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Dedicated to the Memory of Eric K. van Douwen

1. Introduction

A space X is *para-Lindelöf* (resp. *σ-para-Lindelöf*) if every open cover of X has a locally countable (resp. σ-locally countable) open refinement, and X has the *discrete countable chain condition* (dccc) if every discrete family of open subsets of X is countable. *Discrete finite chain condition* (dfcc) is defined analogously.

The following result of Burke and Davis is proved in [Bu, 9.7].

1.1. *Theorem* (Burke and Davis). *Every Tychonoff pseudocompact para-Lindelöf space is Lindelöf.*

Burke and Davis in fact conclude that the space is compact, but our ostensibly weaker form of their theorem is more suggestive of the generalizations considered here.

In 1.1, the hypothesis "pseudocompact" (= dfcc in Tychonoff spaces) can be replaced by "dccc." This follows readily from known results for T_3 spaces (as detailed in 3.4(b) below and also in [Wa, Note 2]). One of our main results is that "Tychonoff pseudocompact" can be replaced in toto by "dccc" (no separation required). Thus we have: Every para-Lindelöf dccc space is Lindelöf (see 3.3). More generally, we extend 1.1 to higher cardinality by way of the following equality among cardinal functions: $L(X) = pL(X) \cdot dc(X)$, where $L(X)$, $pL(X)$, and $dc(X)$ are the Lindelöf number, the para-Lindelöf number, and the discrete cellularity of X, respectively (see 3.2). (For definitions of these, and other, cardinal functions, see §2.)

Analogously, we show also that every σ-para-Lindelöf ccc space is Lindelöf (4.3) and that, more generally, $L(X) \leq \sigma pL(X) \cdot c(X)$, where $\sigma pL(X)$ and $c(X)$ are the σ-para-Lindelöf number and the cellularity of X, respectively (4.2). Furthermore, $L(X) \cdot til(X) = \sigma pL(X) \cdot c(X)$, where $til(X)$ is the tiling number of X (4.6).

The following generalization of 1.1 is announced in [Bu, p. 416] and proved in [BD]:

1.2. *Theorem* (Burke and Davis). *Every Tychonoff pseudo-compact σ-para-Lindelöf space is compact.*

While 1.1 and 1.2 are, at least superficially, very similar, the generalizations to higher cardinality that we obtain for them here are not. In contrast to the generalizations of 1.1 cited above, we show that 1.2 (for T_3 spaces) can be cardinally generalized simply by adding a suitable cardinal parameter to each of the concepts appearing in its statement. We show, in fact, that if κ is a regular cardinal, and if X is a T_3 pseudo-κ-compact σ-para-κ-Lindelöf P_κ-space, then X is $[\kappa,\infty)$-compact (5.5).

The following notation and terminology will be used throughout:

The letter κ will always denote a cardinal.

For a subset A of a space X and a collection $\mathcal{S}$ of subsets of X, we set $\mathcal{S}(A) = \{S \in \mathcal{S}: S \cap A \neq \emptyset\}$, $st(A,\mathcal{S}) = \cup\mathcal{S}(A)$, and $ord(A,\mathcal{S}) = |\mathcal{S}(A)|$. We say that $\mathcal{S}$ is *locally* $\leq\kappa$ (resp. *locally* $<\kappa$) in X if there is an open cover $\mathcal{U}$ of X such that $ord(U,\mathcal{S}) \leq \kappa$ (resp. $ord(U,\mathcal{S}) < \kappa$) for every $U \in \mathcal{U}$.

If $\mathcal{A}$ is another collection of subsets of X, then clearly $\mathcal{S}(\cup\mathcal{A}) = \cup\{\mathcal{S}(A): A \in \mathcal{A}\}$.

Unless specifically mentioned to the contrary, no separation properties will be assumed.

I wish to thank G. M. Reed for a number of very useful conversations on the subject of this paper, and for calling my attention to the relevance of [Re_1], [Re_2], and [Re_3].

I would also like to express my appreciation for the generosity with which Eric van Douwen discussed this paper with me on numerous occasions, and for his substantive contributions recorded in 2.4(b) and 4.7.

2. Some Cardinal Functions

In this section we describe the main cardinal functions of the present paper and record some relationships among them.

A space X is κ-*Lindelöf* if every open cover of X has a subcover of cardinality $\leq\kappa$, and a *cellular family* in X is a pairwise disjoint family of nonempty open subsets of X. The *Lindelöf number* L(X), the *cellularity* c(X), and the *discrete cellularity* dc(X) of X are defined as follows:

$$L(X) = \min\{\kappa: X \text{ is } \kappa\text{-Lindelöf}\} + \omega,$$
$$c(X) = \sup\{|\mathcal{C}|: \mathcal{C} \text{ is a cellular family in } X\} + \omega,$$
$$dc(X) = \sup\{|\mathcal{D}|: \mathcal{D} \text{ is a discrete cellular family in } X\} + \omega.$$

Of these three cardinal functions, L and c are, of course, standard (see e.g. [En, p. 248 and 1.7.12] or [Ju, p. 6]), and dc is essentially so. (For a regular space X, $dc(X) = \sup\{|\mathcal{U}|: \mathcal{U}$ is a locally finite family of open subsets of $X\} + \omega$ (the "pseudocompactness number" of X [CH, 1.1]; see 5.1 below).

We shall call a space X *para-κ-Lindelöf* if every open cover of X has a locally $\leq\kappa$ open refinement, and *σ-para-κ-Lindelöf* if every open cover of X has an open refinement that is the union of $\leq\kappa$ families, each of which is locally $\leq\kappa$. We define the *para-Lindelöf number* pL(X) and the *σ-para-Lindelöf number* σpL(X) of X as follows:

$$pL(X) = \min\{\kappa: X \text{ is para-}\kappa\text{-Lindelöf}\} + \omega,$$
$$\sigma pL(X) = \min\{\kappa: X \text{ is } \sigma\text{-para-}\kappa\text{-Lindelöf}\} + \omega.$$

Next let $U \subset X$. By a *κ-tiling* of U in X we mean a family $\mathcal{T}$ of regularly closed subsets of X such that $|\mathcal{T}| \leq \kappa$ and $\cup\mathcal{T} \subset U \subset \text{cl } \cup\mathcal{T}$. (A subset F of X is *regularly closed* in X if $F = \text{cl int } F$ (equivalently: $F = \text{cl } G$ for some open subset G of X).) We say that U is *κ-tilable* in X if U has a κ-tiling in X.

If U is an open subset of a regular space X, then clearly U is $|U|$-tilable in X, and we define til(U,X) as follows:

$$\text{til}(U,X) = \min\{\kappa: U \text{ is } \kappa\text{-tilable in } X\}.$$

The *tiling number* til(X) of a regular space X is then defined in the following way:

$$\text{til}(X) = \sup\{\text{til}(U,X)\colon U \text{ is an open subset of } X\} + \omega.$$

Then, as is easily seen, X is perfectly wd-normal in the sense of Reed [Re_3, p. 174] (or, in Reed's earlier terminology, "has property J" [Re_2, 1.7]) if and only if $\text{til}(X) = \omega$. (We shall here call X *countably tilable* if $\text{til}(X) = \omega$.)

Obviously every perfectly normal T_1 space is countably tilable.

A space X is *strongly collectionwise Hausdorff* (scwH) if every discrete collection of singletons in X can be separated by a discrete cellular family in X (see [FR, 1.1(d)]).

We note the following inequalities involving the cardinal functions introduced above:

2.1. *Proposition.*

(1) $\sigma pL(X) \leq pL(X) \leq L(X)$.

(2) $dc(X) \leq \min\{L(X), c(X)\}$.

(3) *If* X *is regular, then* $\text{til}(X) \leq c(X)$.

(4) *If a* T_1 *space* X *is* scwH *and perfect, then* $c(X) = dc(X)$.

Proof. (1) and (2) are clear.

To verify (3), consider any open subset U of X. By Zorn's lemma, there exists a maximal pairwise disjoint family $\mathcal{J}$ of nonempty regularly closed subsets of X such that $\cup\mathcal{J} \subset U$. Then, by maximality, $U \subset \text{cl} \cup\mathcal{J}$, and since

$|\mathcal{F}| \leq c(X)$, we conclude that U is $c(X)$-tilable. Thus $til(U,X) \leq c(X)$, and hence $til(X) \leq c(X)$.

For (4), let $\mathcal{G}$ be a cellular family in X, pick $x_G \in G$ for each $G \in \mathcal{G}$, and let $D = \{x_G: G \in \mathcal{G}\}$. Since X is perfect, there exists a countable collection $\mathcal{F}$ of closed subsets of X such that $\cup\mathcal{G} = \cup\mathcal{F}$. Now for each $F \in \mathcal{F}$, $D \cap F$ is discrete and closed in X (since X in T_1), and since X is scwH we then have $|D \cap F| \leq dc(X)$. Then $|\mathcal{G}| = |D| = |\cup\{D \cap F: F \in \mathcal{F}\}| \leq dc(X)$, and we conclude that $c(X) \leq dc(X)$.

2.2. *Remarks.* (a) The first inequality of 2.1(1) can be strict (because there exist σ-para-Lindelöf spaces that are not para-Lindelöf; see e.g. [FR, 2.5 or 2.6]), and the second can also be strict (consider any nonseparable metrizable space).

(b) The inequality of 2.1(2) can be strict (consider the space ω_1 of countable ordinals).

(c) The inequality of 2.1(3) can be strict (again consider any nonseparable metrizable space).

The next result shows that the tiling number is precisely what is needed for a factorization of cellularity. For the case $c(X) = \omega$, this result is due to Reed (who notes it, without proof, in [Re_1, 5.6]).

2.3. *Proposition. If* X *is regular, then* $c(X) = til(X)\cdot dc(X)$.

Proof. Let $\kappa = til(X)$ and $\lambda = dc(X)$, and let $\mathcal{U}$ be a cellular family in X. Since $til(\cup\mathcal{U},X) \leq \kappa$, there exists a κ-tiling $\mathcal{J}$ of $\cup\mathcal{U}$. For each $F \in \mathcal{J}$, let

$$\mathcal{U}_F = \{U \in \mathcal{U}: U \cap \text{int } F \neq \emptyset\}$$

and

$$\mathcal{V}_F = \{U \cap \text{int } F: U \in \mathcal{U}_F\}.$$

For each $F \in \mathcal{J}$, we have $F \subset \cup\mathcal{J} \subset \cup\mathcal{U}$, and it follows easily that $\mathcal{V}_F$ is discrete in X. Since the mapping given by $U \mapsto U \cap \text{int } F$ is a bijection from $\mathcal{U}_F$ onto $\mathcal{V}_F$, we therefore have $|\mathcal{U}_F| = |\mathcal{V}_F| \leq \lambda$. We also have $\cup\mathcal{U} \subset \text{cl } \cup\mathcal{J}$, and from this it follows that $\mathcal{U} = \cup\{\mathcal{U}_F: F \in \mathcal{J}\}$. Then $|\mathcal{U}| \leq \Sigma_{F\in\mathcal{J}}|\mathcal{U}_F| \leq \kappa\cdot\lambda$, and we conclude that $c(X) \leq \kappa\cdot\lambda$. The reverse inequality follows from 2.1.

We conclude this section with some remarks which, although closely related to the foregoing, will not be needed in the sequel.

2.4. *Remarks*. (a) A space X is κ-*paracompact* if every open cover of X of cardinality $\leq\kappa$ has a locally finite open refinement (see [Mor, p. 223]). We note the following sufficient condition for the equality $\sigma pL(X) = pL(X)$: If X is $\sigma pL(X)$-paracompact, then $\sigma PL(X) = pl(X)$. (For the special case $\sigma pL(X) = \omega$, this is due to Tall [Ta_2, I.1.21]. The proof in [FR, 3.1] of this special case generalizes immediately to yield the present result.)

(b) Let us call a subset U of X *disjointly* κ-*tilable* in X if U has a pairwise disjoint κ-tiling in X. An open set U in a regular space X is disjointly $c(X)$-tilable in X (by the proof of 2.1(3)), and we set $\text{dtil}(U,X) = \min\{\kappa$: U is disjointly κ-tilable in X$\}$. The *disjoint tiling* *number* dtil(X) of a regular space X is then defined as follows:

$$\mathrm{dtil}(X) = \sup\{\mathrm{dtil}(U,X)\colon U \text{ is an open subset of } X\} + \omega.$$

(Among known cardinal functions, dtil is unusual in that it exists in ZFC, but (presumably) not in ZF.) Clearly til(X) $\leq$ dtil(X), but we do not know whether this inequality can be strict. Van Douwen has noted (personal communication) that dtil(X) = ω if X is metrizable, and that this is true, more generally, if X is regular and either (1) has a σ-discrete π-base [Ju, 1.4] or (2) is scwH and has a dense σ-closed-discrete subset. The argument is as follows: (1) and (2) each imply that there is a sequence $\langle D_n\colon n \in \omega\rangle$ of subsets of X such that $\cup_{n\in\omega} D_n$ is dense in X and such that, for each $n \in \omega$, the points of D_n can be separated by some discrete cellular family in X. Let U be open in X. By a routine recursion, there exists a sequence $\langle \mathcal{J}_n\colon n \in \omega\rangle$ such that:

(i) For each $n \in \omega$, $\mathcal{J}_n$ is a discrete family of regularly closed subsets of X and $\cup\{D_i \cap U\colon i \leq n\} \subset \cup\{\cup\mathcal{J}_i\colon i \leq n\}$.

(ii) $\langle \cup\mathcal{J}_n\colon n \in \omega\rangle$ is a pairwise disjoint sequence of subsets of U.

Then it is easy to verify that $\{\cup\mathcal{J}_n\colon n \in \omega\}$ is an ω-tiling of U.

3. Para-Lindelöf dccc Spaces are Lindelöf

We begin with a lemma that is an analogue of several known results (see e.g. [Mo, Chap. 1, Theorem 18] and [Bl, 2.3 and 4.3]).

3.1. *Lemma. Let* κ *be a cardinal and let* $\mathcal{W}$ *be a locally* $<\kappa$ *open cover of* X. *Then there exists a discrete collection* $\mathcal{G}$ *of open subsets of* X *such that:*

(1) *For every* $G, G' \in \mathcal{G}$ *with* $G \neq G'$, $G \cap \mathrm{st}(G', \mathcal{W}) = \emptyset$.

(2) *For every* $G \in \mathcal{G}$, $\mathrm{ord}(G, \mathcal{W}) < \kappa$.

(3) $X = \mathrm{cl}\ \mathrm{st}(\cup\mathcal{G}, \mathcal{W})$.

Proof. By Zorn's lemma, there is a collection $\mathcal{G}$ of open subsets of X that is maximal with respect to (1) and (2). Clearly $\mathcal{G}$ is a discrete collection that satisfies (3).

The main result of this section is now as follows:

3.2. *Theorem. If* X *is any space, then* $L(X) = \mathrm{pl}(X) \cdot \mathrm{dc}(X)$.

Proof. Let $\kappa = \mathrm{pL}(X)$ and $\lambda = \mathrm{dc}(X)$, and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a locally $\leq\kappa$ open refinement $\mathcal{V}$, there is an open cover $\mathcal{W}'$ of X such that $\mathrm{ord}(W, \mathcal{V}) \leq \kappa$ for every $W \in \mathcal{W}'$, and $\mathcal{W}'$ has a locally $\leq\kappa$ open refinement $\mathcal{W}$ (and, of course, $\mathrm{ord}(W, \mathcal{V}) \leq \kappa$ for every $W \in \mathcal{W}$). By 3.1, there is a discrete cellular family $\mathcal{G}$ in X such that $\mathrm{ord}(G, \mathcal{W}) \leq \kappa$ for every $G \in \mathcal{G}$ and such that $X = \mathrm{cl}\ \mathrm{st}(\cup\mathcal{G}, \mathcal{W})$. Since $\mathcal{V}$ covers X, this last equality implies that $X = \cup\mathcal{V}(\cup\mathcal{W}(\cup\mathcal{G}))$. But $|\mathcal{W}(\cup\mathcal{G})| = |\cup\{\mathcal{W}(G): G \in \mathcal{G}\}| \leq \kappa\cdot\lambda$, and hence $|\mathcal{V}(\cup\mathcal{W}(\cup\mathcal{G}))| = |\cup\{\mathcal{V}(W): W \in \mathcal{W}(\cup\mathcal{G})\}| \leq \kappa\cdot\lambda$. Since $\mathcal{V}$ refines $\mathcal{U}$, it is clear now that $\mathcal{U}$ has a subcover of cardinality $\leq \kappa\cdot\lambda$, and thus $L(X) \leq \kappa\cdot\lambda$. The reverse inequality follows from 2.1.

3.3. *Corollary. Every para-Lindelöf* dccc *space is Lindelöf.*

3.4. *Remarks.* (a) Wiscamb noted 3.3 with "para-Lindelöf" replaced by "paracompact" [Wi, 2.3], and Juhász obtained 3.3 with "dccc" replaced by "ccc." (Juhász's result appears in [Ta_1, 6.1] and [Ta_2, I.3.13]; cf. [Ta_1, 3.4].) We extend Juhász's result in a different way in 4.3 below.

(b) Actually, for T_3 spaces, 3.3 is already a consequence of known results. To see this, note that if X is a T_3 para-Lindelöf space, then X is scwH [FR, 1.7]. Hence, if X is also dccc, then X is ω_1-compact. But a T_1 para-Lindelöf (in fact, meta-Lindelöf) ω_1-compact space is known to be Lindelöf. (This last fact is implicit in Aquaro [Aq], and is recorded explicitly by Aull in [Au, Corollary 1(c)].) See Watson [Wa, Note 2] for a similar explanation. Watson is assuming X is Tychonoff and also uses [FR, 1.7], for which regularity is necessary.

4. σ-Para-Lindelöf ccc Spaces are Lindelöf

We begin with a lemma similar to 3.1.

4.1. *Lemma. If* κ *and* μ *are cardinals, and if* $\mathcal{W} = \bigcup_{\eta\in\kappa}\mathcal{W}_\eta$ *is an open cover of* X *such that each* $\mathcal{W}_\eta$ *is locally* $<\mu$ *in* X*, then there exists a family* $\langle \mathcal{G}_\eta : \eta \in \kappa \rangle$ *such that:*

(1) *For each* $\eta \in \kappa$, $\mathcal{G}_\eta$ *is a cellular family in* X *such that* $\bigcup\mathcal{G}_\eta \subset \bigcup\mathcal{W}_\eta$.

(2) *For every* $\eta \in \kappa$ *and every* $G \in \mathcal{G}_\eta$, $\operatorname{ord}(G,\mathcal{W}_\eta) < \mu$.

(3) $X = \operatorname{cl} \bigcup_{\eta\in\kappa} \operatorname{st}(\bigcup\mathcal{G}_\eta, \mathcal{W}_\eta)$.

Proof. Let Φ be the set of all families $\langle \mathcal{G}_\eta : \eta \in \kappa \rangle$ that satisfy (1) and (2), and partially order Φ as follows: $\langle \mathcal{G}_\eta : \eta \in \kappa \rangle \leq \langle \mathcal{H}_\eta : \eta \in \kappa \rangle$ if $\mathcal{G}_\eta \subset \mathcal{H}_\eta$ for every $\eta \in \kappa$. By Zorn's lemma, Φ has a maximal member $\langle \mathcal{G}_\eta : \eta \in \kappa \rangle$. Now if $V = X - \mathrm{cl} \bigcup_{\eta\in\kappa} \mathrm{st}(\bigcup\mathcal{G}_\eta, \mathcal{W}_\eta) \neq \emptyset$, then obviously there exist $\gamma \in \kappa$ and $W \in \mathcal{W}_\gamma$ such that $V \cap W \neq \emptyset$, and hence there is a nonempty open set $G^* \subset V \cap W$ such that $\mathrm{ord}(G^*, \mathcal{W}_\gamma) < \mu$. But then $\langle \mathcal{G}^*_\eta : \eta \in \kappa \rangle \in \Phi$, where $\mathcal{G}^*_\eta = \mathcal{G}_\eta$ if $\eta \neq \gamma$ and $\mathcal{G}^*_\gamma = \mathcal{G}_\gamma \cup \{G^*\}$, which contradicts the maximality of $\langle \mathcal{G}_\eta : \eta \in \kappa \rangle$. Thus the latter also satisfies (3).

4.2. *Theorem. If* X *is any space, then* $L(X) \leq \sigma pL(X) \cdot c(X)$.

Proof. Let $\kappa = \sigma pL(X)$ and $\lambda = c(X)$, and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has an open refinement $\mathcal{V} = \bigcup_{\xi\in\kappa} \mathcal{V}_\xi$, where each $\mathcal{V}_\xi$ is locally $\leq\kappa$ in X, and for each $\xi \in \kappa$ there is an open cover $\mathcal{W}'_\xi$ of X such that $\mathrm{ord}(W, \mathcal{V}_\xi) \leq \kappa$ for every $W \in \mathcal{W}'_\xi$. Then for each $\xi \in \kappa$, $\mathcal{W}'_\xi$ has an open refinement $\mathcal{W}_\xi = \bigcup_{\eta\in\kappa} \mathcal{W}_{\xi\eta}$, where each $\mathcal{W}_{\xi\eta}$ is locally $\leq\kappa$ in X (and clearly $\mathrm{ord}(W, \mathcal{V}_\xi) \leq \kappa$ for every $W \in \mathcal{W}_\xi$). Moreover, for every $\xi \in \kappa$, there exists, by 4.1 (applied to $\mathcal{W}_\xi$), a family $\langle \mathcal{G}_{\xi\eta} : \eta \in \kappa \rangle$ of cellular families in X such that, for every $\eta \in \kappa$ and every $G \in \mathcal{G}_{\xi\eta}$, $\mathrm{ord}(G, \mathcal{W}_{\xi\eta}) \leq \kappa$, and such that $X = \mathrm{cl} \bigcup_{\eta\in\kappa} \mathrm{st}(\bigcup\mathcal{G}_{\xi\eta}, \mathcal{W}_{\xi\eta})$. It follows that each nonempty member of $\mathcal{V}$ belongs to $\mathcal{V}_\gamma(\bigcup_{\eta\in\kappa} \mathrm{st}(\bigcup\mathcal{G}_{\gamma\eta}, \mathcal{W}_{\gamma\eta}))$ for some $\gamma \in \kappa$, and hence that $X = \bigcup\mathcal{V}^*$, where

$$\mathcal{V}^* = \bigcup_{\xi\in\kappa} \mathcal{V}_\xi(\bigcup_{\eta\in\kappa} \mathrm{st}(\bigcup\mathcal{G}_{\xi\eta}, \mathcal{W}_{\xi\eta})).$$

Now for every $\xi, \eta \in \kappa$, we have

$$|\mathcal{W}_{\xi\eta}(\bigcup\mathcal{G}_{\xi\eta})| = |\bigcup\{\mathcal{W}_{\xi\eta}(G) : G \in \mathcal{G}_{\xi\eta}\}| \leq \kappa\cdot\lambda.$$

Then $|\mathcal{V}_\xi(\cup_{\eta\in\kappa}\mathrm{st}(\cup\mathcal{G}_{\xi\eta},\mathcal{W}_{\xi\eta}))| = |\mathcal{V}_\xi(\cup_{\eta\in\kappa}(\cup\mathcal{W}_{\xi\eta}(\cup\mathcal{G}_{\xi\eta})))|$

$$= |\mathcal{V}_\xi(\cup(\cup_{\eta\in\kappa}\mathcal{W}_{\xi\eta}(\cup\mathcal{G}_{\xi\eta})))|$$

$$= |\cup\{\mathcal{V}_\xi(W): W \in \cup_{\eta\in\kappa}\mathcal{W}_{\xi\eta}(\cup\mathcal{G}_{\xi\eta})\}| \leq \kappa\cdot\lambda,$$

and hence $|\mathcal{V}^*| \leq \kappa\cdot\lambda$. Since $\mathcal{V}^*$ refines $\mathcal{U}$, it follows that $\mathcal{U}$ has a subcover of cardinality $\leq \kappa\cdot\lambda$, and hence $L(X) \leq \kappa\cdot\lambda$.

As a special case of 4.2, we have:

4.3. *Corollary. Every* σ*-para-Lindelöf* ccc *space is Lindelöf.*

In a context which guarantees that $c(X) = dc(X)$ (for example, if X is either (i) T_1, scwH, and perfect (2.1(4)) or (ii) regular and countably tilable (2.3)), then 4.2 implies that $L(X) = \sigma pL(X)\cdot dc(X)$. In general, however, we do not know whether $c(X)$ can be replaced by $dc(X)$ in 4.2. In particular, we leave open the following problem:

4.4. *Problem.* Is every σ-para-Lindelöf dccc space Lindelöf?[1]

Recall that the *spread* $s(X)$ of a space X is defined as follows: $s(X) = \sup\{|D|: D \text{ is a discrete subspace of } X\} + \omega$ [Ju, 1.9]. For a cardinal function ϕ, we denote its hereditary version by $h\phi$ (i.e., $h\phi(X) = \sup\{\phi(Y): Y \subset X\}$).

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[1]Added in revision: In response to 4.4 (as circulated in an earlier version of this paper), R. W. Heath and G. M. Reed have independently constructed examples of T_3 non-Lindelöf dccc spaces with σ-locally countable bases, thus answering 4.4 in the negative. (Reed's example is actually a Moore space.) These examples were announced at the Spring Topology Conference held at the University of Alabama at Birmingham, March 19-21, 1987.

From 4.2 and the fact that $hc(X) = s(X) \leq hL(X)$, we deduce the following hereditary version of 4.2:

4.5. *Corollary. If* X *is any space, then* $hL(X) = h\sigma pL(X) \cdot s(X)$.

The inequality of 4.2 can obviously be strict (for example, consider any compact space X with $c(X) > \omega$). In the following result we show that, for regular spaces, equality can be achieved by introducing the tiling number as well. (As an equality between two products of cardinal functions, this result is unusual.)

4.6. *Corollary. If* X *is regular, then* $L(X) \cdot til(X) = \sigma pL(X) \cdot c(X)$.

Proof. This follows from 4.2, together with the inequalities $til(X) \leq c(X)$, $L(X) \geq \sigma pL(X)$, and $L(X) \cdot til(X) \geq dc(X) \cdot til(X) = c(X)$ (see 2.1 and 2.3).

4.7. *Remark.* I am indebted to Eric van Douwen for the following observations:

In view of 4.6, for regular spaces X there are exactly six possible sets of inequalities among the four cardinal functions L, til, σpL, and c. These are:

(1) $til(X) = c(X) > L(X) \geq \sigma pL(X)$.

(2) $til(X) \leq c(X) < L(X) = \sigma pL(X)$.

(3) $til(X) = c(X) = L(X) = \sigma pL(X)$.

(4) $til(X) < c(X) = L(X) = \sigma pL(X)$.

(5) $til(X) = c(X) = L(X) > \sigma pL(X)$.

(6) $til(X) < c(X) = L(X) > \sigma pL(X)$.

Moreover, each of these six possibilities can actually occur. (For (1), take X compact with $c(X) > \omega$; for (2), take X separable but not Lindelöf; for (3), take $X = \emptyset$; for (4), take $X = X_1 + X_2$ with X_1 separable but not Lindelöf and with X_2 discrete and $|X_2| = L(X_1)$; for (5), take $X = X_1 + X_2$ with X_1 compact and $c(X_1) > \omega$ and with X_2 discrete and $|X_2| = c(X_1)$; and for (6), take X nonseparable metric.)

5. A Cardinal Generalization of 1.2

Let κ and λ be infinite cardinals. As usual, a space X is $[\kappa,\lambda]$-*compact* if every open cover of X of cardinality $\leq \lambda$ has a subcover of cardinality $<\kappa$, X is $[\kappa,\infty)$-*compact* if X is $[\kappa,\mu]$-compact for every infinite cardinal μ, X is *pseudo*-κ-*compact* if every locally finite family of open subsets of X has cardinality $<\kappa$, X is a P_κ-*space* if the intersection of $<\kappa$ open subsets of X is always open in X, and X is a P-*space* if X is a P_{ω_1}-space.

It is known that a Tychonoff space is pseudo-ω-compact if and only if it is pseudocompact (see [En, 3.10.22]). Moreover, for regular spaces, the following alternative formulation of pseudo-κ-compactness is known (see e.g. [Fr, 2.5.3] or [Wi, 2.6] (the latter for $\kappa = \omega_1$)):

5.1. *Proposition.* A regular space X is pseudo-κ-compact if and only if every discrete cellular family in X has cardinality $<\kappa$.

5.2. *Proposition.* If κ is a regular cardinal and if X is a regular P_κ-space, then the following are equivalent:

(1) X is pseudo-κ-compact.

(2) If $\langle U_\xi: \xi \in \kappa \rangle$ is a decreasing family of nonempty open subsets of X, then $\cap_{\xi\in\kappa} \text{cl } U_\xi \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2). Assume there exists a decreasing family $\langle U_\xi: \xi \in \kappa \rangle$ of nonempty open subsets of X such that $\cap_{\xi\in\kappa} \text{cl } U_\xi = \emptyset$. By recursion, there is a family $\langle G_\xi: \xi \in \kappa \rangle$ such that:

(a) For each $\xi \in \kappa$, G_ξ is a nonempty open subset of X with $\text{cl } G_\xi \subset U_\xi - \text{cl } U_\sigma$ for some $\sigma > \xi$.

(b) For every $\xi, \eta \in \kappa$ with $\xi \neq \eta$, $\text{cl } G_\xi \cap \text{cl } G_\eta = \emptyset$.

(The recursion is as follows: Let $\xi \in \kappa$ and assume that $\langle G_\eta: \eta < \xi \rangle$ is already defined subject to (a) and (b). Then for each $\eta < \xi$, $\text{cl } G_\eta \subset X - \text{cl } U_{\sigma(\eta)}$ for some $\sigma(\eta) > \eta$. Let $\sigma = \sup\{\sigma(\eta): \eta < \xi\}$. Since κ is regular, $\sigma < \kappa$ (and clearly $\xi \leq \sigma$). Pick any $x \in U_\sigma$. Then $x \notin \text{cl } U_\gamma$ for some $\gamma \in \kappa$, and necessarily $\gamma > \xi$. By regularity of X, we then have $\text{cl } G_\xi \subset U_\sigma - \text{cl } U_\gamma \subset U_\xi - \text{cl } U_\gamma$ for some nonempty open subset G_ξ of X.)

Now let $x \in X$ and note that $x \notin \text{cl } U_\xi$ for some $\xi \in \kappa$. Let $A = \{\eta \in \xi: x \notin \text{cl } G_\eta\}$ and let $H = X - (\text{cl } U_\xi \cup \cup_{\eta\in A} \text{cl } G_\eta)$. Then, in view of the fact that X is a P_κ-space, H is a neighborhood of x that meets G_γ for at most one $\gamma \in \kappa$. Thus $\langle G_\xi: \xi \in \kappa \rangle$ is discrete in X, and hence X is not pseudo-κ-compact.

(2) $\Rightarrow$ (1). Assume (1) is false so that, by 5.1, there exists a discrete cellular family $\langle H_\xi: \xi \in \kappa \rangle$ in X. For every $\xi \in \kappa$, let $U_\xi = \cup_{\eta\geq\xi} H_\eta$. Then, by (2), there exists $x \in \cap_{\xi\in\kappa} \text{cl } U_\xi$. But then each neighborhood of x meets H_σ and H_η for distinct σ and η, which is a contradiction.

For the special case of 5.2 for which $\kappa = \omega$ and X is Tychonoff, see for example [En, 3.10.23].

The next proposition generalizes the well-known fact that a normal T_1 space is pseudocompact if and only if it is countably compact [En, 3.10.20 and 3.10.21].

5.3. *Proposition.* Let κ be a regular cardinal. A T_1 normal P_κ-space X is pseudo-κ-compact if and only if it is $[\kappa,\kappa]$-compact.

Proof. If X is not $[\kappa,\kappa]$-compact, then (since κ is regular) there exists $D \subset X$ with $|D| = \kappa$ and such that D has no complete accumulation point in X [AU, Chap. 1, Theorem $3^r_{a,b}$], i.e., each $x \in X$ has a neighborhood U in X such that $|U \cap D| < |D|$. Now since X is a P_κ-space, it follows that D has no accumulation point in X, and thus D is closed discrete. Moreover, since X is also normal, X is κ-collectionwise normal (see [Ta_2, I.2.12]), and thus the points of D can be separated by a discrete cellular family in X. Hence X is not pseudo-κ-compact.

Conversely, if X is not pseudo-κ-compact, then by 5.1 there is a discrete cellular family $\mathcal{G}$ in X with $|\mathcal{G}| = \kappa$. If we pick $x_G \in G$ for each $G \in \mathcal{G}$, then $\{x_G : G \in \mathcal{G}\}$ is closed in X but not $[\kappa,\kappa]$-compact, and hence X is not $[\kappa,\kappa]$-compact.

The proof that every regular Lindelöf space is normal (see e.g. [En, 1.5.14]) generalizes immediately to yield the following:

5.4. *Proposition.* If X is a regular P_κ-space with $L(X) \leq \kappa$, then X is normal.

The main result of this section can now be formulated as follows:

5.5. *Theorem. Assume that* κ *is a regular cardinal. If* X *is a* T_3 *pseudo-*κ*-compact* σ*-para-*κ*-Lindelöf* P_κ*-space, then* X *is* $[\kappa,\infty)$*-compact.*

Proof. By 5.4 and 5.3, it suffices to show that X is κ-Lindelöf. Let $\mathcal{U}$ be an open cover of X, and let $\cup_{\xi\in\kappa}\mathcal{V}_\xi$ be an open refinement of $\mathcal{U}$ such that each $\mathcal{V}_\xi$ is locally $\leq\kappa$. Suppose that $\mathcal{U}$ has no subcover of cardinality $\leq\kappa$. Then clearly there exists $\mu \in \kappa$ such that, whenever $\mathcal{V}' \subset \mathcal{V}_\mu$ with $|\mathcal{V}'| \leq \kappa$, we have $\cup\mathcal{V}' \neq \cup\mathcal{V}_\mu$.

Now there exists an open cover $\mathcal{W}^*$ of X such that $\mathrm{ord}(W,\mathcal{V}_\mu) \leq \kappa$ for every $W \in \mathcal{W}^*$, and $\mathcal{W}^*$ has an open refinement $\mathcal{W} = \cup_{\xi\in\kappa}\mathcal{W}'_\xi$ such that each $\mathcal{W}'_\xi$ is locally $\leq\kappa$ (and clearly $\mathrm{ord}(W,\mathcal{V}_\mu) \leq \kappa$ for every $W \in \mathcal{W}$). For each $\xi \in \kappa$, let $\mathcal{W}_\xi = \cup_{\eta\leq\xi}\mathcal{W}'_\xi$. Since X is a P_κ-space, it is easy to verify that each $\mathcal{W}_\xi$ is locally $\leq\kappa$ in X.

We next claim that (by recursion) there exists a family $\langle G_\xi: \xi \in \kappa\rangle$ such that, for every $\xi \in \kappa$, G_ξ is open in X, $\mathrm{ord}(G_\xi,\mathcal{W}_\xi) \leq \kappa$, and $\emptyset \neq G_\xi \subset \cup\mathcal{V}_\mu - \mathrm{cl}\ \cup_{\eta<\xi}\mathrm{st}(G_\eta,\mathcal{W}_\eta)$. (The recursion is as follows: Let $\xi \in \kappa$ and assume that $\langle G_\eta: \eta \in \xi\rangle$ is already defined subject to the preceding conditions. Let $\mathcal{V}' = \mathcal{V}_\mu(\cup_{\eta<\xi}\mathrm{st}(G_\eta,\mathcal{W}_\eta))$, and note that $\mathcal{V}' \subset \mathcal{V}_\mu$ and that $|\mathcal{V}'| = |\mathcal{V}_\mu(\cup\cup_{\eta<\xi}\mathcal{W}_\eta(G_\eta))| = |\cup\{\mathcal{V}_\mu(W): W \in \cup_{\eta<\xi}\mathcal{W}_\eta(G_\eta)\}| \leq \kappa$. Hence there exists $x \in \cup\mathcal{V}_\mu - \cup\mathcal{V}'$, so $x \in V$ for some $V \in \mathcal{V}_\mu - \mathcal{V}'$. Then $V \cap (\cup_{\eta<\xi} st(G_\eta, \mathcal{W}_\eta)) = \emptyset$, and thus $x \notin cl \cup_{\eta<\xi} st(G_\eta, \mathcal{W}_\eta)$. It follows that there exists an open set G_ξ in X such that $ord(G_\xi, \mathcal{W}_\xi) \leq \kappa$ and $x \in G_\xi \subset \cup\mathcal{V}_\mu - cl \cup_{\eta<\xi} st(G_\eta, \mathcal{W}_\eta)$.)

Now for every $\xi \in \kappa$, let $U_\xi = \cup_{\eta \geq \xi} G_\eta$. By 5.2, there exists $x \in \cap_{\xi\in\kappa} cl\ U_\xi$, and there exist $\sigma \in \kappa$ and $W \in \mathcal{W}_\sigma$ such that $x \in W$. Clearly $W \cap G_\eta \neq \emptyset \neq W \cap G_\xi$ for some $\eta, \xi \in \kappa$ with $\sigma \leq \eta < \xi$. Since $W \in \mathcal{W}_\sigma \subset \mathcal{W}_\eta$, we then have $W \subset st(G_\eta, \mathcal{W}_\eta)$, and hence $G_\xi \cap st(G_\eta, \mathcal{W}_\eta) \neq \emptyset$. This is a contradiction, and we conclude that X is κ-Lindelöf.

For the special case $\kappa = \omega$, the preceding theorem is the Burke-Davis result 1.2 (and the preceding proof, in this special case, reduces to that of [BD]).

For the case $\kappa = \omega_1$ we also have the following corollary:

5.6. *Corollary. If* X *is a* T_3 *para-*ω_1*-Lindelöf* P*-space with* dccc*, then* X *is Lindelöf.*

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