N}$. From $0 \in \overline{|K|}^{c_0}$, we have $x \in |K|$ with $\|x\|_\infty < n^{-1}$. Then $C_x \in K$ and $\dim C_x \geq n$ because

$$1 = \sum_{v \in C_x} x(v) \leq \|x\|_\infty(\dim C_x + 1) < n^{-1}(\dim C_x + 1).$$

Therefore K is infinite-dimensional.

To see the "only if" part, let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ so that $(n+1)^{-1} < \varepsilon$. Since K is infinite-dimensional, we have $A \in K$ with $\dim A = n$. Let $\hat{A}$ be the barycenter of $|A|$, that is,

$$\hat{A}(v) = \begin{cases} (n+1)^{-1} & \text{if } v \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\hat{A}\|_\infty = (n+1)^{-1} < \varepsilon$. Hence $0 \in \overline{|K|}^{c_0}$.

2.2. *Lemma. Let* K *be a simplicial complex and* $x \in \overline{|K|}^{c_0}$. *Then* $x(v) \geq 0$ *for all* $v \in V_K$, $\|x\|_1 = \Sigma_{v \in C_x} x(v) \leq 1$ *and* $F(C_x) \subset K$.

*Proof.* The first and the last conditions can be seen similarly as the "only if" part of Lemma 1.1. To see the second condition, assume $1 < \Sigma_{v \in C_x} x(v) \leq \infty$. Then there are $v_1, \cdots, v_n \in C_x$ such that $\Sigma_{i=1}^n x(v_i) > 1$. Since $x \in \overline{|K|}^{c_0}$, we have $y \in |K|$ with

$$\|x - y\|_\infty < n^{-1}(\textstyle\sum_{i=1}^n x(v_i) - 1).$$

Then it follows that

$$\begin{aligned} \textstyle\sum_{i=1}^n y(v_i) &\geq \textstyle\sum_{i=1}^n x(v_i) - \textstyle\sum_{i=1}^n |x(v_i) - y(v_i)| \\ &\geq \textstyle\sum_{i=1}^n x(v_i) - n \cdot \|x - y\|_\infty > 1. \end{aligned}$$

This is contrary to $y \in |K|$. Therefore $\Sigma_{v \in C_x} x(v) \leq 1$.

Now we prove the "if" part of Proposition 0.3, that is,

2.3. *Proposition. Let* K *be a finite-dimensional simplicial complex. Then* $\overline{|K|}^{c_0} = |K|$, *that is,* $(|K|, d_\infty)$ *is complete.*

*Proof.* Let dim K = n and $x \in \overline{|K|}^{c_0}$. By Proposition 2.1, $x \neq 0$, that is, $C_x \neq \emptyset$. And $C_x$ is finite, otherwise K contains an (n+1)-simplex by Lemma 2.2. Therefore $C_x \in K$ by Lemma 2.2. For any $\varepsilon > 0$, we have $y \in |K|$ with $\|x - y\|_\infty < 2^{-1}(n+1)^{-1}\varepsilon$. Note $C_x \cup C_y$ contains at most

2(n+1) vertices. Then it follows that

$$\begin{aligned} |\textstyle\sum_{v\in C_x} x(v) - 1| &= |\textstyle\sum_{v\in V_K} x(v) - \sum_{v\in V_K} y(v)| \\ &\le \textstyle\sum_{v\in V_K} |x(v) - y(v)| \\ &= \textstyle\sum_{v\in C_x \cup C_y} |x(v) - y(v)| \\ &\le 2(n+1)\cdot\|x - y\|_\infty < \varepsilon. \end{aligned}$$

Therefore $\|x\|_1 = \Sigma_{v\in C_x} x(v) = 1$. By Lemma 2.2, $x(v) \ge 0$ for all $v \in V_K$. Hence $x \in |K|$.

Thus Proposition 0.3 is obtained. As a corollary, we have the following

2.4. *Corollary. Let* L *be a finite-dimensional subcomplex of a simplicial complex* K. *Then* $|L|$ *is closed in* $\overline{|K|}^{c_0}$.

Before proving Theorems 0.4 and 0.5, we decide the difference between the $\ell_1$-completion and the $c_0$-completion as sets. Let K be a simplicial complex and let $A \in K$. The *star* St(A) of A is the subcomplex defined by

$$St(A) = \{B \in K \mid A, B \subset C \text{ for some } C \in K\}.$$

We say that A is *principal* if $A \not\subset B$ for any $B \in K\setminus\{A\}$, that is, A is *maximal* with respect to $\subset$. By Max(K), we denote all of principal simplexes of K. We define the subcomplexes ID(K) and P(K) of K as follows:

$$ID(K) = \{A \in K \mid \dim St(A) = \infty\},$$

$$P(K) = \{A \in K \mid A \subset B \text{ for some } B \in Max(K)\}.$$

Then clearly $K = P(K) \cup ID(K)$. Observe $ID(K) = K$ if and only if $P(K) = \emptyset$, however $P(K) = K$ does not imply $ID(K) = \emptyset$

(the converse implication obviously holds). For example, let

$$K_1 = F(\{0,1\}),\ K_2 = F(\{0,2,3\}),$$
$$K_3 = F(\{0,4,5,6\}),\cdots$$

and let $K = \bigcup_{n\in\mathbb{N}} K_n$. Then $P(K) = K$ but $\dim St(\{0\}) = \infty$. In general, for any $A,B \in K$, $St(A) \subset St(B)$ if and only if $B \subset A$. Then $ID(K) = \emptyset$ if and only if $\dim St(\{v\}) < \infty$ for each $v \in V_K$, that is, $K$ is locally finite-dimensional.

2.5. *Theorem. Let* $K$ *be an infinite-dimensional and locally finite-dimensional simplicial complex, namely* $ID(K) = \emptyset$, *then* $\overline{|K|}^{c_0} = |K| \cup \{0\}$.

*Proof.* By Proposition 2.1, $|K| \cup \{0\} \subset \overline{|K|}^{c_0}$. Let $x \in \overline{|K|}^{c_0} \setminus |K|$. Assume $x \neq 0$, that is, $C_x \neq \emptyset$. From $ID(K) = \emptyset$, $K$ has no infinite full simplicial complex. Then $C_x$ is finite because $F(C_x) \subset K$ by Lemma 2.2. This implies $C_x \in K$. Put $\dim St(C_x) = n$. From $x \notin |K|$, it follows $\sum_{v\in C_x} x(v) < 1$. Let

$$\delta = \min\{(n+1)^{-1}(1 - \textstyle\sum_{v\in C_x} x(v)),\ \min_{v\in C_x} x(v)\} > 0.$$

If $\|x - y\|_\infty < \delta$ then $y(v) > 0$ for all $v \in C_x$, that is, $C_x \subset C_y$. From $\dim St(C_x) = n$, we have $\dim C_y \leq n$. Hence

$$\begin{aligned}\textstyle\sum_{v\in C_y} y(v) &\leq \textstyle\sum_{v\in C_y} x(v) + \sum_{v\in C_y} |x(v) - y(v)| \\ &\leq \textstyle\sum_{v\in C_x} x(v) + (\dim C_y + 1)\cdot\|x - y\|_\infty \\ &< \textstyle\sum_{v\in C_x} x(v) + (n + 1)\delta \\ &\leq \textstyle\sum_{v\in C_x} x(v) + (1 - \sum_{v\in C_x} x(v)) = 1.\end{aligned}$$

This is contrary to $y \in |K|$. Therefore $x = 0$.

2.6. *Lemma. Let* K *be a simplicial complex with no principal simplex, namely* $ID(K) = K$. *Then*

$$\overline{|K|}^{c_0} = I \cdot \overline{|K|}^{\ell_1} = \{tx \mid x \in \overline{|K|}^{\ell_1},\ t \in I\}.$$

*Proof.* Let $x \in \overline{|K|}^{c_0}$. If $x = 0$ then clearly $x \in I \cdot \overline{|K|}^{\ell_1}$. If $x \neq 0$ then $\|x\|_1^{-1} x \in \overline{|K|}^{\ell_1}$ by Lemmas 2.2 and 1.1. Since $\|x\|_1 \leq 1$ by Lemma 2.2, $x = \|x\|_1(\|x\|_1^{-1}x) \in I \cdot \overline{|K|}^{\ell_1}$. Conversely let $x \in \overline{|K|}^{\ell_1}$ and $t \in I$. For any $\varepsilon > 0$, we have $y \in |K|$ with $\|x - y\|_1 < \varepsilon$, hence $\|x - y\|_\infty < \varepsilon$. Choose $n \in \mathbb{N}$ so that $(n+1)^{-1} < \varepsilon$. Since $C_y \in K = ID(K)$ we have $A \in K$ such that $C_y \subset A$ and $\dim A \geq n$. Let

$$z = ty + (1-t)\hat{A} \in |A| \subset |K|,$$

where $\hat{A}$ is the barycenter of $|A|$. Since $\|\hat{A}\|_\infty \leq (n+1)^{-1} < \varepsilon$ (see the proof of Proposition 2.1),

$$\begin{aligned} \|tx - z\|_\infty &= \|tx - ty - (1-t)\hat{A}\|_\infty \\ &\leq t \cdot \|x - y\|_\infty + (1-t) \cdot \|\hat{A}\|_\infty \\ &< t\varepsilon + (1-t)\varepsilon = \varepsilon. \end{aligned}$$

Therefore $tx \in \overline{|K|}^{c_0}$.

In Lemma 2.6, we should remark that $\overline{|K|}^{c_0} \neq I \cdot \overline{|K|}^{\ell_1}$ as spaces. In fact, for each $n \in \mathbb{N}$, let $A_n \in K$ with $\dim A = n$. Then the set $\{\hat{A}_n \mid n \in \mathbb{N}\}$ is discrete in $\overline{|K|}^{\ell_1}$ but has the cluster point 0 in $\overline{|K|}^{c_0}$.

As general case, we have the following

2.7. *Theorem. Let* K *be a simplicial complex with* $ID(K) = \emptyset$. *Then* $\overline{|K|}^{c_0} = |P(K)| \cup I \cdot \overline{|ID(K)|}^{\ell_1}$.

*Proof*. Since $I\cdot\overline{\rceil ID(K)\lceil}^{\ell_1} = \overline{\rceil ID(K)\lceil}^{c_0} \subset \overline{\rceil K\lceil}^{c_0}$ by Lemma 2.5, we have $|P(K)| \cup I\cdot\overline{\rceil ID(K)\lceil}^{\ell_1} \subset \overline{\rceil K\lceil}^{c_0}$. Let $x \in \overline{\rceil K\lceil}^{c_0}\setminus|K|$. If $x = 0$ then clearly $x \in I\cdot\overline{\rceil ID(K)\lceil}^{\ell_1}$. In case $x \neq 0$, if $C_x$ is finite and $C_x \notin ID(K)$, $C_x \in K\setminus ID(K)$ by Lemma 2.2, hence $\dim St(C_x) < \infty$. The arguments in the proof of Theorem 2.5 lead a contradiction. Thus $C_x$ is infinite or $C_x \in ID(K)$. In both cases, clearly $F(C_x) \subset ID(K)$. Then using Lemmas 1.1 and 2.2 as in the proof of Lemma 2.6, we can see $x \in I\cdot\overline{\rceil ID(K)\lceil}^{\ell_1}$. Since $|K| = |P(K)| \cup |ID(K)|$, we have $\overline{\rceil K\lceil}^{c_0} \subset |P(K)| \cup I\cdot\overline{\rceil ID(K)\lceil}^{\ell_1}$.

Next we show that Theorem 0.1 does not hold for the $c_0$-completion.

2.8. *Lemma*. *Let* $X$ *be a dense subspace of a Hausdorff space* $\tilde{X}$. *Then any locally compact open subset of* $X$ *is open in* $\tilde{X}$. *Hence for a locally compact set* $A \subset X$, $\mathrm{int}_{\tilde{X}}A = \mathrm{int}_X A$.

*Proof*. Let $Y$ be a locally compact open subset of $X$ and $y \in Y$. We have an open set $U$ in $X$ such that $y \in U \subset Y$ and $c\ell_Y U$ is compact. Let $\tilde{U}$ be an open set in $\tilde{X}$ with $U = \tilde{U} \cap X$. Since $c\ell_Y U$ is closed in $\tilde{X}$, $\tilde{U}\setminus c\ell_Y U$ is open in $\tilde{X}$. Observe that

$$(\tilde{U}\setminus c\ell_Y U) \cap X = U\setminus c\ell_Y U = \emptyset.$$

Then $\tilde{U}\setminus c\ell_Y U = \emptyset$ because $X$ is dense in $\tilde{X}$. Hence $\tilde{U}\setminus X = \emptyset$, that is, $\tilde{U} = U$. Therefore $Y$ is open in $\tilde{X}$.

Let $K$ be a simplicial complex. Then for each $A \in K$,

$$\mathrm{int}_{\overline{\rceil K\lceil}^{c_0}}|A| = \mathrm{int}_{|K|_m}|A| = |A| \cup \{|B| \mid B \in K, B \not\subset A\}.$$

Thereby abbreviating subscripts, we write $\mathrm{int}|A|$ and also $\mathrm{bd}|A| = |A| \setminus \mathrm{int}|A|$. Notice that $\mathrm{int}|A| \neq \emptyset$ if and only if A is principal. We define the subcomplex BP(K) of P(K) as follows:

$$BP(K) = \{A \in P(K) \mid |A| \subset \mathrm{bd}|B| \text{ for some } B \in \mathrm{Max}(K)\}.$$

By the following proposition, we can see that Theorem 0.1 does not hold for the $c_0$-completion.

2.8. *Proposition. Let* K *be a simplicial complex. If* $\dim P(K) = \infty$ *and* $\dim BP(K) < \infty$ *then* $\overline{|K|}^{c_0}$ *is not locally connected at* 0.

*Proof.* By Corollary 2.4, $|BP(K)|$ is closed in $\overline{|K|}^{c_0}$. Put

$$\delta = d_\infty(0, |BP(K)|) > 0.$$

and let U be a neighborhood of 0 in $\overline{|K|}^{c_0}$ with daim $U > \delta$. Similarly as the proof of Proposition 2.1, we have a principal simplex $A \in K$ with $\hat{A} \in U$. Since $\mathrm{bd}|A| \subset |BP(K)|$, $U \cap \mathrm{bd}|A| = \emptyset$, hence $U \cap |A|$ is open and closed in U. And $\emptyset \neq U \cap |A| \subsetneq U$ because $\hat{A} \in U \cap |A|$ and $0 \notin U \cap |A|$. Therefore U is disconnected.

Now we prove the first statement of Theorem 0.4.

2.9. *Theorem. Let* K *be a simplicial complex with no principal simplex. Then the* $c_0$*-completion* $\overline{|K|}^{c_0}$ *is an* AR.

*Proof.* (Cf. the proof of Theorem 1.3). Define $\mu: c_0(V_K)^2 \to c_0(V_K)$ exactly as Theorem 1.3, that is, as follows:

$$\mu(x,y)(v) = \min\{|x(v)|, |y(v)|\}.$$

Then for each $(x,y),(x',y') \in c_0(V_K)^2$,

$$\|\mu(x,y) - \mu(x',y')\|_\infty \leq \max\{\|x - x'\|_\infty, \|y - y'\|_\infty\},$$

hence $\mu$ is continuous. Here we define an equi-connecting map $\lambda: c_0(V_K)^2 \times I \to c_0(V_K)$ as follows:

$$\lambda(x,y,t) = \begin{cases} (1-2t)x + 2t\mu(x,y) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (2t-1)y + (2-2t)\mu(x,y) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Using Lemmas 1.1 and 2.6, it is easy to see that $\lambda((\lceil K\rceil^{c_0})^2 \times I) \subset \lceil K\rceil^{c_0}$. Let $z \in \lceil K\rceil^{c_0}$ and $\varepsilon > 0$. Then the $\varepsilon$-neighborhood of $z$ is $\lambda$-convex. In fact, let $x,y \in \lceil K\rceil^{c_0}$ such that $\|x - z\|_\infty, \|y - z\|_\infty < \varepsilon$. Observe

$$\begin{aligned}\|\mu(x,y) - z\|_\infty &= \|\mu(x,y) - \mu(z,z)\|_\infty \\ &\leq \max\{\|x - z\|_\infty, \|y - z\|_\infty\} < \varepsilon.\end{aligned}$$

For $0 \leq t \leq 1/2$,

$$\begin{aligned}\|\lambda(x,y,t) - z\|_\infty &= \|(1 - 2t)x + 2t\mu(x,y) - z\|_\infty \\ &\leq (1 - 2t)\|x - z\|_\infty + 2t\|\mu(x,y) \\ &\quad - z\|_\infty < \varepsilon.\end{aligned}$$

For $1/2 \leq t \leq 1$, similarly $\|\lambda(x,y,t) - z\|_\infty < \varepsilon$. By Lemma 1.2, $\lceil K\rceil^{c_0}$ is an AR.

As corollaries, we have the second statement of Theorem 0.4 and the first half of Theorem 0.5.

2.10. *Corollary. Let* K *be a simplicial complex with* dim P(K) $< \infty$. *Then the* $c_0$*-completion* $\lceil K\rceil^{c_0}$ *is an* ANR.

*Proof.* By Corollary 2.4, $|P(K)|$ is closed in $\lceil K\rceil^{c_0}$. Then $\lceil K\rceil^{c_0} = \lceil P(K)\rceil^{c_0} \cup \lceil ID(K)\rceil^{c_0} = |P(K)| \cup \lceil ID(K)\rceil^{c_0}$.

By Theorem 2.9, $\rceil ID(K) \lceil^{c_0}$ is an AR. Since $|P(K)|$ and $|P(K)| \cap \rceil ID(K) \lceil^{c_0} = |P(K) \cap ID(K)|$ are ANR's, so is $\rceil K \lceil^{c_0}$ (cf., [Hu]).

2.11. *Corollary. For any simplicial complex* K, $\rceil K \lceil^{c_0} \setminus \{0\}$ *is an* ANR.

*Proof.* By Theorems 2.5 and 2.7, $\rceil K \lceil^{c_0} \setminus \{0\} = |P(K)| \cup (\rceil ID(K) \lceil^{c_0} \setminus \{0\})$. Then similarly as the above corollary, we have the result.

The following is the second half of Theorem 0.5.

2.12. *Theorem. For any simplicial complex* K, *the inclusion* $i\colon |K|_m \subset \rceil K \lceil^{c_0} \setminus \{0\}$ *is a homotopy equivalence.*

*Proof.* Since both spaces are ANR's, by the Whitehead Theorem [Wh], it is sufficient to see that $i\colon |K|_m \subset \rceil K \lceil^{c_0} \setminus \{0\}$ is a weak homotopy equivalence, that is, i induces isomorphisms

$$i_*\colon \pi_n(|K|_m) \to \pi_n(\rceil K \lceil^{c_0} \setminus \{0\}),\ n \in \mathbb{N}.$$

Let $\mathcal{J}(K)$ be the family of Lemma 1.4. And for each $|L| \in \mathcal{J}(K)$, let $\phi_L\colon \rceil K \lceil^{c_0} \to I$ be the map defined as Lemma 1.4. (Since $V_L$ is finite, the continuity of $\phi_L$ is clear.) Then $\phi_L^{-1}(1) = L$. Let

$$U(L) = \{x \in \rceil K \lceil^{c_0} \mid C_x \cap V_L \neq \emptyset\}.$$

Then U(L) is an open neighborhood of $|L|$ in $\rceil K \lceil^{c_0}$. In fact, for each $x \in U(L)$, choose $v \in C_x \cap V_L$. If $\|x - y\|_\infty < x(v)$ then $v \in C_y \cap V_L$ because $y(v) > 0$, hence $y \in U(L)$. Since $\phi_L(x) \neq 0$ for each $x \in U(L)$, we can define a retraction

$r_L\colon U(L) \to |L|$ similarly as Lemma 1.4. Observe for each $x \in U(L)$ and $t \in I$,

$$C_{(1-t)x + tr_L(x)} \subset C_x.$$

Then using Lemma 1.1 and Theorem 2.7, it is easily seen that $(1-t)x + tr_L(x) \in \overline{|K|}^{c_0}\setminus\{0\}$. Since

$$C_{(1-t)x + tr_L(s)} \cap V_L \neq \emptyset,$$

it follows that $(1-t)x + tr_L(x) \in U(L)$. Thus we have a deformation $h_L\colon U(L) \times I \to U(L)$ defined by

$$h_L(x,t) = (1-t)x + tr_L(x).$$

It is easy to see that $\overline{|K|}^{c_0}\setminus\{0\} = \cup\{U(L) \mid |L| \in \mathcal{J}(K)\}$.

Now we show that $i_*\colon \pi_n(|K|_m) \to \pi_n(\overline{|K|}^{c_0}\ \{0\})$ is an isomorphism. By $S^n$ and $B^{n+1}$, we denote the unit n-sphere and the unit (n+1)-ball. Let $\alpha\colon S^n \to |K|_m$ and $\beta\colon B^{n+1} \to \overline{|K|}^{c_0}\setminus\{0\}$ be maps such that $\beta|S^n = \alpha$. Note $\alpha$ is homotopic to a map $\alpha'\colon S^n \to |K|_m$ such that $\alpha'(S^n) \subset |L'|$ for some $|L'| \in \mathcal{J}(K)$. By the Homotopy Extension Theorem, $\alpha'$ extends to a map $\beta'\colon B^{n+1} \to \overline{|K|}^{c_0}\setminus\{0\}$. From compactness of $\beta'(B^{n+1})$, we have an $|L| \in \mathcal{J}(K)$ such that $|L'| \subset |L|$ and $\beta'(B^{n+1}) \subset U(L)$. Then $\alpha'$ extends to $r_L\beta'\colon B^{n+1} \to |L| \subset |K|_m$. Therefore $i_*$ is a monomorphism. Next let $\alpha\colon S^n \to \overline{|K|}^{c_0}\setminus\{0\}$ be a map. From compactness of $\alpha(S^n)$, we have an $|L| \in \mathcal{J}(K)$ such that $\alpha(S^n) \subset U(L)$. Then $r_L\alpha\colon S^n \to |L| \subset |K|_m$ is homotopic to $\alpha$ in $U(L)$. This implies that $i_*$ is an epimorphism.

## 3. Completions of the Barycentric Subdivisions

By Sd K, we denote the barycentric subdivision of a simplicial complex K, that is, Sd K is the collection of non-empty finite sets $\{A_0, \cdots, A_n\} \subset K = V_{\mathrm{Sd}\,K}$ such that $A_0 \subsetneq \cdots \subsetneq A_n$. We have the natural homeomorphism $\theta\colon |\mathrm{Sd}\,K|_m \to |K|_m$ defined by

$$\theta(\xi)(v) = \sum_{v \in A \in K} \frac{\xi(A)}{\dim A + 1}.$$

The inverse $\theta^{-1}\colon |K|_m \to |\mathrm{Sd}\,K|_m$ of $\theta$ is given by

$$\theta^{-1}(x)(A) = (\dim A + 1)\cdot\max\{\min_{v \in A} x(v) - \max_{v \notin A} x(v), 0\}.$$

In fact, let $x \in |K|$ and write $C_x = \{v_0, \cdots, v_n\}$ so that $x(v_0) \geq \cdots \geq x(v_n)$. For each $v \in V_K$,

$$\theta\theta^{-1}(x)(v) = \sum_{v \in A \in K} \max\{\min_{u \in A} x(u) - \max_{u \notin A} x(u), 0\}.$$

If $v \notin C_x$ then $\min_{u \in A} x(u) = 0$ for $v \in A \in K$, hence $\theta\theta^{-1}(x)(v) = 0$. For $A \in K$, if $A \neq \{v_0, \cdots, v_j\}$ for any $j = 0, \cdots, n$ then $\min_{u \in A} x(u) - \max_{u \notin A} x(u) = 0$. Hence

$$\theta\theta^{-1}(x)(v_i) = \sum_{j=i}^{n-1}(x(v_j) - x(v_{j+1})) + x(v_n) = x(v_i).$$

Therefore $\theta\theta^{-1}(x) = x$.

Conversely let $\xi \in |\mathrm{Sd}\,K|$ and write $C_\xi = \{A_0, \cdots, A_n\}$ so that $A_0 \subsetneq \cdots \subsetneq A_n$. For each $A \in K$,

$$\theta^{-1}\theta(\xi)(A) = (\dim A + 1)\cdot\max\{\min_{v \in A} \theta(\xi)(v) - \max_{v \notin A} \theta(\xi)(v), 0\}.$$

If $A \notin C_\xi$ then $A \not\subset A_n$ or $A_{i-1} \not\supset A \subsetneq A_i$ for some $i = 0, \cdots, n$, where $A_{-1} = \emptyset$. In case $A \not\subset A_n$, we have $v_0 \in A \setminus A_n$. If $v_0 \in B \in K$ then $\xi(B) = 0$ because $B \neq A_i$ for any $i = 0, \cdots, n$. Therefore

$$\theta(\xi)(v_0) = \sum_{v_0 \in B \in K} \frac{\xi(B)}{\dim B + 1} = 0,$$

hence $\theta^{-1}\theta(\xi)(A) = 0$. Observe if $v \in A_i \setminus A_{i-1}$ then

$$\theta(\xi)(v) = \sum_{v \in B \in K} \frac{\xi(B)}{\dim B + 1} = \sum_{j=i}^{n} \frac{\xi(A_j)}{\dim A_j + 1}.$$

In case $A_{i-1} \not\supset A \subsetneqq A_i$ for some $i = 0,\cdots,n$, we have $v_1 \in A \setminus A_{i-1}$ and $v_2 \in A_i \setminus A$. Since

$$\min_{v \in A} \theta(\xi)(v) \leq \theta(\xi)(v_1) = \sum_{j=i}^{n} \frac{\xi(A_j)}{\dim A_j + 1}$$
$$= \theta(\xi)(v_2) \leq \max_{v \notin A} \theta(\xi)(v),$$

it follows $\theta^{-1}\theta(\xi)(A) = 0$. It is easy to see that

$$\min_{v \in A_i} \theta(\xi)(v) = \sum_{j=i}^{n} \frac{\xi(A_j)}{\dim A_j + 1} \text{ and}$$
$$\max_{v \notin A_i} \theta(\xi)(v) = \sum_{j=i+1}^{n} \frac{\xi(A_j)}{\dim A_j + 1}.$$

Thus we have

$$\theta^{-1}\theta(\xi)(A_i) = (\dim A_i + 1)\left(\sum_{j=i}^{n} \frac{\xi(A_j)}{\dim A_j + 1} - \sum_{j=i+1}^{n} \frac{\xi(A_j)}{\dim A_j + 1}\right)$$
$$= (\dim A_i + 1) \frac{\xi(A_i)}{\dim A_i + 1} = \xi(A_i).$$

Therefore $\theta^{-1}\theta(\xi) = \xi$.

3.1. *Theorem. For a simplicial complex* K, *the natural homeomorphism* $\theta\colon |\mathrm{Sd}\ K|_m \to |K|_m$ *induces a homeomorphism* $\bar{\theta}\colon \overline{|\mathrm{Sd}\ K|}^{\ell_1} \to \overline{|K|}^{\ell_1}$.

*Proof.* For each $\xi,\eta \in |\mathrm{Sd}\ K|$,

$$\|\theta(\xi) - \theta(\eta)\|_1 = \sum_{v \in V_K} \left| \sum_{v \in A \in K} \frac{\xi(A)}{\dim A + 1} - \sum_{v \in A \in K} \frac{\eta(A)}{\dim A + 1} \right|$$
$$\leq \sum_{v \in V_K} \sum_{v \in A \in K} \frac{|\xi(A) - \eta(A)|}{\dim A + 1}$$
$$= \sum_{A \in K} |\xi(A) - \eta(A)| = \|\xi - \eta\|_1.$$

Then $\theta$ is uniformly continuous with respect to the metrics $d_1$ on $|\mathrm{Sd}\ K|_m$ and $|K|_m$. Hence $\theta$ induces a map

$\overline{\theta}: \overline{|\mathrm{Sd}\ K|}^{\ell_1} \to \overline{|K|}^{\ell_1}$. (However, we should remark that $\theta^{-1}$ is not uniformly continuous in case dim $K = \infty$. In fact, let $A \in K$ be an n-simplex and $B \subset A$ an (n-1)-face. Then for the barycenters $\hat{A} \in |A|$ and $\hat{B} \in |B|$, we have $\|\hat{A} - \hat{B}\|_1 = 2/n$ but $\|\theta^{-1}(\hat{A}) - \theta^{-1}(\hat{B})\|_1 = \|A - B\|_1 = 2$.) Since $\theta$ is injective, so is $\overline{\theta}$. In order to show that $\overline{\theta}$ is surjective, it suffices to see $\overline{|K|}^{\ell_1} \setminus |K| \subset \overline{\theta}(\overline{|\mathrm{Sd}\ K|}^{\ell_1})$. Let $x \in \overline{|K|}^{\ell_1} \setminus |K|$. Then $C_x$ is infinite. Otherwise $C_x \in |K|$ by Lemma 2.2, so $x \in |K|$ because $x(v) \geq 0$ for all $v \in V_K$ and $\|x\|_1 = 1$. Recall $C_x$ is countable. Then write $C_x = \{v_n | n \in \mathbb{N}\}$ so that $x(v_1) \geq x(v_2) \geq \cdots > 0$. Observe

$$n \cdot x(v_{n+1}) + \sum_{i=n+1}^{\infty} x(v_i) \leq \sum_{i=1}^{\infty} x(v_i) = 1.$$

Moreover $n \cdot x(v_n)$ converges to 0. If not, we have $\varepsilon > 0$ and $1 \leq n_1 < n_2 < \cdots$ such that $n_i x(v_{n_i}) > \varepsilon$ for each $i \in \mathbb{N}$. We may assume $\sum_{n>n_1} x(v_n) < \varepsilon/2$. Since

$$\begin{aligned}(n_{i+1} - n_i) \frac{\varepsilon}{n_{i+1}} &\leq (n_{i+1} - n_i) \cdot x(v_{n_{i+1}}) \\ &\leq \sum_{n=n_i+1}^{n_{i+1}} x(v_n) < \frac{\varepsilon}{2},\end{aligned}$$

$2(n_{i+1} - n_i) < n_{i+1}$ hence $n_{i+1} < 2n_i$. Observe

$$\begin{aligned}&\sum_{n=n_1}^{n_{i+1}-1} \frac{\varepsilon}{2n} \\ &= \left(\frac{1}{2n_1} + \cdots + \frac{1}{2(2n_2-1)}\right)\varepsilon + \cdots + \left(\frac{1}{2n_i} + \cdots + \frac{1}{2(n_{i+1}-1)}\right)\varepsilon \\ &< \frac{n_2-n_1}{2n_1} \cdot \varepsilon + \cdots + \frac{n_{i+1}-n_i}{2n_i} \cdot \varepsilon \\ &< \frac{n_2-n_1}{n_2} \cdot \varepsilon + \cdots + \frac{n_{i+1}-n_i}{n_{i+1}} \cdot \varepsilon \\ &< (n_2-n_1) \cdot x(v_{n_2}) + \cdots + (n_{i+1}-n_i) \cdot x(v_{n_{i+1}})\end{aligned}$$

$$\leq (x(v_{n_1+1}) + \cdots + x(v_{n_2})) + \cdots + (x(v_{n_i+1}) + \cdots + x(v_{n_{i+1}}))$$

$$= \sum_{n=n_1+1}^{n_{i+1}} x(v_n) < \frac{\varepsilon}{2} .$$

This contradicts to the fact $\Sigma_{n=n_1}^{\infty} n^{-1}$ is not convergent.

For each $n \in \mathbb{N}$, let $A_n = \{v_1, \cdots, v_n\}$. Define $\xi_n \in |\mathrm{Sd}\ K|$, $n \in \mathbb{N}$ and $\xi \in \ell_1(K)$ as follows:

$$\xi_n(A) = \begin{cases} i(x(v_i) - x(v_{i+1})) & \text{if } A = A_i,\ i \leq n, \\ (n+1)x(v_{n+1}) + \sum_{i=n+2}^{\infty} x(v_i) & \text{if } A = A_{n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\xi(A) = \begin{cases} n(x(v_n) - x(v_{n+1})) & \text{if } A = A_n,\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $n \cdot x(v_n)$ converges to 0, we have

$$\| \xi_n - \xi \|_1 = 2 \sum_{i=n+2}^{\infty} x(v_i) .$$

Then $\| \xi_n - \xi \|_1$ converges to 0, that is, $\xi_n$ converges to $\xi$. Hence $\xi \in \overline{|\mathrm{Sd}\ K|}^{\ell_1}$. It is easy to see that

$$\theta(\xi_n)(v) = \begin{cases} x(v_i) + \dfrac{\sum_{n+2}^{\infty} x(v_i)}{n+1} & \text{if } v = v_i,\ i \leq n+1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\| \theta(\xi_n) - x \|_1 = 2 \sum_{n+2}^{\infty} x(v_i) .$$

Then $\theta(\xi_n)$ converges to $x$. This implies $\overline{\theta}(\xi) = x$.

Finally, we see the continuity of $\theta^{-1}$. Let $x \in \overline{|K|}^{\ell_1}$, $\xi = \theta^{-1}(x) \in \overline{|\mathrm{Sd}\ K|}^{\ell_1}$ and $\varepsilon > 0$. Write $C_x = \{v_i | i \in \mathbb{N}\}$ so that $x(v_1) \geq x(v_2) \geq \cdots$. Recall $i \cdot x(v_i)$ converges to 0. We can choose $n \in \mathbb{N}$ so that $(n+1) \cdot x(v_{n+1}) < \varepsilon/6$,

$\Sigma_{i=n+2}^{\infty} x(v_i) < \varepsilon/6$ and $x(v_n) > x(v_{n+1})$. Put

$$\delta = \min\{x(v_i) - x(v_{i+1}) \mid x(v_i) > x(v_{i+1}), \quad i = 1,\cdots,n\} > 0.$$

Let $y \in \overline{|K|}^{\ell_1}$ with

$$\|x - y\|_1 < \min\{\frac{\delta}{2}, \frac{\varepsilon}{6n(n+1)}\}$$

and $\eta = \overline{\theta}^{-1}(y) \in \overline{|\mathrm{Sd}\ K|}^{\ell_1}$. Remark that for $1 \le i < j \le n+1$, $x(v_i) > x(v_j)$ implies $y(v_i) > y(v_j)$ because

$$y(v_i) - y(v_j) > (x(v_i) - \frac{\delta}{2}) - (x(v_j) + \frac{\delta}{2})$$
$$= (x(v_i) - x(v_j)) - \delta > 0.$$

Then, reordering $v_1,\cdots,v_n$, we can assume that

$$y(v_1) \ge y(v_2) \ge \cdots \ge y(v_n) > y(v_{n+1}).$$

For each $i \in \mathbb{N}$, let $A_i = \{v_1,\cdots,v_i\}$. Then $C_\xi \subset \{A_i \mid i \in \mathbb{N}\}$,

$$\xi(A_i) = i\cdot(x(v_i) - x(v_{i+1})) \text{ for all } i \in \mathbb{N}, \text{ and}$$
$$\eta(A_i) = i\cdot(y(v_i) - y(v_{i+1})) \text{ for } i = 1,\cdots,n.$$

Therefore

$$\sum_{i=1}^{n} |\xi(A_i) - \eta(A_i)|$$
$$= \sum_{i=1}^{n} |i\cdot(x(v_i) - x(v_{i+1})) - i\cdot(y(v_i) - y(v_{i+1}))|$$
$$\le \sum_{i=1}^{n} i\cdot|x(v_i) - y(v_i)| + \sum_{i=1}^{n} i\cdot|x(v_{i+1}) - y(v_{i+1})|$$
$$\le 2(\sum_{i=1}^{n} i)\cdot\|x - y\|_1 = n(n + 1)\cdot\|x - y\|_1 < \frac{\varepsilon}{6}.$$

Since $i\cdot x(v_i)$ converges to 0,

$$\sum_{i=n+1}^{\infty} \xi(A_i) = (n+1)x(v_{n+1}) + \sum_{i=n+2}^{\infty} x(v_i) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Then $\sum_{i=1}^{n} \xi(A_i) = \|\xi\|_1 - \sum_{i=n+1}^{\infty} \xi(A_i) > 1 - \frac{\varepsilon}{3}$, hence

$$\sum_{i=1}^{n} \eta(A_i) \ge \sum_{i=1}^{n} \xi(A_i) - \sum_{i=1}^{n} |\xi(A_i) - \eta(A_i)|$$
$$> (1 - \frac{\varepsilon}{3}) - \frac{\varepsilon}{6} = 1 - \frac{\varepsilon}{2}.$$

This implies $\Sigma_{A \in K \setminus \{A_1,\cdots,A_n\}} \eta(A) < \frac{\varepsilon}{2}$. Thus we have

$$\|\theta^{-1}(x) - \theta^{-1}(y)\|_1 = \|\xi - \eta\|_1$$
$$\leq \sum_{i=1}^{n} |\xi(A_i) - \eta(A_i)| + \sum_{i=n+1}^{\infty} |\xi(A_i)|$$
$$+ \sum_{A \in K \setminus \{A_1, \ldots, A_n\}} |\eta(A)|$$
$$\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{3} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is completed.

Thus the $\ell_1$-completion well behaves in the barycentric subdivision of a metric simplicial complex. However the $c_0$-completion does not.

3.2. *Proposition. Let* K *be an infinite-dimension simplicial complex. Then there is no homeomorphism* $h: \overline{|\mathrm{Sd}\ K|}^{c_0} \to \overline{|K|}^{c_0}$ *extending the natural homeomorphism* $\theta: |\mathrm{Sd}\ K|_m \to |K|_m$.

*Proof*. Assume there is a homeomorphism $h: \overline{|\mathrm{Sd}\ K|}^{c_0} \to \overline{|K|}^{c_0}$ such that $h||\mathrm{Sd}\ K| = \theta$. For each simplex $A \in K$, we define $A^* \in |\mathrm{Sd}\ K|$ by $A^*(A) = 1$. Note $h(A^*) = \theta(A^*)$ is the barycenter of $\hat{A}$ of $|A|$. For each $n \in \mathbb{N}$, take an n-simplex $A_n \in K$. Then as seen in the proof of Proposition 2.1, $h(A_n^*) = \hat{A}_n$ converges to 0. However $\|A_n^* - A_m^*\|_\infty = 1$ for any $n \neq m \in \mathbb{N}$. This shows that $h^{-1}$ is not continuous at 0.

In the above, $h^{-1}$ is not continuous at $x \neq 0$ either. For example, let $A_0 \in K$ with $\dim \mathrm{St}(A_0) = \infty$ and for each $n \in \mathbb{N}$ take an n-simplex $A_n \in \mathrm{St}(A_0)$. We define $\xi_n = \frac{1}{2} A_0^* + \frac{1}{2} A_n^* \in |\mathrm{Sd}\ K|$, $n \in \mathbb{N}$. Then $h(\xi_n) = \frac{1}{2}\hat{A}_0 + \frac{1}{2}\hat{A}_n$ converges to $\frac{1}{2}\hat{A}_0$ but $\|\xi_n - \xi_m\|_\infty = \frac{1}{2}$ for any $n \neq m \in \mathbb{N}$. This implies $h^{-1}$ is not continuous at $\hat{A}_0$.

## 4. The $\ell_1$-Completion of a Metric Combinatorial $\omega$-Manifold

Let $\Delta^\infty$ be the countable-infinite full simplicial complex, that is, $\Delta^\infty = F(\mathbb{N})$. For the $\ell_1$-completion and the $c_0$-completion of $|\Delta^\infty|_m$, we have

4.1. *Proposition. The pairs* $(\overline{|\Delta^\infty|}^{\ell_1}, |\Delta^\infty|_m)$ *and* $(\overline{|\Delta^\infty|}^{c_0}, |\Delta^\infty|_m)$ *are homeomorphic to the pair* $(\ell_2, \ell_2^f)$.

Using the result of [CDM], this follows from the following

4.2. *Lemma. Let* K *be a simplicial complex with no principal simplex. Then* $\overline{|K|}^{\ell_1}$ *and* $\overline{|K|}^{c_0}$ *are nowhere locally compact.*

*Proof.* Because of similarity, we show only the $\ell_1$-case. Let $x \in \overline{|K|}^{\ell_1}$ and $\varepsilon > 0$. It suffices to construct a discrete sequence $x_n \in \overline{|K|}^{\ell_1}$, $n \in \mathbb{N}$, so that $\|x - x_n\|_1 < \varepsilon$. If $C_x$ is infinite, write $C_x = \{v_n | n \in \mathbb{N}\}$ so that $x(v_1) \geq x(v_2) \geq \cdots$. If $C_x$ is finite, choose a countable-infinite subset V of $V_K$ such that $C_x \subset V$ and $F(V) \subset K$ and then write $V = \{v_n | n \in \mathbb{N}\}$ so that $x(v_1) \geq x(v_2) \geq \cdots$. (Such a V exists because K has no principal simplex.) Note that $x(v_1) > 0$ and $x(v_n) \leq n^{-1}$ for each $n \in \mathbb{N}$. Put

$$\delta = \min\{\tfrac{\varepsilon}{3}, x(v_1), \tfrac{1}{2}\} > 0.$$

By Lemma 1.1, we can define $x_n \in \overline{|K|}^{\ell_1}$, $n \in \mathbb{N}$, as follows:

$$x_n(v) = \begin{cases} x(v_1) - \delta & \text{if } v = v_1, \\ x(v_{n+1}) + \delta & \text{if } v = v_{n+1}, \\ x(v) & \text{otherwise.} \end{cases}$$

Then clearly $\|x - x_n\|_1 = 2\delta < \varepsilon$ for each $n \in \mathbb{N}$ and $\|x_n - x_m\|_1 = 2\delta$ if $n \neq m$.

The second half of Conjecture 0.8 (i.e., Corollary 0.9) is a direct consequence of Theorem 1.5 and the following

4.3. *Proposition. Let* M *be an* $\ell_2^f$*-manifold which is contained in a metrizable space* $\tilde{M}$. *If for each open cover* $\mathcal{U}$ *of* $\tilde{M}$ *there is a map* $f: \tilde{M} \to M$ *which is* $\mathcal{U}$*-near to* id, *then* M *is an* f-d *cap set for* $\tilde{M}$.

*Proof.* By [$Sa_3$, Lemma 2], M has a strongly universal tower $\{X_n\}_{n\in\mathbb{N}}$ for finite-dimensional compact such that $M = \cup_{n\in\mathbb{N}} X_n$ and each $X_n$ is a finite-dimensional compact strong Z-set in M. From the condition, it is easy to see that each $X_n$ is a strong Z-set in $\tilde{M}$. Let $\mathcal{U}$ be an open cover of $\tilde{M}$ and Z a finite-dimensional compact set in $\tilde{M}$. Since M is an ANR, M has an open cover $\mathcal{V}$ such that any two $\mathcal{V}$-near maps from an arbitrary space to M are $\mathcal{U}$-homotopic [Hu, Ch. IV, Theorem 1.1]. For each $V \in \mathcal{V}$, choose an open set $\tilde{V}$ of $\tilde{M}$ so that $\tilde{V} \cap M = V$ and define an open cover $\tilde{\mathcal{V}}$ of $\tilde{M}$ by

$$\tilde{\mathcal{V}} = \{\tilde{V} | V \in \mathcal{V}, V \cap X_n \neq \emptyset\} \cup \{\tilde{M} \setminus X_n\}.$$

Let $\mathcal{W}$ be an open cover of $\tilde{M}$ which refines $\mathcal{U}$ and $\tilde{\mathcal{V}}$. From the condition, there is a map $f: \tilde{M} \to M$ which is $\mathcal{W}$-near to id. Observe that $f|Z \cap X_n: Z \cap X_n \to M$ and the inclusion $Z \cap X_n \subset M$ are $\mathcal{V}$-near, hence $\mathcal{U}$-homotopic. By the Homotopy Extension Theorem [Hu, Ch. IV, Theorem 2.2 and its proof], we have a map $g: Z \to M$ such that $g|A \cap X_n = \text{id}$ and g is

$\mathcal{U}$-homotopic to $f|Z$. From the strong universality of the tower $\{X_n\}_{n\in\mathbb{N}}$, we have an embedding $h: Z \to X_m$ of $Z$ into some $X_m$ such that $h|Z \cap X_n = g|Z \cap X_n = \mathrm{id}$ and $h$ is $\mathcal{U}$-near to $g$, hence st $\mathcal{U}$-near to id.

4.4. *Remark*. In connection with Conjecture 0.8 and our results, one might conjecture more generally that a completion $\tilde{M}$ of an $\ell_2^f$-manifold $M$ is an $\ell_2$-manifold if the inclusion $M \subset \tilde{M}$ is a fine homotopy equivalence. However this conjecture is false. In fact, let $\tilde{M}$ be a complete ANR such that $\tilde{M} \setminus A$ is a $\ell_2$-manifold for some Z-set $A$ in $\tilde{M}$ but $\tilde{M}$ is not an $\ell_2$-manifold. Such an example is constructed in [BBMW]. And let $M$ be an f-d cap set for $\tilde{M} \setminus A$. Then $M$ is also an f-d cap set for $M$ by the same arguments in Proposition 4.4. Using [$Sa_3$, Lemma 5], it is easily seen that the inclusion $M \subset \tilde{M}$ is a fine homotopy equivalence. And $M$ is an $\ell_2^f$-manifold by [$Ch_2$, Theorem 2.15].

*Addendum:* Recently, Conjecture 0.8 has been proved in [$Sa_5$]. In fact, it is proved that $\overline{|K|}^{\ell_1}$ *is an* $\ell_2$*-manifold if and only if* $K$ *is a combinatorial* $\infty$*-manifold.*

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