TOPOLOGY PROCEEDINGS

Volume 11, 1986

Pages 349–383

http://topology.auburn.edu/tp/

HAUSDORFF DIMENSION

by

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Topology Proceedings

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ISSN:	0146-4124

HAUSDORFF DIMENSION

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Introduction

The theory of Hausdorff dimension has come to play an important role in many different areas of mathematics and science. Physicists and biologists among others are using the concept. Mandelbrot ([15] and [16]) lists many diverse areas of application including music, the study of coastlines and landforms, fluid dynamics and turbulence, distribution of stellar matter, Brownian motion, the geometry of soap bubbles and liquid crystals, polymer geometry in chemistry, and the study of word frequencies in language. Not all applications are mathematically rigorous, but some are in classical areas of pure mathematics. One encounters Hausdorff dimension in geometry ([5], [26], [31], and [32]), geometric measure theory ([6] and [7]), calculus of variations ([6] and [7]), ergodic theory ([2] and [35]), stochastic processes ([20] and [29]), partial differential equations ([24] and [30]), dynamical systems theory ([2], [13], and [35]), complex function theory ([2]), dimension theory ([10]), and geometric topology ([33]).

Theorem 7.5 and Note 7.6 are new and answer a question posed by Kaplan and Yorke in [13]. Some other results may be new, but the purpose of the paper is to introduce Hausdorff dimension to topologists, not to present advances in the field. Proofs are included where they are helpful or difficult to locate in the literature.

In the first section of the paper we give the definitions of Hausdorff measure and dimension. In the second section we state elementary properties. In the third section we discuss self-similarity. In the fourth section Hausdorff dimension in Brownian motion is presented. In sections five and six we give some theorems in Hausdorff dimension from a topologist's viewpoint. Applications of these theorems are given in the areas of point set and geometric topology. In section seven mappings and Hausdorff dimension are studied. The mappings of interest in Hausdorff dimension theory are those of Lipschitz class δ. With these mappings one can establish a relationship between the Hausdorff dimension of the domain and range. For maps that are only continuous this is not possible. In section eight we discuss the question whether a compact metric space of infinite Hausdorff dimension has closed subsets of every finite dimension. This problem is related to the topological dimension theory of infinite dimensional spaces. Quite different techniques will be required in solving the problem in Hausdorff dimension theory as is shown by theorems and examples presented in this paper.

The bibliography included here is brief. However, there are extensive bibliographies in the works of Falconer ([5]), Mandelbrot ([15] and [16]), and Rogers ([21]).

Hausdorff measure and Hausdorff dimension are only defined for metric spaces which are separable. All spaces that we consider are separable metric spaces with a specified metric.

1. Basic Definitions

In this section of the paper we give the definition of Hausdorff dimension. We first define Hausdorff measure.

1.1 *Definition.* Let X be a separable metric space and let p be a real number $0 \leq p < \infty$. Let $\varepsilon > 0$. We let

$$\Lambda_{p,\varepsilon}(X) = \inf_{\text{mesh}\mathcal{U}<\varepsilon} \sum_{U_i \in \mathcal{U}} (\text{diam}(U_i))^p$$

We then let

$$\Lambda_p(X) = \sup_{\varepsilon>0} \Lambda_{p,\varepsilon}(X)$$

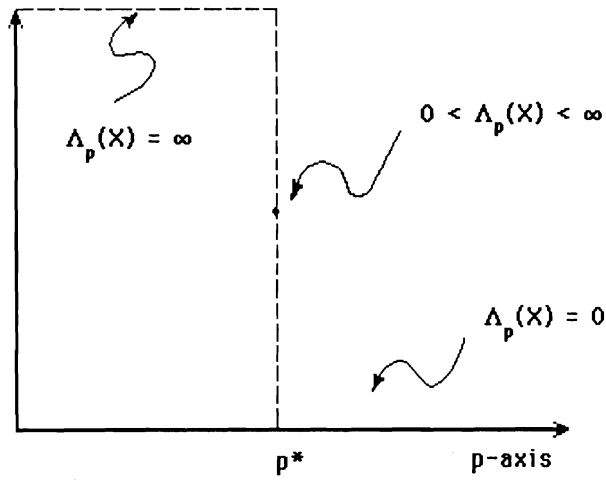
$\Lambda_p(X)$ is the p-*measure of* X. It is not a topological invariant of a separable metric space since it depends on the specific metric used. The following property holds for $\Lambda_p(X)$. If $q > p$ and $\Lambda_p(X)$ is finite, then $\Lambda_q(X) = 0$. We give a quick proof of this fact.

1.2 *Theorem. Suppose that* $q > p$ *and that* $\Lambda_p(X)$ *is finite, then* $\Lambda_q(X) = 0$.

Proof. Suppose that $q > p$ and that $\Lambda_p(X) < \infty$. Then there exists an $M > 0$ such that for all $\varepsilon > 0$ there is a covering $\mathcal{U}_\varepsilon$ of X of mesh less than ε such that $\sum_{U_i \in \mathcal{U}_\varepsilon} (\text{diam}(U_i))^p < M$. But this implies that for this same ε $\sum_{U_i \in \mathcal{U}_\varepsilon} (\text{diam}(U_i))^q = \sum_{U_i \in \mathcal{U}_\varepsilon} (\text{diam}(U_i))^p \times (\text{diam}(U_i))^{q-p} \leq \sum_{U_i \in \mathcal{U}_\varepsilon} (\text{diam}(U_i))^p \times \varepsilon^{q-p} < M \times \varepsilon^{q-p}$. But this implies that as $\varepsilon \searrow 0$, $\sum_{U_i \in \mathcal{U}_\varepsilon} (\text{diam}(U_i))^q \to 0$ as well. Thus $\Lambda_q(X) = 0$.

Now we are prepared to give the definition of Hausdorff dimension.

1.3 *Definition.* Let $p^* = \inf\{p > 0 | \Lambda_p(X) = 0\}$. Then $\Lambda(X) = p^*$ is the *Hausdorff dimension of* X. Theorem 1.2 shows that the p-measure of X can be finite for at most one value of p, namely for $p = \Lambda(X)$. For all other values $\Lambda_p(X)$ is either zero of infinity. It may happen that $\Lambda_{p^*}(X)$ may also be zero or infinity. Note also that $\Lambda_{p^*}(X)$ has not been defined if $p^* = \infty$.



Graph of Hausdorff p-Measure

There are other concepts related to Hausdorff dimension which are sometimes assumed to be equivalent. Several of these are not equivalent to Hausdorff dimension and one needs to note the precise definition used by a particular author. We use the above definition throughout.

2. Elementary Properties

In this section we state some elementary properties of Hausdorff dimension. They are part of the lore that helps

to justify the definition and motivate further research. The first property is trivial.

2.1 *Proposition.* *If* X *is a separable metric space and* $A \subset X$, *then* $\Lambda(A) \leq \Lambda(X)$.

The next property is not so obvious, but is extremely useful. It allows us to make use of measure theory in studying Hausdorff dimension.

2.2 *Theorem.* *Let* X *be a separable metric space and* $0 \leq p < \infty$. *Then* $\Lambda_p(A)$ *is a countably-additive measure on the Borel subsets* A *of* X. *If* $A \subset X$ *is any subset, then there is a* G_δ*-set* $G \supset A$ *in* X *such that* $\Lambda_p(G) = \Lambda_p(A)$.

The next result shows that $\Lambda(P)$ is the appropriate number for polygons with linear metrics.

2.3 *Theorem.* *Let* I^n *be the unit* n*-cube in* R^n *with the usual Euclidean metric.* *Then* $\Lambda(I^n) = n$.

The following result shows that contrary to naive expectation we do not have $\Lambda_n(I^n) = 1$. Thus Hausdorff n-dimensional measure and Lebesgue measure differ in R^n by a constant factor.

2.4 *Theorem.* *Let* R^n *have the usual Euclidean metric and let* $A \subset R^n$. *Let* λ^* *denote Lebesgue outer measure on the subsets of* R^n. *Then* $\Lambda_n(A) = \alpha_n \times \lambda^*(A)$.

The factor $\alpha_n = 1/V_n$ where V_n is the n-dimensional volume of the ball of diameter one. If we let Γ denote

the gamma function, then V_n is given by the following formula.

$$V_n = \frac{\Gamma(1/2)^n \times (1/2)^n}{\Gamma(n/2 + 1)}$$

Note that as $n \to \infty$, $V_n \to 0$.

2.5 *Corollary. Let* λ *denote Lebesgue* n*-dimensional measure on* R^n *and let* $C \subset R^n$ *a Cantor set with* $\lambda(C) > 0$. *Then* $\Lambda(C) = n$.

Cantor sets have topological dimension zero. Since there are Cantor sets in R^n with positive n-dimensional Lebesgue measure, these Cantor sets are examples of separable metric spaces whose Hausdorff dimension is n and whose topological dimension is zero. By Theorem 2.6 below, the Hausdorff dimension is always greater than or equal to the topological dimension.

2.6 *Theorem. Let* X *be a separable metric space. Then* $\dim X \leq \Lambda(X)$.

2.7 *Theorem. Let* C *be the Cantor middle-third set. Then* $\Lambda(C) = \frac{\ln(2)}{\ln(3)}$.

This result was first discovered by Hausdorff and together with the next example shows that Hausdorff dimension need not be an integer.

2.8 *Theorem. Let* K *be the Koch Curve. Then* $\Lambda(K) = \frac{\ln(4)}{\ln(3)}$.

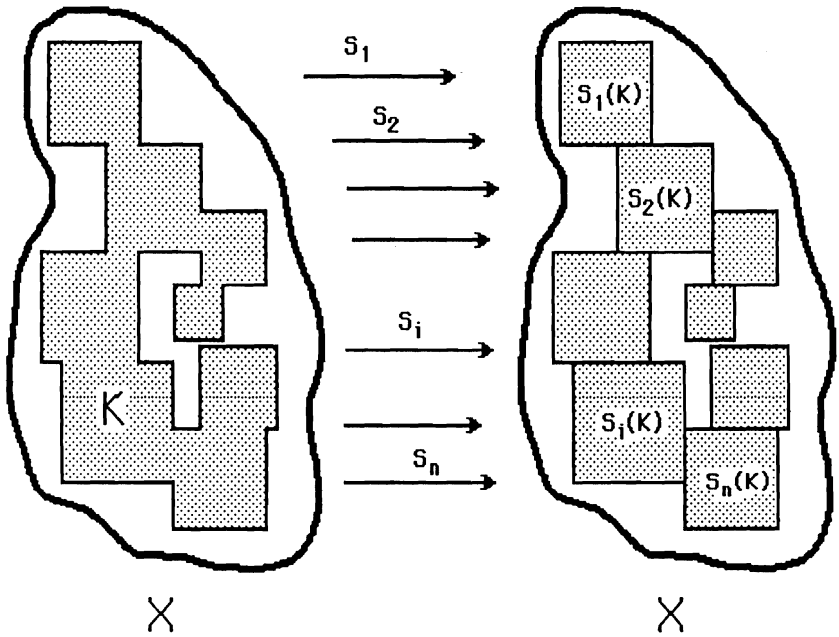
The Koch Curve will be described in the next section as well as a method for computing the Hausdorff dimension of it and the Cantor set.

3. Self-Similarity

There is a difficulty in determining the Hausdorff dimension of a space. The topological dimension of the space is a lower bound by Theorem 2.6. Suppose that $p > 0$ is a fixed number. Suppose that there is a sequence of covers of the space $\{\mathcal{U}_i \mid i=1,2,\cdots\}$ such that the mesh of the covers goes to zero and $\lim_{i\to\infty} \sum_{U_j\in\mathcal{U}_i} (\text{diam}(U_j))^p \to 0$. Then from the definition of Hausdorff dimension one must have $\Lambda(X) \leq p$. The difficulty is that given $\Lambda(X) \leq p$, how can one determine when $\Lambda(X) = p$? If it happens that $p = \dim X$, then $\Lambda(X) = p$ is clear. Also, if $X \subset R^n$ and X has positive n-dimensional Lebesgue outer measure, then $\Lambda(X) = n$. Apart from these two cases the most useful and easily applied tool for determining the precise Hausdorff dimension of spaces is *self-similarity*. We give a brief description of the concept here. For details consult the paper of Hutchinson ([11]). See also the paper by Moran ([18]).

Assume that X is a complete separable metric space. A *contraction mapping* of X to itself is a map $S: X \to X$ such that there exists a $K < 1$ such that for all x and y in X, $d(S(x),S(y)) \leq K \times d(x,y)$. A *similitude* of X to itself is a map $S: X \to X$ such that there is a number $0 < t_S < 1$ with $d(S(x), S(y)) = t_S \times d(x,y)$ for all x and y in X. Let $\{S_i \mid i=1,2,\cdots,n\}$ be a collection of contraction

mappings of X to itself. Let S be a function defined by $S(A) = \cup\{S_i(A) \mid i=1,2,\cdots,n\}$. Let $\mathcal{C}(X)$ denote the space of all compact subsets of X with the Hausdorff metric. Then $S: \mathcal{C}(X) \to \mathcal{C}(X)$ is a contraction mapping with $\mathcal{C}(X)$ a complete metric space. Thus S has a unique fixed point. That is, there is a unique compact set $K \subset X$ such that $K = \cup\{S_i(K) \mid i=1,2,\cdots,n\}$. The set K is the *invariant set* of S and K is said to be *self-similar*.



Suppose that the contraction mappings are actually simulitudes with $0 < t_i < 1$ the number associated with the similitude S_i. Then the following formula determins the *similarity dimension* D of K.

$$\sum_{i=1}^{n} t_i^D = 1$$

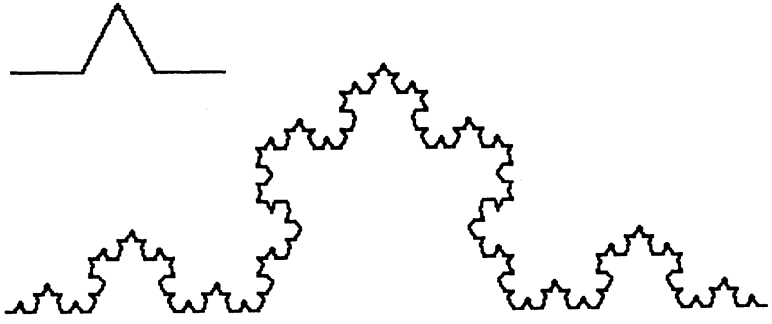
When certain conditions are met the Hausdorff dimension of this compact invariant set K is given by $\Lambda(K) = D$. The conditions are not difficult, but we do not go into detail. The basic idea is that the Hausdorff D-measure of K should be the sum of the Hausdorff D-measures of the subsets $S_i(K)$ and that this number should be finite and non-zero. Clearly, the Cantor middle-third set is self-similar. There are two similitudes on the real line $\{S_1, S_2\}$ that define the Cantor set C such that $C = S_1(C) \cup S_2(C)$. The S_i's are such that $t_i = 1/3$ for $i = 1$ and 2. The required conditions are met so that the Hausdorff dimension of C is given by D where

$$(1/3)^D + (1/3)^D = 1.$$

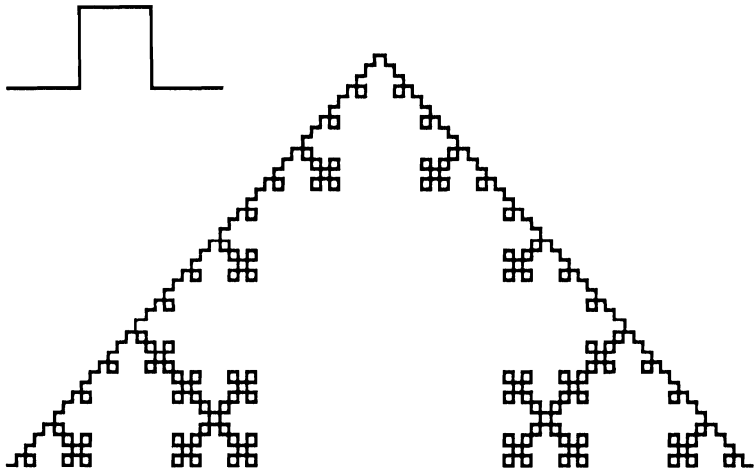
The solution for D is clearly $D = \frac{\ln(2)}{\ln(3)}$.

The Koch Curve, K, is determined by four similitudes in R^2. For each of them $t_i = 1/3$. The conditions are met so that its similarity dimension is equal to its Hausdorff dimension which is thus given by $\Lambda(K) = \frac{\ln(4)}{\ln(3)}$. An approximation of this curve is given below.

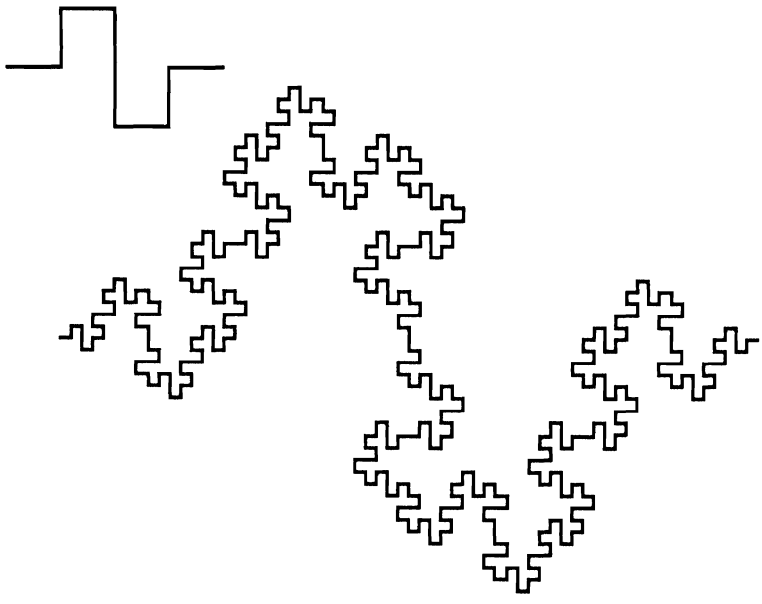
The polygonal line helps to visualize the relationships between the four similitudes that define the Koch Curve.



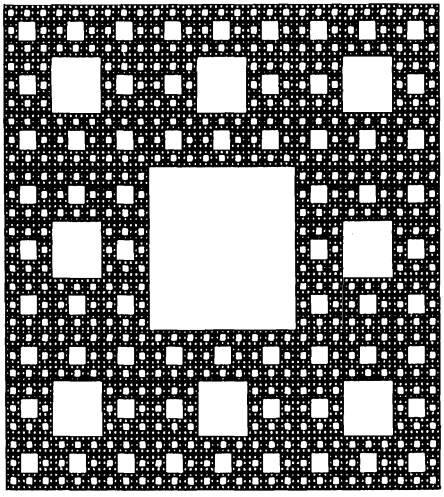
The Koch Curve, $\Lambda(K) = 1.2619$



Self-Similar Set K in R^2 Determined by
Five Similitudes, $\Lambda(K) = 1.4650$



Self-Similar Set in R^2 Determined by
Eight Similitudes, $\Lambda(K) = 1.5000$



The Sierpinski Plane Curve Is Determined by Eight Similitudes, $\Lambda(K) = 1.8928$

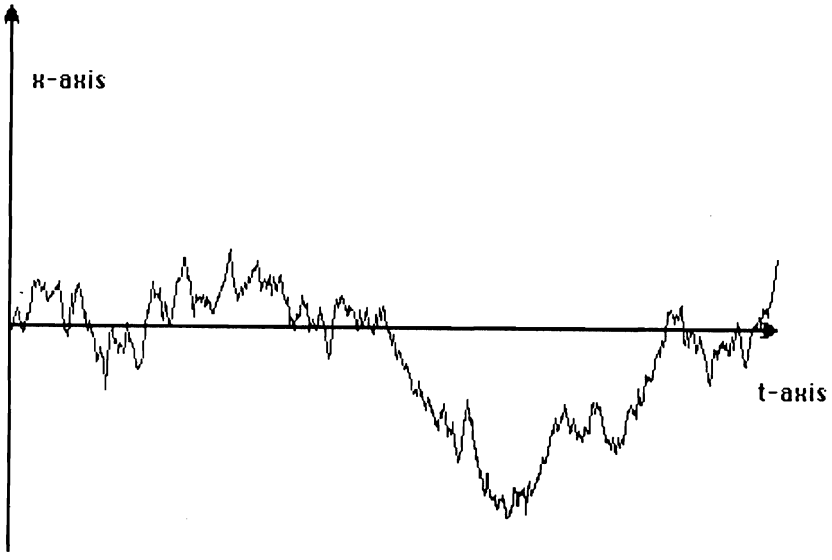
Computer graphics programs have been written which will generate similar examples of plane sets generated by such similitudes. Such programs can be made interactive so that the first polygonal approximation of the set can be drawn and the computer generates several iterations of the function S to further approximate the self-similar set. The first three examples above were produced in this fashion. A similar program produced the familiar Sierpinski Curve.

The books by Mandelbrot [15] and [16] and the book by Peitgen and Richter [19] have beautiful graphics illustrating what intricate geometric patterns can be generated in fractal theory using very simple algorithms.

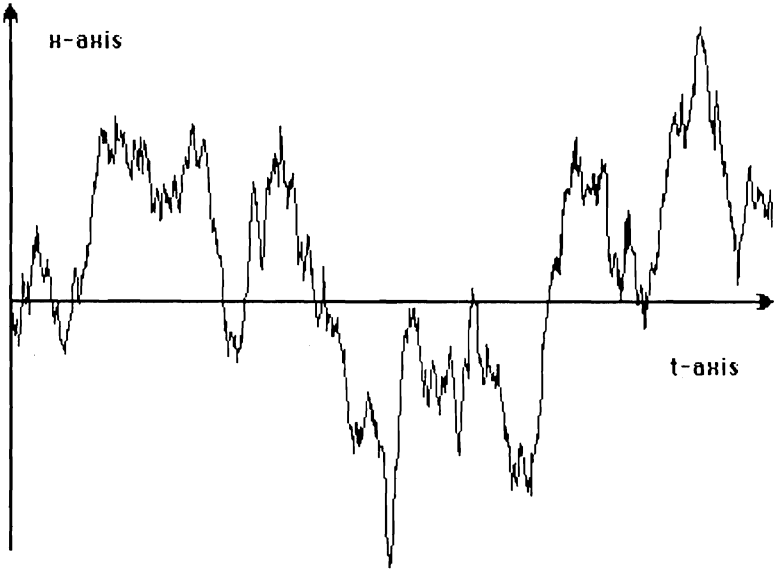
4. Brownian Motion

The mathematical theory of Brownian motion has played an important role in motivating the use of Hausdorff

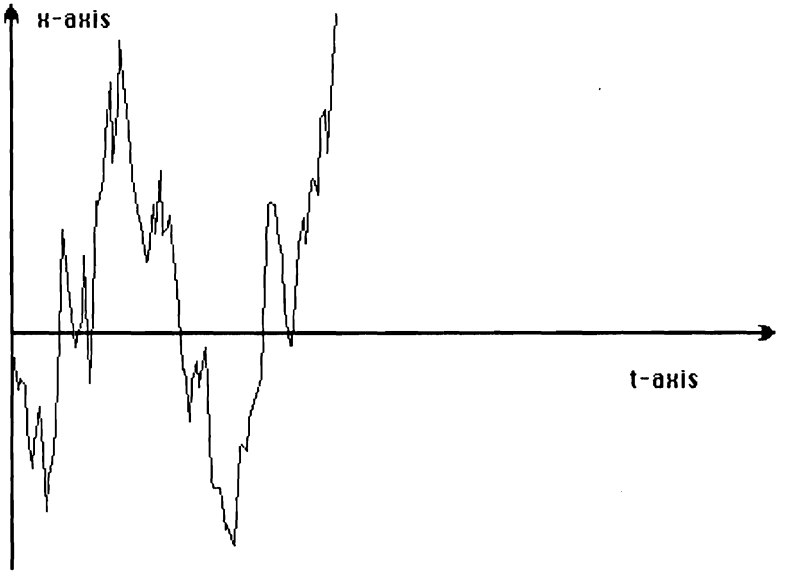
dimension in studying stochastic phenomena. We illustrate this theory in this section of the paper. The following graphs were produced by simulating a one-dimensional Brownian path using the principles of Brownian motion. A program producing random numbers from a normal distribution was used to produce a sample path. The first illustration is the graph so produced.



Graph of a Sample Path of One-Dimensional Brownian Motion



First Part of the Preceding Graph
Magnified Four Times



First Part of the Preceding Graph
Magnified Four Times

As can be readily seen from the successive magnifications of the graph, the appearance is becoming more jagged. Just how jagged things are becoming is brought out by the following computation. Let $t > 0$ be a time chosen at random and let M^+ and M^- be computed as follows.

$$M^+ = \limsup_{\Delta t \searrow 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$M^- = \liminf_{\Delta t \searrow 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Then with probability one $M^+ = +\infty$ and $M^- = -\infty$. It is also true that with probability one the graph is continuous and nowhere differentiable. See the paper by Dvoretski, Erdös, and Kakutani ([4]) for elegant proofs of these facts and other unexpected properties of Brownian paths. What is interesting from the standpoint of Hausdorff dimension is that if P is the graph of a one-dimensional Brownian path, then $\Lambda(P) = 1.5$ with probability one. Let x_0 be given. Let $P_{x_0} = \{t | x(t) = x_0\}$. Then $\Lambda(P_{x_0}) = .5$ with probability one. In any fixed interval of time $[t_1,t_2]$, $P_{x_0} \cap [t_1,t_2]$ is either empty or a Cantor set with probability one. See Taylor ([27]) and Orey ([20]). These results motivated researchers to look into other stochastic processes that produced sets with nonintegral Hausdorff dimension.

5. Sum Theorems

The theorems in this section are a direct result of the countable additivity of Hausdorff measure on its measurable sets. The results are straightforward from a measure-theoretic standpoint. However, they invite

comparison with the Decomposition Theorem and Sum Theorem in topological dimension theory. They also lead to some interesting applications.

5.1 *Theorem. Let* X *be a separable metric space with* $X = \cup\{A_i \mid i=1,2,\cdots\}$. *Then* $\Lambda(X) = \sup\{\Lambda(A_i) \mid i=1,2,\cdots\}$.

5.2 *Theorem. Suppose that* $\Lambda(X) = p < \infty$ *with* $\Lambda_p(X) > 0$. *Suppose that* $X = \cup\{A_i \mid i=1,2,\cdots\}$. *Then there is an* i_0 *such that* $\Lambda(A_{i_0}) = p$.

5.3 *Theorem (Decomposition Theorem for Topological Dimension). Suppose that* X *is a separable metric space with* $\dim X \leq n < \infty$. *Then* $X = \cup\{A_i \mid i=0,1,\cdots,n\}$ *with* $\dim A_i \leq 0$ *for each* i.

5.4 *Theorem (Sum Theorem for Topological Dimension). Suppose that* X *is a separable metric space and that* $X = \cup\{A_i \mid i=1,2,\cdots\}$ *with each* A_i *closed in* X. *Then* $\dim X = \sup\{\dim A_i \mid i=1,2,\cdots\}$.

A comparison of Theorems 5.1 and 5.2 with Theorems 5.3 and 5.4 shows that there is no possibility for topological dimension and Hausdorff dimension to coincide on all spaces. Note also that the sum theorem for Hausdorff dimension (Theorem 5.1) does not require that the subsets A_i of X be closed in X as they are in the Sum Theorem for topological dimension (Theorem 5.4). They can be any arbitrary subsets.

6. The Boundary Theorem

In topological dimension for separable metric spaces, dimension can be defined inductively. The empty set has dimension minus one and a space X has dim $X \leq n$ if and only if for every point $x \in X$ and every open $U \subset X$ containing x, there is an open set $V \subset U$ containing x such that the boundary of V has $\dim(\mathrm{Bdy}(V)) \leq n-1$. This definition of dimension was important in the development of the theory for separable metrizable spaces. See Hurewicz and Wallman ([10]) for details concerning this definition and its equivalence with other definitions of topological dimension for separable metric spaces. From this definition it is straightforward to show that if X is a separable metric space with $\dim X = n < \infty$, then for any $0 \leq m < n$ there is a closed subset $K \subset X$ with $\dim K = m$. The next theorem is due to Marczewski (alias Szpilrajn) and relates the Hausdorff dimension of the space X with the boundaries of ε-neighborhoods of the points of X. For a simple proof of Theorem 6.1 see Hurewicz and Wallman ([10], p. 104).

6.1 *Theorem.* *Let* $0 < p < \infty$ *and suppose that* $\Lambda_{p+1}(X) = 0$. *Let* $x \in X$. *Then for almost all* $\varepsilon > 0$ $\Lambda_p(S_\varepsilon(x)) = 0$.

One can modify the proof of Theorem 6.1 in Hurewicz and Wallman to obtain the following useful version of the theorem.

6.2 *Theorem.* *Let* $p \geq 0$ *and* $\varepsilon > 0$ *be real numbers and* k *a positive integer. Suppose that* $A \subset R^n$ *is any subset with* $\Lambda_{p+k}(A) = 0$. *Then there is a collection of hyperplanes* $\mathcal{H}(n,p,k,\varepsilon) = \{H_{i,j} \mid i=1,2,\cdots,n;\ j=0,\pm 1,\pm 2,\cdots\}$ *such that* (1) $H_{i,j}$ *is perpendicular to the* x_i*-axis for each* j; (2) *for all* j $H_{i,j}$ *and* $H_{i,j+1}$ *are distance at most* ε *apart; and* (3) *for any* $1 \leq m \leq k$ *and any* m *distinct numbers* $\{i_1,\cdots,i_m\} \subset \{1,\cdots,n\}$, $\Lambda_{p+k-m}(H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m} \cap A) \leq 0$.

If $\Lambda(X) = p > 0$ and $q \leq p - 1$, then one would like to assert that there is a point $x \in X$ and an $\varepsilon > 0$ such that for almost all $\delta < \varepsilon$ the δ-sphere centered at x, $S_\delta(x)$, has $\Lambda(S_\delta(x)) \geq q$. However, this fails to hold. Let $p > 0$. In this section we shall give an example of a space X_p such that $\Lambda(X_p) = p$, but for any subset $A \subset X$ such that $\Lambda(A) < p$, $\Lambda(A) = 0$. We assume the Continuum Hypothesis for this construction. In the examples it follows that almost all ε-spheres have $\Lambda(S_\varepsilon(x)) = 0$.

The next result shows that inside every compact space of Hausdorff dimension r, one can find a Cantor set of Hausdorff dimension r. In particular, there is a Cantor set L in the Koch curve such that L has the same Hausdorff dimension as the Koch curve. Thus the geometry of the Koch curve is not an obligate property of its Hausdorff dimension, since the Cantor set L has the same dimension and shares very little of the geometry. The rule of thumb is that if a counterexample exists for a conjecture in the theory of

Hausdorff dimension, then one can be found which is a Cantor set. One may as well begin by looking at Cantor sets.

6.3 *Theorem.* *Let* X *be a compact metric space with* $\Lambda(X) = r > 0$. *Then there exists a Cantor set* $C \subset X$ *with* $\Lambda(C) = r$.

Proof. Let X be a compact metric space satisfying the hypotheses of Theorem 6.2. For simplicity assume that $r < \infty$ and that $\Lambda_r(X) > 0$. By Theorem 6.1 there is a countable basis of ε-balls for the topology of X such that the corresponding ε-spheres, $\{S_i \mid i=1,2,\cdots\}$, containing the boundaries of these ε-balls each have $\Lambda_r(S_i) = 0$. Now let $B = \cup\{S_i \mid i=1,2,\cdots\}$. Now Λ_r is a countably additive measure on the Borel subsets of X by Theorem 2.2. Since each S_i is contained in the completion of this measure, $\Lambda_r(B) = 0$. Again by Theorem 2.2 let $G \supset B$ be a G_δ-set such that $\Lambda_r(G) = 0$. Now let $X \setminus G = \cup\{F_i \mid i=1,2,\cdots\}$ with each F_i closed in X. By Theorem 5.2 there must be an i_0 such that $\Lambda(F_{i_0}) = r$. However, F_{i_0} has the property that $F_{i_0} \subset X \setminus \cup\{S_i \mid i=1,2,\cdots\}$. Thus F_{i_0} is a compact zero-dimensional set. It may not be a Cantor set since there may be isolated points in F_{i_0}. However, by removing a countable set of points from F_{i_0} we can get a Cantor set having the same Hausdorff dimension.

Handling the cases $r = \infty$ and $\Lambda_r(X) = 0$ is not difficult using the same ideas.

6.4 *Theorem. Assume the Continuum Hypothesis. Let* $\Lambda(X) = r$ *and suppose that* $\Lambda_r(X) > 0$. *Then there is a* $B \subset X$ *such that* $\Lambda(B) = r$ *such that if* $A \subset B$ *with* $\Lambda(A) < r$, *then* A *is countable. In particular,* $\Lambda(A) \leq 0$.

Proof. Let $\mathcal{G} = \{G_\alpha | \alpha < c\}$ be the G_δ-subsets of X which have Huasdorff r-measure zero. Since there are c of these, assume that they are indexed by the countable ordinals. Think of the initial ordinals as cardinals, so that $\{\alpha | \alpha < c\}$ denotes the countable ordinals.

For each $\beta < c$, let x_β be chosen from $X \setminus \cup\{G_\alpha | \alpha < \beta\}$. Note that $\{\alpha | \alpha < \beta\}$ is countable. Thus $\cup\{G_\alpha | \alpha < \beta\}$ has r-measure zero. Since $\Lambda_r(X) > 0$, the choice of x_β is possible for each $\beta < c$. Let $B = \{x_\beta | \beta < c\}$.

Claim. $\Lambda(B) = r$.

Proof of Claim. Suppose that $\Lambda(B) < r$. Then $\Lambda_r(B) = 0$. Thus there must be an $\alpha < c$ such that $B \subset G_\alpha$ by Theorem 2.2. However, there must be a β with $\alpha < \beta < c$. For this β, $x_\beta \in B$ by definition. However, $x_\beta \notin G_\alpha$, a contradiction of the fact that $B \subset G_\alpha$. This proves the Claim.

Returning to the proof of Theorem 6.4, we now wish to show that if $A \subset B$ has the property that $\Lambda_r(A) = 0$, then A is countable. Suppose that $\Lambda_r(A) = 0$. Then there is a $G_\alpha \supset A$ such that $\Lambda_r(G_\alpha) = 0$. Now $G_\alpha \cap B \supset A$. However, $x_\beta \in G_\alpha \cap B$ only if $\beta < \alpha$. This implies that $G_\alpha \cap B$ is a countable set. Thus A is also countable. Thus also $\Lambda(A) = 0$.

6.5 *Note.* The above proof is very similar to that used by Hurewicz ([9]) to construct the first known example of an infinite-dimensional space having no positive finite-dimensional subsets. Using the Continuum Hypothesis the proof of Theorem 6.4 can be modified to give an example of a separable metric space X_∞ such that $\Lambda(X_\infty) = \infty$ such that if $A \subset X_\infty$ with $\Lambda(A) < \infty$, then A is countable and hence $\Lambda(A) = 0$. One can replace the Continuum Hypothesis by Martin's Axiom in Theorem 6.4 using an adaptation of Theorem 3 and its proof in [25]. Thus Theorem 6.4 is consistent with the negation of the Continuum Hypothesis as well.

6.6 *Theorem. Let* $A \subset R^n$ *be any compact set and suppose* $P^k \subset R^n \setminus A$ *is a* k*-dimensional polyhedron which is not contractible to a point in* $R^n \setminus A$. *Then* $\Lambda_{n-k-1}(A) > 0$ *and thus* $\Lambda(A) \geq n - k - 1$.

Proof. The proof is by contradiction. Suppose the theorem false. Then let $A \subset R^n$ be a compact set and $P^k \subset R^n \setminus A$ be a k-dimensional polyhedron which cannot be contracted to a point in $R^n \setminus A$. Let $m = n - k - 1$ and suppose that $\Lambda_m(A) = 0$. Let the distance between P^k and A in R^n be less than $(\varepsilon^n)^{1/2} > 0$. Let $\mathcal{H}(n,0,m,\varepsilon)$ be the collection of hyperplanes given by Theorem 6.2 for the given parameters. Let C be any PL contraction of P^k to a point in R^n. Then C will have topological dimension $\leq k + 1$. We must have that $C \cap A \neq \emptyset$. If $m = 0$, then $\Lambda_m(A) = 0$ implies that A is empty. Thus C is a contraction of P^k in $R^n \setminus A$, a contradiction. Thus we must have that $m \geq 1$ and

thus that $k + 1 < n$. Thus C has no interior points in R^n. Let $\{B_j | j=1,2,\cdots,q\}$ be a collection of rectangles in R^n such that (1) for $1 \leq i \leq n$, each B_j is bounded in the i-direction by a pair of hyperplanes $H_{i,p}$ and $H_{i,p+1}$ from $\mathcal{H}(n,0,m,\varepsilon)$; (2) $B_j \cap A \neq \emptyset$ for each j; and (3) the interior of $\cup\{B_j | j=1,2,\cdots,q\}$ in R^n contains A. By the choice of ε, we have that $(\cup\{B_j | j=1,2,\cdots,q\}) \cap P^k = \emptyset$ or the distance between P^k and A would be less than $(\varepsilon^n)^{1/2}$. For each $j \in \{1,2,\cdots,q\}$ let x_j be in the interior of B_j with $x_j \notin C$. From these points we can project $C \cap \text{int}\{\cup\{B_j | j=1,\cdots,q\})$ into the boundaries of the B_j's so that we obtain a homotopy C' having the properties that (1) C' coincides with C on $R^n \setminus \text{int}\{\cup\{B_j | j=1,\cdots,q\})$; (2) C' has topological dimension at most k + 1 and (3) $C' \cap \text{int}\{\cup\{B_j | j=1,\cdots,q\})$ is contained in the union of the hyperplanes which form the boundaries of the B_j's. The topological dimension of the union of the hyperplanes is at most n - 1. If m = 1, then $\Lambda_0(A \cap H_{i,j}) = 0$ for each $H_{i,j}$ in $\mathcal{H}(n,0,m,\varepsilon)$ and thus $A \cap H_{i,j} = \emptyset$ for each $H_{i,j}$ in $\mathcal{H}(n,0,m,\varepsilon)$. This implies that $A \cap C' = \emptyset$. Thus we have a homotopy in $R^n \setminus A$ which contracts P^m to a point. Thus one must have that $k \geq 2$.

Consider a hyperplane $H_{i,r}$ whose intersection with the interior of $\cup\{B_j | j=1,2,\cdots,q\}$ is nonempty. Then for each $j \in \{1,\cdots,q\}$ there must be a point $x_{i,r,j}$ in the interior (relative to $H_{i,r}$) of $(H_{i,r} \cap B_j) \setminus C'$ whenever this interior is contained in the interior of $\cup\{B_j | j=1,2,\cdots,q\}$ in R^n. By projection from the points $x_{i,r,j}$ we get a homotopy C" such that $C" \cap \text{int}(\cup\{B_j | j=1,2,\cdots,q\}) \subset$

$\cup\{H_{i,j} \cap H_{r,s} | H_{i,j}$ and $H_{r,s}$ distinct in $\mathcal{H}(n,0,m,\varepsilon)\}$. If $m = 2$, then $\Lambda_0(A \cap H_{i,r} \cap H_{r,s}) = 0$ for each pair of distinct $H_{i,j}$ and $H_{r,s}$ in $\mathcal{H}(n,0,m,\varepsilon)$ which implies that $A \cap H_{i,j} \cap H_{r,s} = \emptyset$ for each pair of distinct $H_{i,j}$ and $H_{r,s}$ in $\mathcal{H}(n,0,m,\varepsilon)$. Thus $A \cap C'' = A \cap C'' \cap \text{int}(\cup\{B_j | j=1,\cdots,q\}) \subset A \cap (\cup\{H_{i,j} \cap H_{r,s} | H_{i,j}$ and $H_{r,s}$ distinct in $\mathcal{H}(n,0,m,\varepsilon)\}) = \emptyset$. Thus we have a homotopy C'' contained in $R^n \setminus A$ in this case also.

Proceeding in this fashion we finally obtain a homotopy $C^{(m)}$ which is the same as C on $R^n \setminus \text{int}(\cup\{B_j | j=1,\cdots,q\})$ and such that $C^{(m)} \cap A \cap \text{int}(\cup\{B_j | j=1,\cdots,q\}) \subset A \cap (\cup\{H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m} | H_{i_1,j_1},\cdots,H_{i_m,j_m}$ are distinct hyperplanes from $\mathcal{H}(n,0,m,\varepsilon)\})$. By the definition of $\mathcal{H}(n,0,m,\varepsilon)$ $\Lambda_0(A \cap H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m}) = 0$ and thus $A \cap H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m} = \emptyset$ for each m distinct $H_{i_1,j_1},\cdots,H_{i_m,j_m}$ from $\mathcal{H}(n,0,k,\varepsilon)$. Thus $C^{(m)} \cap A \cap \text{int}(\cup\{B_j | j=1,\cdots,q\}) = \emptyset$. This $C^{(m)}$ is a homotopy which has the property that $C^{(m)} \subset R^n \setminus A$. This is a contradiction. To prevent this contradiction we must have that $\Lambda_{n-k-1}(A) > 0$. This completes the proof of the theorem.

6.7 *Definition.* Let $A \subset R^3$ be any Cantor set having the property that $R^3 \setminus A$ is not simply connected. Then A is an *Antoine's Necklace*. Such Cantor sets in R^3 are known to exist.

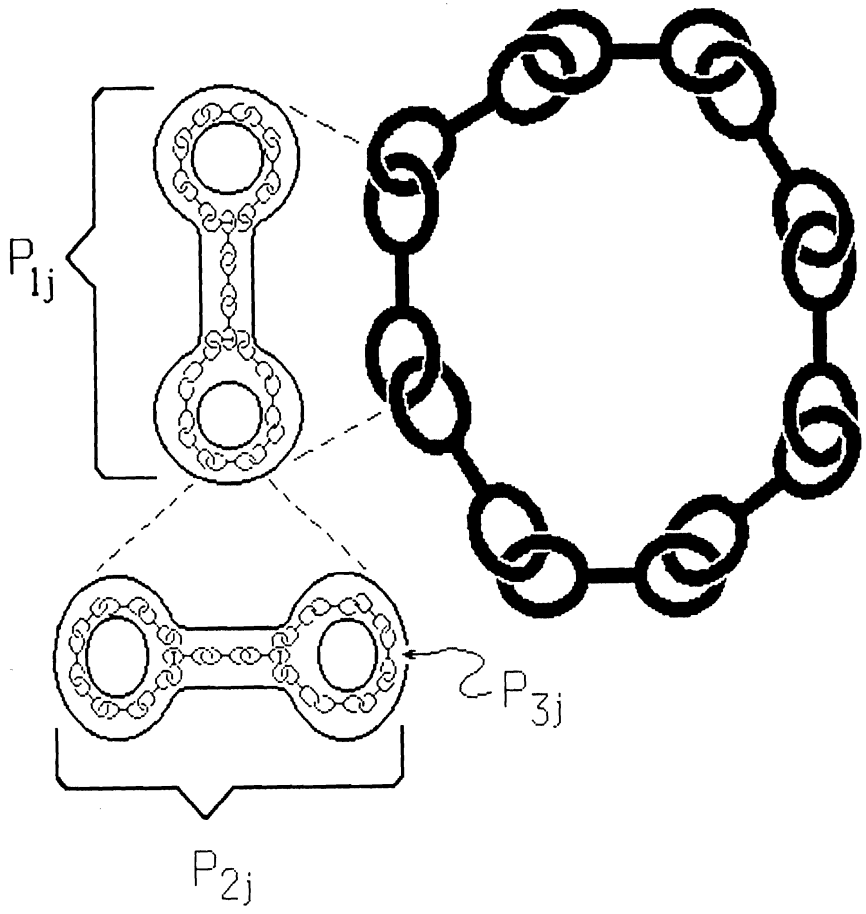
The following corollary shows that any construction of Antoine's Necklace in R^3 must be an example of a Cantor set whose Hausdorff dimension is at least one.

6.8 *Corollary.* *Let* $A \subset R^3$ *be any Antoine's necklace. Then* $\Lambda_1(A) > 0$. *Thus* $\Lambda(A) \geq 1$.

As a matter of fact one can construct examples of Antoine's Necklaces with $\Lambda_1(A) < \varepsilon$ for any $\varepsilon > 0$. Instead of using linked tori as is usual, one can use linked eyebolts. The heads on the bolts are chosen to be very small relative to the length of the body of the eyebolt so that the sum of the lengths of the eyebolts is very nearly the sum of the lengths of the eyebolts less the heads. The following illustration shows how the construction proceeds, but the heads of the eyebolts are not drawn to scale to show the linking detail. We let $\mathcal{U}_i = \{P_{ij} | j=1,2,\cdots,n_i\}$ be the eyebolts at the ith stage of the construction. Let $A_i = \cup\{P_{ij} | j=1,2,\cdots,n_i\}$ and let Antoine's Necklace be $A = \cap\{A_i | i=1,2,\cdots\}$.

Let $0 < M < \infty$ be given. Let $\mathcal{U}_i = \{P_{ij} | j=1,\cdots,j_i\}$ be the eyebolts in the ith stage of the construction described above, then one can choose the lengths of the eyebolts at each stage so that $\lim_{i\to\infty} \sum_{P_{ij}\in\mathcal{U}_i} \operatorname{diam}(P_{ij}) = M$ with $\operatorname{mesh}(\mathcal{U}_i) \to 0$ also. This implies that $\Lambda_1(A) \leq M$. Since Corollary 6.6 implies that $\Lambda_1(A) > 0$, we must have that $\Lambda(A) = 1$.

These results are related to the concept of demension due to Stanko. See Väisälä ([33]) for a precise statement of the relationship of demension and Hausdorff measure.



A Construction of
Antoine's Necklace

7. Lipschitz Mappings

The kinds of mappings that are important in the study of Hausdorff measure and Hausdorff dimension are Lipschitz mappings of class δ. If $f(X) = Y$ is such a map, then one can say something about the relationship between $\Lambda(X)$ and $\Lambda(Y)$. If f is simply continuous, there is no useful relationship between $\Lambda(X)$ and $\Lambda(Y)$. There is no useful

relationship between $\Lambda(X)$ and $\Lambda(Y)$ even when X and Y are related by a homeomorphism.

7.1 *Definition.* Let $\delta > 0$. Then $f: X \to Y$ is said to be of *Lipschitz class* δ, $f \in \mathrm{Lip}(\delta)$, provided there exists a $K > 0$ such that for all $x,y \in X$, $d(f(x),f(y)) \leq K \times d(x,y)^{\delta}$. If $\delta = 1$, then f is said to be a *Lipschitz map*.

The Theorem 7.2 and 7.3 are straightforward applications of the definition of Hausdorff dimension.

7.2 *Theorem. Suppose that* $f(X) = Y$ *is of Lipschitz class* δ. *Then* $\Lambda(Y) \leq 1/\delta \times \Lambda(X)$.

7.3 *Theorem. Let* $\delta > 0$. *Suppose that there exists an* $\varepsilon > 0$ *and* $K > 0$ *such that for all* x *and* y *in* X *with* $d(x,y) < \varepsilon$, $d(f(x),f(y)) \geq K \times d(x,y)^{\delta}$. *Then* $\Lambda(Y) \geq 1/\delta \times \Lambda(X)$.

7.4 *Theorem. Let* $f(X) = Y$ *be a Lipschitz map. Then for all* $p \geq 0$, $\Lambda_p(Y) \leq \Lambda_p(X)$.

We now give several applications of these mapping theorems. The first answers a question posed by Kaplan and Yorke ([13]). They asked whether every space X with integral Hausdorff dimension n has the property that there is a Lipschitz mapping $f(X) = I^n$. We show that for each n a space X_n exists such that $\Lambda(X_n) = n$ with $0 < \Lambda_n(X_n) < \infty$ such that there is no Lipschitz map of X_n onto the interval [0,1]. In fact we show that there is no continuous map of X_n onto [0,1].

7.5 *Theorem. Assume the Continuum Hypothesis. Let* n *be any positive integer. Then there exists a separable metric space* X_n *such that* $\Lambda(X_n) = n$ *with* $0 < \Lambda_n(X_n) < \infty$, *but with the property that there is no continuous map* $f: X_n \to [0,1]$ *which is onto.*

Proof. Let $\mathcal{G} = \{G \subset I^n | G \text{ is a } G_\delta \text{ in } I^n \text{ with } \lambda(G) = 0\}$. There are c such subsets of I^n and, assuming the Continuum Hypothesis, we can index $\mathcal{G}$ with the countable ordinals, $\mathcal{G} = \{G_\alpha | \alpha < c\}$. We now define $X_n \subset I^n$. Suppose that $\alpha < c$ and suppose x_β has been chosen for all $\beta < \alpha$. The set $\{\beta < \alpha\}$ is countable and thus $I^n \setminus \cup\{G_\beta | \beta < \alpha\} \neq \emptyset$ since $\lambda(I^n) = 1 > 0$. Thus we can choose $x_\alpha \in I^n \setminus \cup\{G_\beta | \beta < \alpha\}$. Let $X_n = \{x_\alpha | \alpha < c\}$. We first need to show that $0 < \Lambda_n(X_n) < \infty$.

Claim 1. $0 < \Lambda_n(X_n) < \infty$ and thus $\Lambda_n(X_n) = n$.

Proof. Let λ^* be Lebesgue n-dimensional outer measure. Then $\Lambda_n(X_n) = \alpha_n \times \lambda^*(X_n)$ by Theorem 2.4. Thus we only need to show that $\lambda^*(X_n) > 0$. Suppose not. Then $\lambda^*(X_n) = 0$ and there is a G_δ set G containing X_n such that $\lambda^*(G) = 0$ also. However, this implies that $G \in \mathcal{G}$ and that $G = G_\alpha$ for some $\alpha < c$. But $x_\alpha \notin G_\alpha$ and supposedly $X_n \subset G_\alpha$. This contradiction proves Claim 1.

We now proceed to show that if $f: X_n \to 1$ is any continuous map, then $f(X_n)$ cannot be onto. This will complete the proof of the theorem.

Claim 2. Let $f: X_n \to I$ be any continuous map. Then f cannot be onto.

Proof. Suppose $f: X_n \to I$ is a continuous map. There is a continuous extension $g: G \to I$ of f to a G_δ-set containing X_n in I^n. There is a collection of c pairwise disjoint Cantor sets in I. This is because the Cantor set C is homeomorphic to $C \times C$. Let $D_\alpha = g^{-1}(C_\alpha)$ for each α. Each D_α is closed in G and thus each D_α is a G_δ-set as well. Thus each D_α is measurable and the collection $\{D_\alpha | \alpha < c\}$ is pairwise disjoint. Thus there must be an α with $\lambda(D_\alpha) = 0$. This D_α is a G_δ-set in I^n and thus is in $\mathcal{G}$. Let $G_\beta = D_\alpha$ for some $\beta < c$. Then $x_\gamma \in G_\beta$ implies that $\gamma < \beta$. Since $\{\gamma < \beta\}$ is a countable set we have that $X_n \cap G_\beta = X_n \cap D_\alpha$ is a countable set. Since $f^{-1}(C_\alpha) \subset g^{-1}(C_\alpha) \cap X_n = D_\alpha \cap X_n$, we must have that $f^{-1}(C_\alpha)$ is countable. This is a contradiction of the assumption that f is onto since C_α is uncountable and f cannot map a countable set onto an uncountable one. Therefore no continuous function can be onto. This proves Claim 2 and the proof of the theorem is complete.

7.6 *Note*. Theorem 7.5 is true assuming Martin's Axiom by Theorem 3 of [25]. Thus the theorem is consistent with the negation of the Continuum Hypothesis. The example in Theorem 7.5 would be more interesting if it were compact. Let n be a positive integer. Here we give an example of a compact space X_n such that $\Lambda(X_n) = n$, but such that there is no Lipschitz map $f: X_n \to I^n$ which is onto. Let Z_i be any compact space having $\Lambda(Z_i) = n - 1/i$. We can suppose

that the metric of Z_i is bounded by ε_i with $\varepsilon_i \to 0$. Then let X_n be the union of the Z_i's together with a point z such that $Z_i \to z$ in X_n. The resulting space can be metrized in such a fashion that for x and y in Z_i, the distance between x and y in X_n is just the distance between x and y in Z_i. By Theorem 5.1 $\Lambda(X_n) = \sup\{\Lambda(Z_i) \mid i=1,2,\cdots\} = n$. However, if $f: X_n \to I^n$ is any Lipschitz map, then $\Lambda(f(Z_i)) < n$ because Lipschitz maps cannot raise Hausdorff dimension. Thus the topological dimension of $f(Z_i)$ must be less than n also. Thus $f(X_n) = \cup\{f(Z_i) \mid i=1,2,\cdots\} \cup f(z)$ must have topological dimension less than n by Theorem 5.4. Thus $f(X_n)$ cannot be all of I^n. Note that although $\Lambda(X_n) = n$, $\Lambda_n(X_n) = 0$ in this example.

7.7 *Example.* Let X be any metric space and let X_δ be given the metric the function $D(x,y) = d(x,y)^\delta$ for some fixed $0 < \delta \leq 1$. Then $\Lambda(X_\delta) = 1/\delta \times \Lambda(X)$.

7.8 *Example.* Let X be any metric space with metric bounded by $e^{-3/2}$. Let X_D be given the metric $D(x,y) = \{\ln(d(x,y)^{-1})\}^{-1/2}$. In this example, if $L \subset X$ has $0 < \Lambda(L) < \infty$, then in X_D $\Lambda(L) = \infty$. Thus if $L \subset X_D$ has $\Lambda(L) < \infty$, then $L \subset X$ has $\Lambda(L) = 0$.

8. A Problem

This section deals with an open problem in the theory of Hausdorff dimension. The question is whether a compact space X can have $\Lambda(X) > 0$ and not have any closed subsets $A \subset X$ with $0 < \Lambda(A) < \Lambda(X)$. We shall not go into too much detail about this problem, but a counterexample must have

$\Lambda(X) = \infty$. There are compact metric X in the topological theory of dimension with dim $X = \infty$, such that if $A \subset X$, then dim $A = 0$ or ∞. See Walsh ([34]) for the most general and readable example of this sort. Such a space as is described in [34] cannot be a counterexample to the above question as we shall show. Such a space must contain a nontrivial continuum. In every nontrivial continuum, there are closed subsets of Hausdorff dimension one. We give a simple proof of this. For a more general result see Larman ([14]). Thus, any counterexample to the above question must have topological dimension zero.

On the other hand, for finite-dimensional topological spaces, there are always subsets of every lesser topological dimension. This is not true for Hausdorff dimension, at least not for noncompact spaces. Let $p > 0$. Then Theorem 6.4 shows that there are examples X_p which have the property that $\Lambda(X_p) = p$ such that if $A \subset X_p$, then $\Lambda(A) = p$ or 0. The examples use the Continuum Hypothesis in their construction (actually Martin's Axiom is all that is required as pointed out in Note 6.5). They definitely are not compact.

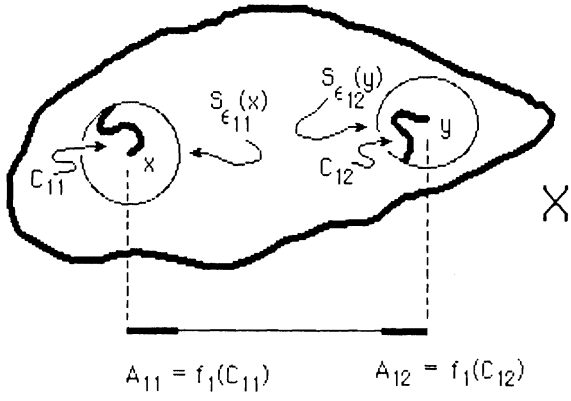
8.1 *Theorem. Let* X *be a nontrivial metric continuum. Then there exists a Cantor set* $C \subset X$ *with* $\Lambda(C) = 1$.

Proof. Let X be a nontrivial metric continuum. Let x and y be any two distinct points in X. Let $d(x,y) = \varepsilon > 0$. We will now construct a Cantor set $C \subset X$ having the following properties: (1) C has a sequence of open covers $\{\mathcal{U}_i \mid i=1,2,\cdots\}$ such that $\sum_{P_{ij} \in \mathcal{U}_i} \operatorname{diam}(P_{ij}) < \varepsilon$ for every i

and (2) there is a Lipschitz map $f: C \to [0,\varepsilon]$ such that $f(C)$ has positive Lebesgue measure. Before we construct C, we show that these properties imply that $\Lambda(C) = 1$. In property (1) above, the given sequence of covers shows that $\Lambda_1(C) \leq \varepsilon$. On the other hand, the Lipschitz map $f: C \to [0,\varepsilon]$ maps C onto a set of positive Lebesgue measure in the interval. Thus the set $f(C)$ must have $\Lambda(f(C)) = 1$. However, a Lipschitz map cannot lower Hausdorff dimension. Thus, $\Lambda(C) \geq 1$. Thus we have $\Lambda(C) = 1$.

We now proceed with the construction of C in X. First let $\{M_i \mid i=1,2,\cdots\}$ be a sequence of positive real numbers such that $\varepsilon > M_1 > M_2 > \cdots > \varepsilon/2$. Then let $\{\varepsilon_{ij} \mid j=1,2,\cdots 2^i\}$ be such that $\varepsilon_{ij} = M_i/2^i$ so that $\sum\{\varepsilon_{ij} \mid j=1,2,\cdots,2^i\} = M_i$. Let B_1 and B_2 be the open ε_{11}-ball about x and the open ε_{12}-ball about y, respectively. Let C_{11} be the closure of the component of x in B_1 and C_{12} be the closure of the component of y in B_2. From a well-known theorem in general topology, C_{11} must meet the boundary of B_1 and C_{12} must meet the boundary of B_2. Let $D_1 = C_{11} \cup C_{12}$ and $f_1: D_1 \to [0,\varepsilon]$ be defined by $f_1(z) = d(x,z)$ if $z \in C_{11}$ and $f_1(z) = \varepsilon - d(y,z)$ if $z \in C_{12}$. Clearly, f_1 is Lipschitz from D_1 to $[0,\varepsilon]$ with constant $K = 1$. Now the diameter of C_{11} is at most $2 \times \varepsilon_{11}$ and the diameter of C_{12} is at most $2 \times \varepsilon_{12}$. Also, the diameter of $f_1(C_{11}) = A_{11}$ is exactly ε_{11} and the diameter of $f_1(C_{12}) = A_{12}$ is exactly ε_{12}.

Now let $x_{21} = x$ in C_1 and x_{22} be any point in $C_{11} \cap (\mathrm{Bdy}(B_1))$. Let x_{23} be y in C_{12} and x_{24} be any point in $C_{12} \cap (\mathrm{Bdy}(B_2))$. Let C_{21} be the closure component of x_{21}



First Stage in the Construction

in $C_{11} \cap B_{\varepsilon_{21}}(x_{21})$ and C_{22} be the closure of the component of x_{22} in $C_{11} \cap B_{\varepsilon_{22}}(x_{22})$. Let C_{23} be the closure of the component of x_{23} in $C_{12} \cap B_{\varepsilon_{23}}(x_{23})$ and let C_{24} be the closure of the component of x_{24} in $C_{12} \cap B_{\varepsilon_{24}}(x_{24})$. Let $D_2 = C_{21} \cup C_{22} \cup C_{23} \cup C_{24}$ and let $f_2: D_2 \to [0,\varepsilon]$ be defined as follows. For $z \in C_{21}$, define $f_2(z) = d(x_{21},z)$. For C_{22}, define $f_2(z) = \varepsilon_{11} - d(x_{22},z)$. For C_{23} let $f_2(z) = \varepsilon - \varepsilon_{12} + d(x_{23},z)$. Lastly, for C_{24} let $f_2(z) = \varepsilon - d(x_{24},z)$. Note that f_2 is a Lipschitz map with constant $K = 1$ and if $A_{2j} = f_2(C_{2j})$, then the diameter of C_{2j} is at most $2 \times \varepsilon_{2j}$ and the diameter of A_{2j} is ε_{2j}. Proceeding in this fashion we obtain collections of disjoint continua $\mathcal{U}_i = \{C_{ij} | j=1,2,\cdots,2^i\}$ in X such that the diameter of C_{ij} is at most $2 \times \varepsilon_{ij}$. We define $D_i = \cup\mathcal{U}_i$. For each i we also have a Lipschitz map $f_i: D_i \to [0,\varepsilon]$ with $K = 1$ such that if $A_{ij} = f_i(C_{ij})$, then $A_{ij} \cap A_{im} = \emptyset$ for all

$j \neq m$ and the diameter of A_{ij} is ε_{ij}. Also, $f_i(D_i) \supset f_{i+1}(D_{i+1})$ for all i.

We define the Cantor set C to be the intersection of the sets $\{D_i \mid i=1,2,\cdots\}$ in the space X. One can verify that the sequence of mappings $f_i|C$ forms a Cauchy sequence in the supremum metric. Thus we can define a map $f: C \to [0,\varepsilon]$ to be the limit of the functions $f_i|C$. The map $f: C \to [0,\varepsilon]$ will also be Lipschitz with constant $K = 1$. Now the set $f(C)$ will be the intersection of the sets $f_i(D_i) = f(D_i)$ and the Lebesgue measure of each of these sets is M_i by construction. This implies that the Lebesgue measure of $f(C)$ is the limit of the M_i's. Thus $\lambda(f(C)) \geq \varepsilon/2 > 0$. This implies that $\Lambda(f(C)) \geq 1$. Thus $\Lambda(C) \geq 1$ since a Lipschitz map cannot raise dimension. Thus $\Lambda(C) = 1$ and this is the required Cantor set of Hausdorff dimension one that was to be constructed.

8.2 *Corollary. Let* X *be a metric continuum. Then for every* $r \in [0,1]$, *there exists a Cantor set* $C \subset X$ *with* $\Lambda(C) = r$.

Proof. This is a simple application of Theorem 8.2 together with Example 7.6 from the previous section.

8.3 *Corollary. Let* $r \in (0,1]$. *Then there is a Cantor set* $C \subset [0,1]$ *such that* $\Lambda(C) = r$.

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