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THE ROLE OF REFINABLE MAPS—A SURVEY

Hisao Kato and Akira Koyama

In [15], Heath and Rogers defined a generalization of the notion of near-homeomorphisms called refinable maps. Refinable maps are interesting and useful in continuum theory, dimension theory, shape theory and ANR theory, and many authors have investigated them. The Japanese School has made many contributions in the study of refinable maps. The presented paper is based on a discussion by the authors at the 6th Geometric Topology Summer Seminar (July 21-25, 1985, Yamanaka-Lake). Most results presented here are published elsewhere or are forthcoming. A few observations seem to be new. Open problems from other sources are included.

1. Refinable Maps and Other Classes of Maps

Throughout this paper a map is a continuous function, and, unless there is some indication to the contrary, $f: X \to Y$ means that f maps X *onto* Y. Except in the last section, all spaces considered are *compact metric spaces*. A *continuum* is a connected compact metric space. For $\varepsilon > 0$, a map $f: X \to Y$ is said to be an ε-*map* if $\operatorname{diam}[f^{-1}(y)] < \varepsilon$ for each $y \in Y$. A map $r: X \to Y$ is *refinable* if for each $\varepsilon > 0$, there is an ε-map $f: X \to Y$ such that

$$d(r,f) = \sup\{d(r(x),f(x)) \mid x \in X\} < \varepsilon.$$

Such a map f is called an ε-*refinement* of r. Equivalently, r is a uniform limit of ε-maps for every $\varepsilon > 0$. Refinable maps clearly include near-homeomorphisms but, by easy examples, the notions are not equivalent even when both the range and domain are polyhedra. On the otherhand, under some conditions on the domains, refinable maps are near-homeomorphisms. That is,

1.1. *Let* $r: X \to Y$ *be a refinable map. If any one of the following conditions is satisfied, then* r *is a near-homeomorphism.*

(i) X *is a graph* ([17]).

(ii) X *is a closed* 2-*manifold* ([15] *and* [43]).

(iii) X *is a closed* n-*manifold*, $n > 4$, *and* Y *is an* ANR ([13] *and* 1.6).

(iv) *Both* X *and* Y *are* Q-*manifolds* ([10] *and* 1.6).

(v) *Either* X *or* Y *is the Cantor set* ([26]).

(vi) *Either* X *or* Y *is the pseudo-arc* ([26]).

A map $f: X \to Y$ is *weakly confluent* provided that for each continuum $K \subset Y$, there exists a component C of $f^{-1}(K)$ such that $f(C) = K$. The following was obtained by Heath and Rogers [15]:

1.2. *Every refinable map is weakly confluent.*

Next, by adding some conditions on the range, we will consider interesting relationships between refinable maps and known kinds of maps in continuum theory and shape theory.

A map $f: X \to Y$ is said to be *confluent* if for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$, we have $f(C) = K$; and f is *monotone* if $f^{-1}(K)$ is connected for each connected subset $K \subset Y$. A space X has *property* [k] provided that, for each $\varepsilon > 0$, ther exists $\delta > 0$ such that, if $a, b \in X$, $d(a,b) < \delta$, and A is a continuum in X with $a \in A$, then there exists a continuum B in X such that $b \in B$ and $d_H(A,B) < \varepsilon$, where d_H is the Hausdorff metric induced by a metric d on X. Property [k] was defined by Kelley [27] and he proved that if a continuum X has property [k], the hyperspace C(X) of subcontinua of X is contractible. Note that every locally connected continuum has property [k] but, by an easy example, the converse is not valid.

1.3. *Let* $r: X \to Y$ *be a refinable map. If* Y *has property* [k], *then* r *is confluent* [26] *(see* §6 *for a generalization).*

Note that 1.3 is an extension of [36] and that the condition "Y has property [k]" is essential (see also [26]).

1.4. *Let* $r: X \to Y$ *be a refinable map. If* Y *is locally connected, then* r *is monotone* ([15]).

We will consider circumstances under which monotone maps are refinable maps. Let $\mathcal{P}$ be a collection of spaces. A space X is said to be *$\mathcal{P}$-like* if for each $\varepsilon > 0$, there are an element P of $\mathcal{P}$ and an ε-map $f: X \to P$. If $\mathcal{P}$ consists of a single element P, then we say X is P-like if it is $\mathcal{P}$-like.

1.5. *Let* $f: X \to Y$ *be a monotone map. If any one of the following conditions is satisfied, then* f *is refinable.*

(i) X *is a chainable continuum and* Y *is an arc* ([15]).

(ii) X *is a* $S_1 \vee \cdots \vee S_n$*-like continuum and* Y *is* $S_1 \vee \cdots \vee S_n$, $n \geq 1$, *where* $S_1 \vee \cdots \vee S_n$, $n \geq 1$, *denotes a one-point union of* n *circles* ([22]).

By 1.5, we can see that hereditarily decomposable chainable continua and hereditarily decomposable circle-like continua admit refinable maps onto an arc and a circle, respectively ([15] and [22]).

A space X is *locally* n*-connected*, $X \in LC^n$, if for each point $x \in X$ and each neighborhood U of x in X, there is a neighborhood V of x such that $V \subset U$ and every map $h: S^k \to V$, $k \leq n$, is null-homotopic in U; and X is *locally contractible*, $X \in LC$, if for each point $x \in X$ and each neighborhood U of x in X, there is a neighborhood V of x such that $V \subset U$ and V is contractible in U. Clearly LC^0 is equivalent to being locally connected and every locally contractible continuum is locally n-connected for every $n \geq 0$. A space X in the Hilbert cube Q is *approximately* n*-connected* if for each neighborhood U of X in Q, there is a neighborhood V of X such that $V \subset U$ and every map $h: S^k \to V$, $k \leq n$, is null-homotopic in U; and X has *trivial shape* (or *the shape of a point*) if for every neighborhood U of X in Q, there is a neighborhood V of X such that $V \subset U$ and V is contractible in U. Note that the properties of being approximatively n-connected and of having trivial shape are not dependent on the embedding of X in Q, and therefore we denote those by $X \in AC^n$ and $sh(X) = 0$, respectively. Our notations and terminologies in shape theory are due to [35]. We refer readers to see [35] for shape theory.

A map $f: X \to Y$ is UV^n-*map* if $f^{-1}(y) \in AC^n$ for each $y \in Y$; and f is a CE-*map* (*cell-like*) if $sh(f^{-1}(y)) = 0$ for each $y \in Y$. UV^n-maps and CE-maps are interesting and useful concepts in ANR theory, shape theory, and dimension theory, and have been investigated by many authors. The following fact suggests that refinable maps may be useful in those theories.

1.6. *Let* $r: X \to Y$ *be a refinable map. If* $Y \in LC^n$ *and* $n \geq 0$, *then* r *is a* UV^n-*map*.

Moreover, if $Y \in LC$, *then* r *is a* CE-*map* ([21]).

Finally, we show some of the relationships among the above types of spaces and maps. The converses of the implications are false.

ANR ⇒ LC ⇒ LC^n ⇒ locally connected ⇒ property [k]

CE ⇒ UV^n ⇒ monotone ⇒ confluent

⇒ weakly confluent

2. Refinable Maps in Continuum Theory

In this section we will consider a fixed but arbitrary refinable map $r: X \to Y$ between continua. Since X is Y-like and, by 1.2, r is weakly confluent, some properties in continuum theory can be considered. In this section we summarize interesting properties and make a table, which may clarify the role of refinable maps in continuum theory. The image (domain) column answers the question, does the image set (domain) of a refinable map have the indicated property, whenever the domain (image set, respectively) has that property?

Property	Image	Domain	Reference
decomposable	Yes	Yes	[15]
hereditarily decomposable	Yes	No	[26]
hereditarily indecomposable	Yes	Yes	[26]
irreducible	Yes	Yes	[26]
atriodic	Yes	Yes	[26]
property [k]	Yes	No	[26]
span of zero	Yes	Yes	[31]
aposyndetic	Yes	No	[20]
strongly aposyndetic	Yes	No	[45]
$\mathcal{P}$-like (not specified)	Yes	No	
$\mathcal{P}$-like ($\mathcal{P}$ consists of ANR's)	Yes	Yes	[15]
the pseudo-arc	Yes	Yes	[26]

Table

In [46] Wardle proved that confluent maps preserve property [k], and in [37], Question (16.38), Nadler asked: *What kinds of maps preserve property* [k]*?* Refinable maps give a partial answer to his question [26]. On the other hand, refinable preimages of continua having property [k] need not have property [k].

2.1. *Example* ([26]). In the plane R^2, put

$$X = \{(x, \sin \tfrac{1}{x}) \mid 0 < x \leq 1\} \cup \{(0,y) \mid -1 \leq y \leq \tfrac{3}{2}\} \cup \{(x, \tfrac{1}{2} + \sin \tfrac{1}{x}) \mid -1 \leq x < 0\},$$

$$Y = \{(x,0) \mid -1 \leq x \leq 1\}.$$

Define a map $r: X \to Y$ by $r(x,y) = (x,0)$ for $(x,y) \in X$. Then r is refinable and Y has property [k], but X does not have property [k].

Refinable maps preserve the contractibility of $C(X)$ for continua X that have property [k], since they preserve property [k] [26] and property [k] implies contractibility of the hyperspace [27]. It is not known whether contractibility of the hyperspace is preserved in the absence of property [k], so we have the following question. Note that in Example 2.1, $C(Y)$ is contractible but $C(X)$ is not contractible.

Question 1 ([26]). Let $r: X \to Y$ be a refinable map between continua. If $C(X)$ is contractible, then is $C(Y)$ contractible?

Other questions are listed as follows:

Question 2 ([26]). Let $r: X \to Y$ be a refinable map between continua. If X is irreducible between a and b, then is Y irreducible between $r(a)$ and $r(b)$?

A partial answer of Question 2 has appeared in [20].

Question 3 ([15]). Do refinable maps preserve the property of being rational?

3. Refinable Maps in Dimension Theory

A space X is *weakly infinite-dimensional* if for any countable family $\{(A_i,B_i) \mid i = 1,2,3,\cdots\}$ of pairs of disjoint closed subsets of X, there are separators S_i between A_i and B_i in X ($i = 1,2,3,\cdots,$) such that $\cap_{i\geq 1}S_i = \emptyset$. A space is *strongly infinite-dimensional* if it is not weakly infinite-dimensional. A space X is *countable-dimensional* if $X = \cup_{i \geq 1} X_i$ with $\dim X_i \leq 0$ for each $i \geq 1$.

3.1. (1) *Let* $r: X \to Y$ *be a refinable map. Then*

(i) $\dim X = \dim Y$ ([22] *and* [39], c.f. 6.7), *and*

(ii) *if* X *is weakly infinite-dimensional, then* Y *is also weakly infinite-dimensional* ([24]).

(2) *There exists a refinable map from a strongly infinite-dimensional* AR *onto a countable-dimensional* AR ([24]).

(3) *There exists a refinable map from a weakly infinite-dimensional, not countable-dimensional* AR *onto a countable-dimensional* AR ([24]).

It is well-known that if spaces X and Y are *quasi-homeomorphic*, i.e., X is Y-like and Y is X-like, then $\dim X = \dim Y$. In the constructions of 3.1 (2) and (3), the spaces are quasi-homeomorphic (c.f. 5.1 and 5.2 below). Hence some notions of infinite-dimensional spaces are not invariant under quasi-homeomorphisms.

We know that between countable-dimensionality and weak infinite-dimensionality there is another notion called property C. A space X has *property* C if for each sequence $\{\mathcal{U}_i \mid i \geq 1\}$ of open covers of X, there is an open cover $\mathcal{V} = \cup\{\mathcal{V}_i \mid i \geq 1\}$ of X such that for each $i \geq 1$, $\mathcal{V}_i$ is a pairwise disjoint collection which refines $\mathcal{U}_i$. Property C is investigated in ANR theory and shape theory and has proved to be useful and interesting in these theories.

Ancel [1] found a kind of map which preserves property C. A map $f: X \to Y$ is *approximatively invertible* [2] if some embedding $i: X \to Z$ has the following property: For every collection $\mathcal{W}$ of open subsets of Z which is refined by the family $\{i(f^{-1}(y)) \mid y \in Y\}$, there is a map $g: Y \to Z$ (not necessarily surjective) such that $\{\{gf(x), i(x)\} \mid x \in X\}$ refines $\mathcal{W}$.

3.2. *Approximatively invertible maps preserve property* C [2].

We can easily see that every refinable map is approximatively invertible. So we have

3.3. *Refinable maps preserve property* C.

Concerning these notions we know the following implications.

finite-dimension $\Rightarrow$ countable-dimension $\Rightarrow$ property C
$\Rightarrow$ weak infinite-dimension

So, related to our results, the next problem is interesting.

Question 4 ([24]). Do refinable maps preserve countable-dimension?

We note that example 3.1 (3) is based on R. Pol's example [41], which, in fact, has property C but is not countable-dimensional. It is still unknown *whether there exists a weakly infinite-dimensional space which does not have property* C. This is a big and important problem in dimension theory.

Next, we discuss an extension property which was motivated by dimension theory and was introduced in [30].

A space X is said to be *extendable with respect to a class* $\mathcal{K}$ *of* ANR's [30] if for any closed subset A of X, any element K of $\mathcal{K}$, and any map $f: A \to K$, there is a continuous extension of f over X. For example, for a normal space X, $\dim X \leq n$ if and only if X is extendable with respect to $\{S^n\}$.

A space X has *small cohomological dimension* $\leq$ n *with respect to an abelian group* G, written $d(X:G) \leq n$, provided that each map $f: A \to K(G:n)$, from a closed subset A of X into an Eilenberg-MacLane complex $K(G:n)$, extends to a map $\tilde{f}: X \to K(G:n)$. That is, $d(X:G) \leq n$ if and only if X is extendable with respect to $\{K(G:n)\}$. Concerning extendability we have

3.4. *Let* $r: X \to Y$ *be a refinable map. If* X *is extendable with respect to a class* $\mathcal{K}$ *of* ANR's, *then so is* Y ([30]).

3.5. *Let* $r: X \to Y$ *be a refinable map. Then* $d(X:G) \geq d(Y:G)$ *for every abelian group* C.

Question 5 ([30]). Let $r: X \to Y$ be a refinable map and let $\mathcal{K}$ be a class of ANR's. If Y is extendable with respect to $\mathcal{K}$, then is X also extendable with respect to $\mathcal{K}$?

Especially, does the inequality $d(X:G) \leq d(Y:G)$ hold?

4. Refinable Maps in ANR Theory

In this section we will consider refinable maps defined on ANR's. The following lemma from [33] is a key tool for investigating such refinable maps.

4.1. *Let* $f: X \to P$ *be a map from a space* X *to an* ANR P. *Then for every* $\varepsilon > 0$, *there is* $\delta > 0$ *such that if* $g: X \to Y$ *is a* δ*-map, there is a map* $h: Y \to P$ *such that* $d(hg,f) < \varepsilon$.

A space X is an *approximative polyhedron* (AP) [34] if for every $\varepsilon > 0$, there is a polyhedron P and there are maps $f: X \to P$ and $g: P \to X$ such that $d(gf,1_X) < \varepsilon$. Every ANR is clearly an AP. In [34], Mardešić showed that the notions of compact metric AP's, Borsuk's NE-sets [6] and Clapp's AANR's [11] are equivalent. Then by 4.1, we have interesting properties of refinable maps defined on AP's and ANR's.

4.2. *If* $r: X \to Y$ *is a refinable map and* X *is an* AP, *then for each* $\varepsilon > 0$, *there is a map* $g: Y \to X$ *such that* $d(gr,1_X) < \varepsilon$.

In particular, X *and* Y *are quasi-homeomorphic* ([15] and [40]).

4.3. *If* $r: X \to Y$ *is a refinable map and* X *is an* ANR, *then* X *is homotopy dominated by* Y. *Moreover, if* Y *is an* ANR, *then* r *is a homotopy equivalence* ([15]).

Now we notice the known results concerning quasi-homeomorphic ANR's. Eilenberg [12] showed that if X and Y are quasi-homeomorphic ANR's, then X homotopically dominates Y and Y homotopically dominates X. In particular, if X and Y are 2-dimensional planar ANR's, then X and Y are quasi-homeomorphic if and only if they are homotopy equivalent; and if X is an n-dimensional ANR and X is M-like for some closed n-dimensional manifold M, then X is homotopy equivalent to M ([16]). If the 2-dimensional ANR X is M-like for some closed 2-dimensional manifold M, then X is homeomorphic to M (see [16] and see [38] for a generalization). However, we do not know *whether every pair of quasi-homeomorphic* ANR*'s are homotopy equivalent.*

On the other hand, Borsuk [3] constructed a 3-dimensional continuum which is quasi-homeomorphic to the 3-ball but is not an ANR. In fact, it is not locally 1-connected and does not have the fixed point property. Hence, by the above discussion, the following problem seems to be the biggest one about refinable maps.

Question 6 ([15]). If $r: X \to Y$ is a refinable map and X is an ANR, need Y also be an ANR?

In particular, consider the case where X is the n-sphere or an n-manifold, $n \geq 3$.

We have partial answers as follows:

4.4. *Let* $r: X \to Y$ *be a refinable map. Then*

(i) *if* X *is a* 1-*dimensional* ANR, *then* Y *is also a* 1-*dimensional* ANR ([22] *and* [39]), *and*

(ii) *if* X *is an* ANR *which is embedded in a* 2-*manifold* M, *then* Y *is an* ANR *which is also embeddable in* M ([14] *and* [38]).

Heath and Kozlowski [14] have some interesting results related to Question 6, namely:

4.5. *Let* $r: X \to Y$ *be a refinable map defined on a finite-dimensional* ANR X. *If any one of the following conditions is satisfied, then* Y *is an* ANR:

(1) $r^{-1}(y)$ *is locally connected for each* $y \in Y$,

(2) $r^{-1}(y)$ *is nearly* 1-*movable for each* $y \in Y$,

(3) $r^{-1}(y) \in AC^1$ *for each* $y \in Y$,

(4) $Y \in LC^1$, *or*

(5) r *has a monotone* ε-*refinement for each* $\varepsilon > 0$.

Conversely, if Y *is an* ANR, *then, by* 1.6, *conditions* (1)-(4) *are satisfied.*

4.6. *Let* $r: X \to Y$ *be a refinable map. Then if* X *is an* ANR, $\check{H}^k(r^{-1}(y)) = 0 = \check{H}_k(r^{-1}(y))$ *for all* $k \geq 0$.

If it is possible to construct a refinable map on a finite-dimensional ANR whose image is not an ANR, each fiber must be acyclic and not nearly 1-movable. Continua having such properties are rare; the Case-Chamberlin curve is a typical one. So the example will be very complicated if it exists.

The following strong result and problem are also in [14].

4.7. *Let* X *be a compactum in* S^3. *Then the projection* $p: S^3 \to S^3/X$ *is refinable if and only if* X *is cellular, if and only if* r *is a near-homeomorphism.*

Question 7. Let X be a compactum in S^n, $n \geq 4$. If the projection $p: S^n \to S^n/X$ is refinable, then must X be cellular?

Concerning generalized ANR's, we have the following.

4.8. *If* r: X $\to$ Y *is a refinable map and* X *is an* AP, *then* Y *is also an* AP.

See [7] and [40] for other results concerning generalized ANR's and refinable maps.

5. Refinable Maps in Shape Theory

Related to 4.2, Heath and Rogers [15] posed the natural question: Do refinable maps preserve shape? First, Watanabe pointed out the negative answer using Borsuk's example [5]. In Borsuk's construction, decomposing spaces into their components played an important role. Hence his compacta are, unfortunately, not connected. Here we will construct locally connected continua X and Y in $\mathbb{R}^3$, and a refinable map r: X $\to$ Y such that sh(X) $\not\leq$ sh(Y), and X and Y are quasi-homeomorphic.

From [24], we obtain a method of constructing refinable maps.

5.1. *For an arbitrary space* Z, *there exist compacta* X, Y, *and a refinable map* r: X $\to$ Y *such that*

(1) Z *is a retract of* X,

(2) Y *is countable-dimensional, movable and locally connected, and*

(3) *if* Z *is a locally connected continuum,* X *and* Y *are quasi-homeomorphic.*

Moreover, if dim Z $\leq$ n, *we can construct* X *and* Y *so that* dim X $\leq$ n *and* dim Y $\leq$ n.

5.2. *Example* ([23]). Let Z be the non-movable 2-dimensional locally connected continuum in $\mathbb{R}^3$ defined by Borsuk [4]. Then there is an inverse sequence $((Z_n,z_n),f_n)$ of closed surfaces Z_n of genus n and surjective bonding maps f_n such that $\varprojlim (Z_n,f_n) = Z$. By the construction in 5.1, we obtain locally connected continua X and Y, and a refinable map $r: X \to Y$ such that X and Y are embeddable in $\mathbb{R}^3$. Then, by 5.1, (1) and (2), X is non-movable but Y is movable. Hence $sh(X) \not\leq sh(Y)$.

Concerning compacta in $\mathbb{R}^n$, we have the following ([23] and [42]).

5.3. *If* X *and* Y *are compacta in* $\mathbb{R}^n$ *and there is a refinable map from* X *onto* Y*, then* $\mathbb{R}^n - X$ *and* $\mathbb{R}^n - Y$ *have the same number of components.*

Therefore if X *and* Y *are in* $\mathbb{R}^2$*, then* $sh(X) = sh(Y)$.

In spite of the example, refinable maps play an interesting part in shape theory, and under some conditions the question of shape preservation has an affirmative answer. For this purpose we introduce some notions in shape theory.

Let $\mathcal{K}$ be an arbitrary category. A system map $F = \{f,f_b,B\}: \mathcal{X} = \{X_a,p_{aa'},A\} \to \mathcal{Y} = \{Y_b,q_{bb'},B\}$ is a *pseudo-isomorphism* [22] if for each $b \in B$ and each $a \geq f(b)$, there exist $g(a,b) \geq b$ and a morphism $g_{(a,b)}$: $Y_{g(a,b)} \to X_a$ such that for every $b' \geq g(a,b)$, there exist $h(b') \geq a$ and a morphism $h_{b'}: X_{h(b')} \to Y_{b'}$ such that

$$f_b p_{f(b)a} g_{(a,b)} = q_{bg(a,b)}, \text{ and}$$

$$g_{(a,b)} q_{g(a,b)b'} h_{b'} = p_{ah(b')}.$$

A morphism f: $\mathsf{X} \to \mathsf{Y}$ in pro-K is called a *pseudo-isomorphism* if it has a pseudo-isomorphism F: $\mathsf{X} \to \mathsf{Y}$ as its representation.

An inverse system $\mathsf{X} = \{X_a, p_{aa'}, A\}$ in pro-K is *calm* [8] if there exists $a_0 \in A$ such that

for any $a \geq a_0$, there is an $a^* \geq a$ such that if morphisms $f,g: Y \to X_{a^*}$ in K satisfy the condition $p_{a_0a^*}f = p_{a_0a^*}g$, then $p_{aa^*}f = p_{aa^*}g$.

5.4. *Let* f: $\mathsf{X} \to \mathsf{Y}$ *be a pseudo-isomorphism in pro-*K. *Then*

(1) *if* X *is movable, then* Y *is movable,*

(2) $\mathsf{X} \cong *$ *in pro-*K *if and only if* $\mathsf{Y} \cong *$, *where* $*$ *is the trivial system in pro-*K,

(3) *if* Y *is calm, then* f *is an isomorphism* ([22] *and* [25]).

5.5. *An inverse system* X *in pro-*K *is strongly movable if and only if it is movable and calm. Therefore, if* X *is a tower,* X *is stable in pro-*K *if and only if it is movable and calm* ([9]).

Let HCW be the category of spaces having the homotopy type of CW-complexes and homotopy classes of maps (not necessarily surjective). A map f: X $\to$ Y is a *pseudo-isomorphism* if there is a pseudo-isomorphism f: $\mathsf{X} \to \mathsf{Y}$ in pro-HCW which is associated with the map f.

A space X is *calm* if there exists a calm inverse system $\mathbf{X}$ in pro-HCW associated with X. Hence a space X is an FANR if and only if it is movable and calm.

5.6 ([22]). *If* $r: X \to Y$ *is a refinable map, then the shape morphism* $S(r)$ *induced by* r *is a pseudo-isomorphism. Therefore we have*

(1) *if* X *is movable, then* Y *is movable,*

(2) $X \in AC^n$ *if and only if* $Y \in AC^n$*,*

(3) $\operatorname{ddim} X = \operatorname{ddim} Y$*, and*

(4) X *is an* FAR *if and only if* Y *is an* FAR.

5.7 ([25]). *If* $r: X \to Y$ *is a refinable map and* Y *is calm, then* r *is a shape equivalence.*

5.8 ([25]). *If* $r: X \to Y$ *is a refinable map, and if* X *or* Y *is* S^n*-like,* $n \geq 1$*, then* r *is a shape equivalence.*

Related to the above results the following questions remain open.

Question 8 ([22]). If $r: X \to Y$ is a refinable map and X is an FANR, then need Y also be an FANR?

Question 9 ([25]). Do refinable maps preserve calmness?

An affirmative answer of Question 9 implies an affirmative answer of Question 8. Moreover, the next pro-group-theoretic problem may imply the answer to Question 8.

Question 10. Let $f: G \to \mathcal{H} = (H_n, h_n)$ be a morphism of pro-groups. If all groups are countable and f is a pseudo-isomorphism, then is the limit group $\varprojlim \mathcal{H}$ countable?

Some partial answers to Question 8 have been obtained by Kato [22] as follows:

5.9. *Let* $r: X \to Y$ *be a refinable map defined on an* FANR X. *If* $\operatorname{ddim} X \leq 1$ *or* $X \in AC^1$, *then* r *is a shape equiva-elnce. Therefore* Y *is an* FANR.

5.10. *If* $r: X \to Y$ *is a refinable map defined on a movable continuum* X *and* $\operatorname{ddim} X \leq 1$, *then* $\operatorname{sh}(X) = \operatorname{sh}(Y)$.

6. Generalizations (I)

In recent papers, [18] and [19], Grace introduced two generalizations of the notion of a refinable map called proximately refinable map and weakly refinable map. A function $f: X \to Y$ is ε-*continuous*, $\varepsilon > 0$, if for each $x \in X$, there is a neighborhood U of x in X such that $f(U) \subset B(f(x);\varepsilon)$, where $B(y;\varepsilon)$ is the open ε-ball around y. A surjective function $f: X \to Y$ is an ε-*function*, $\varepsilon > 0$, if $\operatorname{diam}[f^{-1}(y)] < \varepsilon$ for each $y \in Y$; and f is a *strong* ε-*function* if for each $y \in Y$, there is a neighborhood V of y in Y such that $\operatorname{diam}[f^{-1}(V)] < \varepsilon$.

Let $f: X \to Y$ be a surjective function. Then a surjective function $g: X \to Y$ is a *proximate* ε-*refinement* of f provided that g is ε-continuous, g is a strong ε-function, and $d(f,g) < \varepsilon$. A function $g: Y \to X$ is an *inverse* ε-*refinement* of f, if f is continuous, g is ε-continuous, and $d(1_Y, fg) < \varepsilon$.

A surjective function $f: X \to Y$ is *proximately refinable* if for each $\varepsilon > 0$, there is a proximate ε-refinement of f; and f is *weakly refinable* if f is continuous and for each $\varepsilon > 0$, there is an inverse ε-refinement of f. We can easily see that proximately refinable maps are continuous. Moreover, refinable maps are proximately refinable, and they in turn are weakly refinable. In [18] Grace extended the results in §1 as follows:

6.1. *Every proximately refinable map is weakly confluent.*

6.2. *If* $f: X \to Y$ *is a proximately refinable map and* Y *is locally connected, then* f *is monotone.*

Next, we show the following (c.f., 1.3).

6.3. *If* $f: X \to Y$ *is a proximately refinable map and* Y *has property* [k], *then* f *is confluent.*

Proof. Let K be a subcontinuum of Y and let C be a component of $f^{-1}(K)$. We will show that $f(C) = K$. For $i \geq 1$, let f_i be a proximate $1/i$-refinement of f. Let $x_0 \in C$. Then we have

(1) $\lim\limits_{\leftarrow i} f_i(x_0) = f(x_0) \in K$.

Since Y has property [k], (1) implies that there is a subsequence $\{i_n\}_{n\geq 1}$ of $\{i\}_{i\geq 1}$ such that for each $n \geq 1$, we have a subcontinuum K_{i_n} of Y satisfying the following conditions

(2) $f_{i_n}(x_0) \in K_{i_n}$, and

(3) $d_H(K, K_{i_n}) < \frac{1}{n}$.

Since 2^X is a compact metric space, we may assume that

(4) $\lim_{\vec{n}} \overline{f_{i_n}^{-1}(K_{i_n})} = D \in 2^X$.

Note that $x_0 \in D$.

We will show that D is a subcontinuum of X. Suppose, on the contrary, that $D = D_1 \cup D_2$, where D_1 and D_2 are disjoint closed non-empty subsets of D. Choose $\delta > 0$ such that $0 < 3\delta < d(D_1,D_2)$. By (4), there is a sufficiently large number n such that

(5) f_{i_n} is a strong δ-function, and

(6) $d_H(\overline{f_{i_n}^{-1}(K_{i_n})},D) < \delta$.

Set $D_1' = \{x \in f_{i_n}^{-1}(K_{i_n}) \mid d(x,D_1) \leq \delta\}$, and

$D_2' = \{x \in f_{i_n}^{-1}(K_{i_n}) \mid d(x,D_2) \leq \delta\}$.

Note that $D_1' \cap D_2' = \emptyset$. Set, also,

$K_1' = \{y \in K_{i_n} \mid f_{i_n}^{-1}(y) \subset D_1'\}$, and

$K_2' = \{y \in K_{i_n} \mid f_{i_n}^{-1}(y) \subset D_2'\}$.

By (5), K_1' and K_2' are non-empty open subsets of K_{i_n}. Clearly, $K_1' \cup K_2' = K_{i_n}$ and $K_1' \cap K_2' = \emptyset$. This is a contradiction. Hence D is connected.

On the other hand, since f_{i_n} is a proximate $\frac{1}{i_n}$-refinement of f, we have

(7) $f(D) = f(\lim_{\vec{n}} \overline{f_{i_n}^{-1}(K_{i_n})}) = \lim_{\vec{n}} \overline{f(f_{i_n}^{-1}(K_{i_n}))}$

$\subset \lim_{\vec{n}} B(K_{i_n};\frac{2}{i_n}) = K$, and

$$(8)\quad f(D) = \lim_{\vec{n}} \overline{f(f_{i_n}^{-1}(K_{i_n}))} = \lim_{\vec{n}} \overline{f_{i_n}(\overline{f_{i_n}^{-1}(K_{i_n})})}$$

$$\supset \lim_{\vec{n}} K_{i_n} = K.$$

Hence $f(D) = K$. Since C is a component of $f^{-1}(K)$ and $C \cap D \neq \emptyset$, $C \supset D$. Therefore $f(C) = K$. This completes the proof.

Klee [28] introduced a notion related to the fixed point property (f.p.p.). A metric space X has the *proximate fixed point property* (*p.f.p.p.*) if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that every δ-continuous function $f: X \to X$ has a point x such that $d(x,f(x)) < \varepsilon$. It is easily seen that a compactum with the p.f.p.p. has the f.p.p., and the converse is not valid. It is known that refinable maps do not preserve the f.p.p. On the other hand, Grace [19] showed the following:

6.4. *If* $f: X \to Y$ *is a weakly refinable map and* X *has the p.f.p.p., then* Y *has the p.f.p.p.*

Next, we examine the relationships of these generalizations of refinability to some of the properties of compacta that we have studied in previous sections. Grace [18] has proved the next fact.

6.5. *If* $f: X \to Y$ *is a weakly refinable map and* X *is a graph, then* Y *is a graph.*

In the same way as in [26], we can prove the following theorem. The details are left to the reader.

6.6. *If* $f: X \to Y$ *is a proximately refinable map and* X *is irreducible, then* Y *is irreducible.*

The following generalizes part of 3.1.

6.7. *If* $f: X \to Y$ *is a proximately refinable map, then* $\dim X \geq \dim Y$.

Proof. It suffices to consider the case where $\dim X = k < \infty$. Let $\mathcal{V} = \{V_1, V_2, \cdots, V_m\}$ be a given finite open cover of Y, and let $\mathcal{G} = \{G_1, G_2, \cdots, G_m\}$ be a finite open cover of Y such that $\overline{G}_j \subset V_j$ for each j, $1 \leq j \leq m$. Since $\dim X = k$, there exists a finite open cover $\mathcal{U} = \{U_1, U_2, \cdots, U_n\}$ and a finite closed cover $\mathcal{H} = \{H_1, H_2, \cdots, H_n\}$ of X such that

(1) $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{G} = \{f^{-1}(G_1), \cdots, f^{-1}(G_m)\}$,

(2) $\operatorname{ord} \mathcal{U} \leq k+1$, and

(3) $U_i \supset H_i$ for each i, $1 \leq i \leq n$.

Choose $\delta > 0$ such that

(4) $V_j \supset B(\overline{G_j}; \delta)$ for $j = 1, 2, \cdots, m$, and

(5) $U_i \supset B(H_i; \delta)$ for $i = 1, 2, \cdots, n$.

Since f is proximately refinable, there is a proximate δ-refinement g of f. Then g satisfies the following conditions.

(6) $d(f, g) < \delta$,

(7) g is δ-continuous, and

(8) for each $y \in Y$, there exists an open neighborhood $W(y)$ of y in Y such that $\operatorname{diam}[g^{-1}(W(y))] < \delta$.

Now we define open sets W_i, for $i = 1, 2, \cdots, n$, in Y as follows:

(9) $W_i = \cup\{W(y) \mid y \in g(H_i)\}$.

Then by (6) and (8),

(10) $H_i \subset g^{-1}(W_i) \subset B(H_i;\delta) \subset U_i$ for $i = 1,2,\cdots,n$.

Hence we have an open cover $\mathcal{W} = \{W_1, W_2, \cdots, W_n\}$ of Y with ord $\mathcal{W} \leq$ ord $\mathcal{U} \leq k+1$. Therefore, it suffices to show that $\mathcal{W}$ refines $\mathcal{V}$. For each $i = 1,2,\cdots,n$, by (1) and (10), there is a member G_j of $\mathcal{G}$ such that $g^{-1}(W_i) \subset f^{-1}(G_j)$. Then by (4) and (6),

$$W_i \subset g(f^{-1}(G_j)) \subset B(\overline{G_j};\delta) \subset V_j.$$

Hence $\mathcal{W}$ is a refinement of $\mathcal{V}$. This completes the proof.

In the case of refinable maps the converse of 6.6 and 6.7 with the inequality reversed are valid (see [26], [22] and [39]). However, we will construct a proximately refinable map in which they do not hold.

6.8. *Example*. First choose AR's in the plane R^2

$$X = \{(x,y) \mid \text{Either } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \text{ or } 1 \leq x \leq 2 \text{ and } y = 0\},$$

$$Y = \{(x,y) \mid 1 \leq x \leq 2 \text{ and } y = 0\},$$

and define a map $f\colon X \to Y$ by

$$f(x,y) = \begin{cases} (1,0) & \text{if } x \leq 1 \\ (x,0) & \text{if } x \geq 1. \end{cases}$$

Then f is proximately refinable. However, X is not irreducible, although Y is irreducible. Moreover, $\dim X = 2 > \dim Y = 1$.

Next, we will consider some properties which are preserved by refinable maps but are not preserved by proximately refinable maps. First, we construct a proximately refinable map which does not preserve property [k].

6.9. *Example.* Let p be the pole (i.e., the origin) of the polar coordinate system in the plane $\mathbb{R}^2$. Consider the following points, represented in polar coordinates.

$$a = (2,0),\ a_n = (2,-\tfrac{1}{n}),$$
$$b = (1,1),\ b_n = (1,1+\tfrac{1}{n}),$$
$$c = (1,0), \text{ and } c_n = (1,\tfrac{1}{n}),\ n = 1,2,3,\cdots,$$

and let

$$X = \overline{pa} \cup (\cup_{n\geq 1}\overline{pa_n}) \cup \overline{pb} \cup (\cup_{n\geq 1}\overline{pb_n}), \text{ and}$$
$$Y = \overline{pa} \cup (\cup_{n\geq 1}\overline{pa_n}) \cup \overline{pc} \cup (\cup_{n\geq 1}\overline{pc_n}),$$

where $\overline{xy}$ stands for the straight line segment joining x and y. Now define a map $f: X \to Y$ by the formula;

$$f(r,\theta) = \begin{cases} (r,\theta) & \text{if } (r,\theta) \in \overline{pa} \cup (\cup_{n\geq 1}\overline{pa_n}) \\ (r,\theta-1) & \text{if } (r,\theta) \in \overline{pb} \cup (\cup_{n\geq 1}\overline{pb_n}). \end{cases}$$

For each $n \geq 1$, we define a surjective function $f_n: X \to Y$ by the following:

$$f_n(r,\theta) = \begin{cases} (r,\theta) & \text{if } (r,\theta \in \overline{pa} \cup (\cup_{i=1}^{n} pa_i), \\ (r,-\frac{1}{n+i}) & \text{if } (r,\theta) \in \overline{pa}_{n+(2i-1)},\ i \geq 1, \\ (r,\frac{1}{n+i}) & \text{if } (r,\theta) \in \overline{pa}_{n+2i},\ i \geq 1, \text{ and} \\ & \qquad 0 \leq r \leq 1, \\ (r,0) & \text{if } (r,\theta) \in \overline{pa}_{n+2i},\ i \geq 1, \text{ and} \\ & \qquad 1 < r \leq 2, \\ (r,\theta-1) & \text{if } (r,\theta) \in \cup_{i=1}^{n}\overline{pb}_i, \\ (r,\frac{1}{n}) & \text{if } (r,\theta) \in (\cup_{i\geq n+1}\overline{pb}_i) \cup \overline{pb} \end{cases}$$

Then we can easily see that each f_n is a proximate $\frac{2}{n}$-refinement of f. Hence f is proximately refinable. Clearly, X has property [k] but Y does not have property [k]. Therefore we have the desired proximately refinable map defined on the fan.

6.10. *Remark*. A map $f: X \to Y$ is *approximately right invertible (ARI)* if for each $\varepsilon > 0$, there exists a map $g: Y \to X$ (not necessarily surjective) such that $d(fg, 1_Y) < \varepsilon$. Weak refinability generalizes ARI in the same way as proximate refinability generalizes refinability. In [32], Boxer posed *problem* 57: *Do ARI-maps preserve property* [k]? Here we show that Example 6.9 implies the negative answer. For each $n \geq 1$, define a map $g_n: Y \to X$ by

$$g_n(r,\theta) = \begin{cases} (r,\theta) & \text{if } (r,\theta) \in \overline{pa} \cup (\cup_{i\geq 1}\overline{pa}_i), \\ (r,0) & \text{if } (r,\theta) \in \cup_{i\geq n+1}\overline{pc}_i \\ (r,\theta+1) & \text{if } (r,\theta) \in \cup_{i=1}^{n}\overline{pc}_i. \end{cases}$$

Then it is easily seen that for each $\varepsilon > 0$, there exists a sufficiently large $n \geq 1$ such that $d(fg_n, 1_Y) < \varepsilon$. Hence the map f is the ARI-map which does not preserve property [k].

Finally, we give a proximately refinable map from a 1-dimensional ANR onto a 1-dimensional AR which is neither a UV^1-map nor a pseudo-isomorphism.

6.11. *Example*. We use the polar coordinate system in $\mathbb{R}^2$ as in Example 6.9. Define

$$X = \{(r,\theta) \mid \text{Either } r = 1 \text{ and } 0 \leq \theta \leq 2\pi \text{ or } 1 \leq r \leq 2 \text{ and } \theta = 0\},$$

$$Y = \{(r,\theta) \mid 1 \leq r \leq 2 \text{ and } \theta = 0\},$$

and let $f: X \to Y$ be a mpa defined by

$$f(r,\theta) = \begin{cases} (r,\theta) & \text{if } (r,\theta) \in Y, \\ (1,0) & \text{otherwise.} \end{cases}$$

For each $0 < \varepsilon < \frac{1}{2}$, we define a surjective function $g: X \to Y$ by the formula:

$$g(r,\theta) = \begin{cases} (-\frac{\varepsilon}{2\pi-\varepsilon}\theta + 1 + 2\varepsilon, 0) \\ \qquad \text{if } r = 1,\ 0 \leq \theta \leq 2\pi - \varepsilon, \\ (\frac{1+\varepsilon}{\varepsilon}(-2\theta + 4\pi - \varepsilon), 0) \\ \qquad \text{if } r = 1,\ 2\pi - \varepsilon \leq \theta \leq 2\pi - \frac{\varepsilon}{2}, \\ (1,0) \text{ if } r = 1,\ 2\pi - \frac{\varepsilon}{2} \leq \theta < 2\pi, \\ ((1-2\varepsilon)r + 4\varepsilon, 0) \text{ if } \theta = 0. \end{cases}$$

Then g is a proximate 3ε-refinement of f. Hence f is proximately refinable. Clearly, the map f satisfies the desired conditions.

7. Generalizations (II)

In this section all spaces are assumed only to be Hausdorff, and a map means a continuous function from a Hausdorff space onto another one. Here we will try to extend the results in previous sections. A map $f: X \to Y$ is said to be a *$\mathcal{U}$-map* for an open cover $\mathcal{U}$ of X provided that, for every $y \in Y$, there is a $U \in \mathcal{U}$ such that $f^{-1}(y) \subset U$. Two maps $f,g: X \to Y$ are said to be *$\mathcal{V}$-near* for an open cover $\mathcal{V}$ of Y provided that for every $x \in X$, there is $V \in \mathcal{V}$ which contains both $f(x)$ and $g(x)$. Watanabe [47] defined refinable maps for arbitrary spaces: A map $r: X \to Y$ is *refinable* if for every normal open cover $\mathcal{U}$ of X and every normal open cover $\mathcal{V}$ of Y, there is a $\mathcal{U}$-map $f: X \to Y$ such that r and f are $\mathcal{V}$-near. We call such a map f a *$(\mathcal{U},\mathcal{V})$-refinement* of r. In the case that both X and Y are compact metric spaces, Watanabe's definition of refinable maps clearly coincides with that given by Heath and Rogers.

A map $f: X \to Y$ is *proper* if for every compact subset $C \subset Y$, $f^{-1}(C)$ is compact; and f is *perfect* if it is closed and proper. Related to the results in §5 and §6, we have

7.1 ([29]). *If* r: X → Y *is a proper refinable map between locally compact paracompact spaces, then* r *is a pseudo-isomorphism.*

Therefore, if Y *is calm, then* r *is a shape equivalence.*

7.2 ([29]). *If* r: X → Y *is a proper refinable map between locally compact paracompact spaces and* X *is an* AP, *then* Y *is also an* AP.

Refinable maps play an interesting role in dimension theory for a large class of spaces which are neither compact nor metrizable. For instance, we have

7.3 ([30]). *Let* r: X → Y *be a refinable map between compact spaces and let* $\mathcal{K}$ *be a class of* ANR's. *Then if* X *is extendable with respect to* $\mathcal{K}$, Y *is extendable with respect to* $\mathcal{K}$. *Therefore it follows that*

(1) dim X = dim Y, *and*

(2) d(X:G) $\geq$ d(Y:G) *for every abelian group* G.

7.4 ([30]). *If* r: X → Y *is a refinable map between compact spaces and* X *is weakly infinite-dimensional, then* Y *is also weakly infinite-dimensional.*

In order to study non-compact spaces we introduce a special class of refinable maps. A map r: X → Y is said to be *closed-refinable* (abbreviated, c-*refinable*) if for every normal open cover $\mathcal{U}$ of X and every normal open cover $\mathcal{V}$ of Y, there is a $(\mathcal{U},\mathcal{V})$-refinement of r which is a closed map. We call such a map a c-$(\mathcal{U},\mathcal{V})$-*refinement* of r.

We note that a refinable map between locally compact paracompact spaces is c-refinable if and only if it is proper.

7.5 ([30]). *If a map* $r: X \to Y$ *between normal spaces is* c-*refinable, then the extension* $\beta f: \beta X \to \beta Y$ *is refinable, where* βZ *is the Stone-Čech compactification of a completely regular space* Z. *Therefore, it follows that*

(1) $\dim X = \dim Y$, *and*

(2) $d_F(X:G) \geq d_F(Y:G)$ *for every abelian group* G, *where* $d_F(-:G)$ *is the small cohomological dimension based on all finite open covers.*

A space X is S-*weakly infinite-dimensional* if for any countable family $\{(A_i,B_i) | i = 1,2,3,\cdots\}$ of pairs of disjoint closed subsets of X, there are separators S_i between A_i and B_i in X, $i = 1,2,3,\cdots$, such that $\cap_{i=1}^n S_i = \emptyset$ for some integer $n \geq 1$.

7.6 ([30]). *If a map* $r: X \to Y$ *between normal spaces is* c-*refinable and* X *is* S-*weakly infinite-dimensional, then* Y *is* S-*weakly infinite-dimensional.*

In [30], the second author posed the problem: *Is there a proper map that is refinable but not* c-*refinable for which the above results are not valid?* Recently, E. van Douwen communicated that for each $n = 1,2,\cdots,\infty$, there is a normal space X_n having the property that $\dim X_n = n$ and every compact subset of X_n is finite. Independently, Tamano [44] constructed a Lindelöf non-zero-dimensional space X having the property that every compact subset of X is finite. Hence considering the identity maps $1_{X_n} : kX_n \to X_n$ and $1_X : kX \to X$, where kZ is the k-leader of a space Z (see [44]), the problem has a negative answer. Note that van Douwen's examples are not Lindelöf and ind $X_n = 0$ for $n \geq 1$.

The following problems are open:

Question 11. Can 7.5 and 7.6 be generalized to apply to any perfect, refinable map r?

Question 12. Do c-refinable maps between normal spaces preserve property C?

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