TOPOLOGY PROCEEDINGS
Volume 11, 1986
Pages 239–245

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

ON A POSSIBLE PROPERTY OF FAR POINTS OF $\beta[0,\infty)$

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We are interested in the Stone-Čech compactification of a space X. If X is completely regular then let βX denote the Stone-Čech compactification of X, and let $X^* = \beta X - X$. A far point of βX is defined (see [vD]) to be a point of X^* which is not in the closure in βX of any closed (in X) and discrete subset of X. Let A denote $[0,\infty)$ the nonnegative reals. It has been shown that βA has far points (see [vD]). It follows from a result of Smith [S] that every point of A^* which is not a far point is a cut point of a subcontinuum of A^*. It is the purpose of this paper to show that under Martin's Axiom the converse is not true.

Definitions and Notations. If X is a metric space then the Stone-Čech compactification of βX can be identified with the space of ultrafilters of closed subsets of X [Wa]. If $x \in X$ then x is identified with the ultrafilter that contains $\{x\}$. If U is open in X then $\{x \in X^* | U$ contains an element of $x\}$ is a basic open set. The collection of all basic open sets forms a basis for the topology of βX. If $Y \subset X$ then $Cl_X(Y)$ denote the closure of Y in X and $Int_X(Y)$ denotes the interior of Y with respect to X (the subscript X may be omitted if the meaning is clear from the context). Let N denote the positive integers.

A point $p \in \beta X$ is called a *far point* of X if $p \in X^*$ but $p \notin Cl_{\beta X}B$ for every discrete subset B of X which is closed in X. A point $p \in \beta X$ is called a *remote point* of X if $p \in X^*$ and $p \notin Cl_{\beta X}B$ for every nowhere dense subset B of X.

Let $A = [0,\infty)$. The fact that βA has remote points is well known [FG,vD]. It is easy to see that every remote point of βA is a far point of βA. So βA has far points.

We will outline the argument that every point of A^* which is not a far point is a cut point of a subcontinuum of A^*. The reader may wish to consult [S] for the details which are skipped.

Theorem 1. *If* $p \in A^*$ *and* p *is not a far point then* p *is a cut point of a subcontinuum of* A^*.

Proof. If $p \in A^*$ and p is not a far point of A then there is a discrete subset D of A which is closed in A so that $p \in Cl_{\beta X}D$. Since A is $[0,\infty)$, D must be countable. Let $D = \{s_i\}_{i=1}^{\infty}$. Let $\{I_n\}_{n=1}^{\infty}$ be a collection of disjoint intervals so that for each $n \in N$ the point s_n lies in the interior of I_n.

If $H \subset \cup_{n=1}^{\infty} I_n$ then let $D_H = \{n | I_n \cap H \neq \emptyset\}$. If $x \in Cl_{\beta X}(\cup_{n=1}^{\infty} I_n)$ then let $u_x = \{K | K = D_H$ for some $H \in x\}$. Then u_x is an ultrafilter in N^* (Lemma 1.1 [S]). Let $L = \{x | x \in Cl_{\beta X}(\cup_{n=1}^{\infty} I_n)$ and $u_x = u_p\}$. Then L is a subcontinuum of A^* and $p \in L$ (Theorem 1 [S]).

Let B be the set to which x belongs if and only if $x \in L$ and there is an element $H \in x$ so that every point of $H \cap I_n$ precedes s_n in $[0,\infty)$ for all $n \in D_H$. Let C be the set to which x belongs if and only if $x \in L$ and there is an element $H \in x$ so that every point of $H \cap I_n$ follows s_n in $[0,\infty)$ for all $n \in D_H$. Then B and C are mutually separated, and $L - \{x\} = B \cup C$ (Theorem 2 [S]). Therefore x is a cut point of L.

The reader who is unfamiliar with Martin's Axiom may wish to consult Jech [J]. Let I denote the interval, let N denote the positive integers and let X denote the space $\mathbb{N} \times I$. If $n \in N$ let $I_n = \{n\} \times [0,1]$. Let c denote the cardinality of the reals.

Notice that X is a subspace of A and hence that X* is a compact subspace of A* with interior. We will examine X* rather than A*.

The original version of this paper used Martin's Axiom, and we would like to thank the referee for pointing out that a weaker axiom suffices, namely MAC, or Martin's Axiom restricted to countable sets (see [We]).

Theorem 2. MAC *implies that there exists a component* L *of* X* *and a cut point of* L *which is a remote point of* X.

Proof. Let Z(X) denote the collection of closed subsets of $X = \cup_{n=1}^{\infty} I_n$. Let $\langle M_\alpha \rangle_{\alpha<c}$ be a well ordering of the elements of Z(X) which are nowhere dense in X. Let $\mathcal{S}$ be the subset of Z(X) to which S belongs if and only if the set $\{n | S \cap I_n$ contains an interval$\}$ is infinite and every open set in X which intersects S contains an interval which is a subset of S. Let $\langle S_\alpha \rangle_{\alpha<c}$ be a well ordering of the elements of $\mathcal{S}$.

If $K \subset X$ then define D_K as above, $D_K = \{n | K \cap I_n \neq \emptyset\}$.

We will construct an ultrafilter in $\beta X - X$ by transfinite induction. Let $\lambda < c$ and suppose that $F_\lambda = \{H_\alpha | \alpha < \lambda\}$ has been defined so that for every $\delta < \lambda$:

1) $H_\delta \in Z(X)$ and $H_\delta \cap M_\delta = \emptyset$,

2) if $H_{\gamma_1}, H_{\gamma_2}, \cdots, H_{\gamma_k}$ is a finite collection of elements of F_λ then $\cap_{i=1}^k H_{\gamma_i} \cap I_n$ is an interval or $\emptyset$ for each n and $\{n | \cap_{i=1}^k H_{\gamma_i} \cap I_n$ is an interval$\}$ is infinite, and

3) $H_\delta \subset S_\delta$ or $H_\delta \subset Cl_X(X - S_\delta)$.

We will now construct H_λ.

Let C be dense in $[0,1]$ such that for all positive integers n and $\alpha < \lambda$, (n,r) is not an end point of $H_\alpha \cap I_n$ for any $r \in C$.

Case 1. For each positive integer k and each finite sequence $H_{\gamma_1}, H_{\gamma_2}, \cdots, H_{\gamma_k}$ of elements of F_λ the set $\{n | \cap_{i=1}^k H_{\gamma_i} \cap I_n \cap S_\lambda$ contains an interval$\}$ is infinite.

Let P be the set to which (A,n) belongs iff

a) $A \in Z(X)$, $A \subseteq S_\lambda - M_\lambda$,

b) $D_A \subseteq \{1,2,\cdots,n\}$,

c) $A \cap I_n$ is an interval with endpoints in C for each $k \in D_A$.

Partially order P by $(A,n) \leq (A',n')$ iff

1) $n' \leq n$,

2) if $k \leq n'$ then $A \cap I_k = A' \cap I_k$.

It is clear that P is a countable partial ordering, so MAC applies.

For each finite subset W of λ and each integer k let $D(W,k) = \{(A,n)\colon \text{for some } m > k \ \cap_{\gamma \in W} H_\gamma \cap A \cap I_m \neq \emptyset\}$. By the hypothesis of Case 1, D(W,k) is dense in P. By MAC, let $G \subseteq P$ be a filter intersecting all D(W,k). Let $H_\lambda = \cup\{A\colon (A,n) \in G\}$.

Claim 1. $\{H_\alpha\colon \alpha \leq \lambda\}$ satisfies the induction hypothesis.

Proof of Claim. (1) and (3) of the induction hypothesis are obvious. To see (2) let $\gamma_1 < \gamma_2 < \cdots < \gamma_\ell \leq \lambda$. By the induction hypothesis we may assume $\gamma_\ell = \lambda$. Let $W = \{\gamma_1, \cdots, \gamma_{\ell-1}\}$. Then for any k, since $D(W,k) \cap G \neq \emptyset$, $\cap_{i=1}^{\ell} H_{\gamma_i} \cap I_n \neq \emptyset$ for some $n > k$. Thus $\{n\colon \cap_{i=1}^{\ell} H_{\gamma_i} \cap I_n$ is an interval$\}$ is infinite. Claim 1 establishes the induction step in Case 1.

Case 2. There exists a positive integer k and a finite sequence $\{H_{\gamma_i}\}_{i=1}^{k}$ of elements of F_λ so that $\{n \mid \cap_{i=1}^{k} H_{\gamma_i} \cap I_n \cap S_\lambda$ contains an interval$\}$ is finite.

Then by condition (2) in the induction hypothesis for $\{H_\delta \mid \delta < \lambda\}$ it is easy to see that if $\delta_1, \delta_2, \cdots, \delta_m$ are ordinals less than λ then $\{n \mid \cap_{i=1}^{m} H_{\delta_i} \cap I_n \cap Cl_X(X - S_\lambda)$ contains an interval$\}$ is infinite.

Now we can reapply the proof of Case 1 with S_λ replaced by $Cl_X(X - S_\lambda)$ and we can construct H_λ. Therefore by induction we can construct $\{H_\alpha\}_{\alpha < c}$ which satisfy conditions 1, 2, and 3 of the induction hypothesis.

Let z = be the filter generated by $\{H_\alpha \mid \alpha < c\}$.

Claim 2. z is an ultrafilter of elements of Z(X).

Proof. Suppose that $S \in Z(X)$. Let $S_1 = Cl_X(Int_X S)$ and let $S_2 = S - Int_X S$. Then S_1 and S_2 are closed, S_2 is nowhere dense in X, every open set intersecting S_1 contains an interval lying in S_1, and $S = S_1 \cup S_2$. Then $S_1 = S_\alpha$ for some $\alpha < c$. If $H_\alpha \subset S_\alpha$ then $S_1 \in z$ so $S \in z$. If $H_\alpha \subset Cl_X(X - S_\alpha)$ then $M_\beta = S_2$ for some $\beta < c$ and $M_\delta = Bd_X(Cl_X(X - S_1))$ for some $\delta < c$. Thus $(H_\alpha \cap H_\beta \cap H_\delta) \cap S = \emptyset$, so $S_2 \in z$. This establishes the claim.

The remainder of the proof precedes as in Theorem 1. If $x \in X^*$ let $u_x = \{K | K = D_H \text{ for some } H \in x\}$. Then u_x is an ultrafilter in N^*. Let $L = \{x \in X^* | u_x = u_z\}$. Then L is a subcontinuum of X^* and $z \in L$ (Theorem 1 [S]). Let B be the set to which x belongs if and only if there is an element $H \in x$ and an element $K \in z$ so that every point of $I_n \cap H$ precedes every point of $I_n \cap K$ for all $n \in D_H$ (with respect to the order on [0,1]).

Let C be the set to which x belongs if and only if $x \in L$ and there is an element $H \in x$ and an element $K \in z$ so that every point of $I_n \cap H$ follows every point of $I_n \cap K$ for all $n \in D_H$. Then B and C are mutually separated and $L - \{x\} = B \cup C$ (see Theorem 2 [S] for details). Therefore z is a cut point of L. The point z is a remote point since by condition 1 no nowhere dense closed subset of X is a member of z. Therefore z is also a far point.

Question. Is there a set theoretic axiom under which it follows that every point of A^* which is not a far point of βA is a cut point of a subcontinuum of A^*? If the answer is yes then this would yield a characterization of far points of $\beta[0,\infty)$ in terms of the topology of A^*.

Question. Let $\{I_n\}_{n=1}^{\infty}$ be a discrete collection of intervals in the reals as above and $X = \cup_{i=1}^{\infty} I_n$. Can one prove in ZFC that there is an ultrafilter $u \in X^*$ having the property that for every $S \in u$ there exists $S' \in u$ with $S' \subset S$ so that $S' \cap I_n$ is an interval or $\emptyset$ for all n?

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