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THE SUBCONTINUA OF $\beta[0,\infty) - [0,\infty)$

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Michel Smith

If we let $A = [0,\infty)$ then it has been shown by Bellamy [B] that the remainder $\beta A - A$ of the Stone-Čech compactification of A is an indecomposable continuum. Also it has been shown by Smith [S1] that $\beta A - A$ does not contain non-degenerate hereditarily indecomposable continua. Thus $\beta A - A$ contains decomposable continua. We are interested in studying the structure of $\beta A - A$ by examining the properties of the subcontinua of $\beta A - A$. We show that there are at least six different decomposable subcontinua of $\beta A - A$. Each such continuum admits a monotone decomposition into an arc and has a dense set of points of local connectivity which are also P-points of that continuum. These continua have been studied extensively by Mioduszewski [Mi]. We state a number of his results which we will use. Also we will point out a number of corollaries to his work to give us a more complete picture of these continua. Mioduszewski also used the ordering "<" defined below.

It was pointed out by S. Baldwin to the author that the set $A u$ defined below is used as a model for non-standard analysis. (But with $[0,1]$ replaced by $(-\infty,\infty)$ in the construction.) So we are interested in the topological properties of these models.

It was learned by the author that some of these results were discovered independently by Eric van Douwen (related in conversation, Topology Conference, April 1986, Lafayette, Louisiana.) In particular van Douwen was aware of the upper semi-continuous structure of the subcontinua of A^* and knew of five different subcontinua of A^*. It was also shown by van Douwen [vDl] that if E^2 denotes the Euclidean plane then there are 2^c different subcontinua of $\beta E^2 - E^2$, this is a generalization of a result of Winslow [Wi] who used homological techniques to show that $\beta E^3 - E^3$ has 2^c different subcontinua where E^3 denotes 3-dimensional Euclidean space. Whether A^* has 2^c different continua or not is still an open question.

In the second portion of this paper we examine the structure of some of the indecomposable subcontinua of A^*. It was shown by M. E. Rudin [R] that the continuum hypothesis implies that A^* has 2^c composants. It was suggested by Mioduszewski [Mi] and it was proven by Blass [Bℓ] that the near coherence of filters axiom implies that A^* has only one composant. We show that there is a certain class of indecomposable subcontinua of A^* any one of which has 2^c composants under the continuum hypothesis and has only one composant under the near coherence of filters axiom.

Definitions and Notation

If X is a completely regular space then the Stone-Čech compactification βX can be identified with the set of ultrafilters of zero sets of X [GJ]. If X is a metric space then βX can be identified with the set of ultrafilters of closed sets of X. If $x \in X$ then x is identified with the ultrafilter that contains $\{x\}$. A basis for the topology of βX is the collection to which J belongs if and only if there is an open set R in X so that $J = \{u \in \beta X | R$ contains an element of $u\}$. Elements of this basis are called regions of βX and if U is open in X let $\mathrm{Rgn}(U)$ denote $\{u \in \beta X | U$ contains an element of $u\}$ which is the region determined by U. If $Y \subset X$ then $\mathrm{Cl}_X(Y)$ denotes the closure of Y in X. Let $X^* = \beta X - X$. Let $Z(X)$ denote the collection of zero sets in X; so $Z(X)$ is the collection to which K belongs if and only if there is a continuous function $f: X \to [-1,1]$ so that $K = f^{-1}(0)$.

Let N denote the positive integers. For the remainder of the paper let $X = [0,1] \times N$. Then X is homeomorphic to a subset of A; so X^* is homeomorphic to a subset of A^*. Let $I_n = [0,1] \times \{n\}$; we have $X = \cup_{n=1}^{\infty} I_n$.

Let Ω denote the set of sequences of numbers in $[0,1]$, $\Omega = \{\{s_n\}_{n=1}^{\infty} | s_n \in [0,1], n \in N\}$. Suppose that $s = \{s_n\}_{n=1}^{\infty} \in \Omega$ and u is an ultrafilter is N^*. Then define $A^N(u,s) = \{\{(s_n,n) | n \in H\} | H \in u\}$. Since $\{(s_n,n)\}_{n=1}^{\infty}$ is homeomorphic to N, then $A^N(u,s)$ is an ultrafilter of $\{(s_n,n)\}_{n=1}^{\infty}$. Let $A(u,s) = \{H \in Z(X) | $there exists $\hat{H} \in A^N(u,s)$ such that $\hat{H} \subset H\}$. Then $A(u,s)$ is an ultrafilter in X^*.

Suppose $H \subset X$ or $H \subset \beta X$ then define $D_H = \{n | H \cap I_n \neq \emptyset\}$. Suppose that $x \in X^*$. Then define $u_x = \{D_H | H \in x\}$.

Following are some theorems which we wish to use which are due to Mioduszewski [Mi] and we will state most of them without proof. The theorems which are labeled M- are due to Mioduszewski, though the notation may be slightly modified.

Theorem M1. *If* $x \in X^*$ *then* $u_x \in N^*$.

Suppose $u \in N^*$. Then define $L(u) = cl_{\beta X}(\{A(u,s) | s \in \Omega\})$.

Theorem M2. *The set* $L \subset X^*$ *is a component of* X^* *if and only if* $L = L(u)$ *for some* $u \in N^*$.

Theorem M3. *If* $x \in X^*$ *and* $u \in N^*$ *then* $x \in L(u)$ *if and only if* $u_x = u$.

Definition. Let $u \in N^*$. Then let $0_u = A(u,\{0\}_{n=1}^{\infty})$ and let $1_u = A(u,\{1\}_{n=1}^{\infty})$. There is a natural order induced on each of the components of X^* by the order on $[0,1]$. Suppose $n \in N$, $A \subset I_n$ and $B \subset I_n$. Then $A \underset{n}{<} B$ means that if $x = (a,n) \in A$ and $y = (b,n) \in B$ then $a < b$. Suppose $u \in N^*$ and x and y are points of $L(u)$ then $x \underset{u}{<} y$ means that there exists $H \in x$, $K \in y$, and $D \in u$ so that $H \cap I_n \underset{u}{<} K \cap I_n$ for all $n \in D$. The subscripts n and u in "$\underset{n}{<}$" and "$\underset{u}{<}$" will be omitted if it is clear from the context what is meant. As we will see not all points of $L(u)$ can be related by "$\underset{u}{<}$". However, the points of $\{A(u,s) | s \in \Omega\}$ are linearly ordered by "$\underset{u}{<}$".

Although Theorem M4 follows from work in [Mi] we include a proof with our notation for completeness.

Theorem M4. *Let* $u \in N^*$. *If* $s \in \Omega$ *and* $A(u,s)$ *is neither* 0_u *nor* 1_u *then* $A(u,s)$ *is a cut point of* $L(u)$ *which separates* 1_u *from* 0_u *in* $L(u)$.

Proof. Let $s = \{s_n\}_{n=1}^{\infty} \in \Omega$ be such that $A(u,s)$ is neither 0_u nor 1_u. So there exists $J \in u$ such that $0 < s_n < 1$ for all $n \in J$. Let P be the set to which x belongs if and only if $x \in L(u)$ and $x \underset{u}{<} A(u,s)$ and let Q be the set to which y belongs if and only if $y \in L(u)$ and $A(u,s) \underset{u}{<} y$. Clearly $P \cup Q \subset L(u)$.

Claim 2.1. $P \cap Q = \emptyset$

Suppose that $P \cap Q \neq \emptyset$ and $x \in P \cap Q$. Then $x < A(u,s)$ and $A(u,s) < x$. So there exist $W_1 \in x$, $W_2 \in x$ and $J_1 \in u$, $J_2 \in u$ so that

$$W_1 \cap I_n < s_n \quad \text{for all } n \in J_1 = D_{W_1}$$

$$W_2 \cap I_n > s_n \quad \text{for all } n \in J_2 = D_{W_2}.$$

But then $W_1 \cap W_2 \cap I_n < s_n$ and $W_1 \cap W_2 \cap I_n' > s_n$ for all $n \in J_1 \cap J_2 \neq \emptyset$, and $W_1 \cap W_2 \in x$. This is a contradiction.

Claim 2.2. $P \cup Q = L(u) - \{A(u,s)\}$.

Suppose $x \in L(u) - \{A(u,s)\}$. Then $u_x = u$ by corollary 1.1. Since $x \neq A(u,s)$ there exists $W \in x$ and $K \in A^N(u,s)$ so that $W \cap K = \emptyset$. That is K is chosen so that there is a set $J \in u$ so that $K = \{(S_n,n) | n \in J\}$. Since $D_W \in u$ and $J \in u$ then $D_W \cap J \in u$. So $\hat{K} = \{(s_n,n) | n \in D_W \cap J\} \in A(u,s)$. Also $W \cap (\{I_n | n \in D_W \cap J\}) \in x$. Let $W_P = \cup\{W \cap ([0,s_n) \times \{n\}) | n \in D_W \cap J\}$ and $W_Q = \cup\{W \cap ((s_n,1] \times \{n\} | n \in D_W \cap J\}$. So $W_P \cup W_Q \in x$ and both W_P and W_Q are closed in X. Therefore either $W_P \in x$ or $W_Q \in x$. If $W_P \in x$ then $x \in P$ and if $W_Q \in x$ then $x \in Q$.

Claim 2.3. P and Q are mutually separated.

Suppose $p \in P$. Then there exists $W \in p$ and $H = D_W \in u$ so that $W \cap I_n < s_n$ for all $n \in H$. The set $W \cap I_n$ is closed so there exists an open set $O_n \subset I_n$ in X so that

$W \cap I_n \subset O_n$ and $O_n < s_n$ for all $n \in H$. Thus $W \subset \cup_{n \in H} O_n$. Also $p \in \mathrm{Rgn}(\cup_{n \in H} O_n)$ and if $x \in \mathrm{Rgn}(\cup_{n \in H} O_n)$ then there exists $K \in x$ so that $K \cap I_n \subset O_n < s_n$ for all $n \in H$. Therefore $x \in P$. So no point of P is a limit point of Q. Similarly no point of Q is a limit point of P.

From the definitions of P and Q it is easy to see that $0_u \in P$ and $1_u \in Q$.

From the proof of theorem M4 we can make the following definitions: if $s \in \Omega$ and $u \in N^*$ then let

$$P(u,s) = \{x \in L(u) \mid x \underset{u}{<} A(u,s)\},$$

$$Q(u,s) = \{x \in L(u) \mid A(u,s) \underset{u}{<} x\},$$

$$\overline{P(u,s)} = P(u,s) \cup \{A(u,s)\}, \text{ and}$$

$$\overline{Q(u,s)} = Q(u,s) \cup \{A(u,s)\};$$

and notice that

$P(u,s)$ and $Q(u,s)$ are mutually separated,

$$L(u) - A(u,s) = P(u,s) \cup Q(u,s),$$

$$\overline{P(u,s)} \cap \overline{Q(u,s)} = \{A(u,s)\},$$

$$\overline{P(u,s)} = \mathrm{Cl}_{\beta X}(P(u,s)), \text{ and}$$

$$\overline{Q(u,s)} = \mathrm{Cl}_{\beta X}(Q(u,s)).$$

Furthermore $\overline{P(u,s)}$ and $\overline{Q(u,s)}$ are both continua (see [M] or [K]).

Since $L(u)$ is defined to be $\mathrm{Cl}_{\beta X}(A(u,s) \mid s \in \Omega)$ then $\{A(u,s) \mid s \in \Omega\}$ is dense in $L(u)$. A continuum I is *irreducible* from the point P to the point Q means that P and Q are points of I and no proper subcontinuum of I contains P and Q. If the continuum I is irreducible from the point P to the point Q then the end of I from P denoted by End(I,P) is the set to which x belongs if and only if I is irreducible from P to x. The set E is called an end of I if it is an end of I from some point of I. Ends of continua are complements of composants of continua.

Theorem M5. *Let* $u \in N^*$ *and* $s \in \Omega$. *Then the continuum* $\overline{P(u,s)}$ *is irreducible from* 0_u *to* $A(u,s)$, *and the continuum* $\overline{Q(u,s)}$ *is irreducible from* 1_u *to* $A(u,s)$.

Proof. Suppose that I is a proper subcontinuum of $\overline{P(u,s)}$ containing 0_u and $A(u,s)$. Then by the observation that $\{A(u,r) | r \in \Omega\}$ is dense in $L(u)$ there exists a point $A(u,r) \in \overline{P(u,s)} - I$. So there exists $J \in A(u,r)$ such that $D_J \in u$ and $J \cap I_n < s_n$ for all $n \in D_J$. But $L(u) - A(u,r) = P(u,r) \cup Q(u,r)$ and by the definitions of $P(u,r)$ and $Q(u,r)$, $0_u \in P(u,r)$ and $A(u,s) \in Q(u,r)$. But $A(u,r) \notin I$ so either $I \subset P(u,r)$ or $I \subset Q(u,r)$ which contradicts the connectedness of I. Similarly $\overline{Q(u,s)}$ is irreducible from I_u to $A(u,s)$.

Following are some corollaries to Mioduszewski's work which we will need.

Corollary M6. *Suppose* $u \in N^*$, s^1 *and* s^2 *are elements of* Ω, *and* $A(u,s^1) < A(u,s^2)$. *Then* $\{y \in L(u) | A(u,s^1) \leq y \leq A(u,s^2)\}$ *is a subcontinuum of* $L(u)$ *irreducible from* $A(u,s^1)$ *to* $A(u,s^2)$. *Furthermore this continuum is* $\overline{P(u,s^2)} \cap \overline{Q(u,s^1)}$.

Proof. The second conclusion follows from the definitions of P and Q. The argument for the corollary is exactly like the argument for theorem M5 but with $A(u,s^1)$ replacing 0_u.

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Corollary M8 should read: *Let* $u \in N^*$. *If* $y \in A_u$ *and* O *is an open set in* βX *containing* y *then there exist* s^1 *and* s^2 *in* Ω *so that* $A(u,s^1) < y < A(u,s^2)$ *and*

$$L(u) - (\overline{P(u,s^1)} \cup \overline{Q(u,s^2)}) \subset O \cap L(u),$$

equivalently $P(u,s^2) \cap Q(u,s^1) \subset O \cap L(u)$.

Corollary M10, proof, line 4 should read:

$$L(u) - (\overline{P(u,s^1)} \cup Q(\overline{u,s^2})) \subset O \cap L(u).$$

Corollary M12: "$t < s$" should read "$A(u,t) < A(u,s)$" in lines 2 and 3.

Corollary M7. *Let* $u \in N^*$, $s \in \Omega$, *and* $A(u,s) \notin \{0_u, 1_u\}$. *Then* $\{A(u,s)\}$ *is an end of* $\overline{P(u,s)}$ *and of* $\overline{Q(u,s)}$.

Proof. Let us first consider $\overline{P(u,s)}$ and suppose $y \in \overline{P(u,s)}$ and $y \neq A(u,s)$. Then there exists $W \in A(u,s)$, $W_y \in y$, and $H \in u$ so that $W = \{(s_n, n) | n \in H\}$ and $W_y \cap I_n < s_n$ for all $n \in H$ and $H = D_{W_y}$. Then let $r \in \Omega$ be chosen such that $W_y \cap I_n < r_n < s_n$ for all $n \in H$. So $\overline{P(u,r)}$ is a subcontinuum of $\overline{P(u,s)}$ containing y and 0_u, and $A(u,s) \in \overline{P(u,r)}$. So if E is the end of $\overline{P(u,s)}$ from 0_u then $y \in E$. Therefore $E = \{A(u,s)\}$. Similarly $\{A(u,s)\}$ is the end of $\overline{Q(u,s)}$ from 1_u.

Suppose $u \in N^*$. Then let A_u denote the set $\{A(u,s) | s \in \Omega\}$.

Corollary M8. *Let* $u \in N^*$. *If* $y \in A_u$ *and* O *is an open set in* βX *contianing* y *then there exist* s^1 *and* s^2 *in* Ω *so that* $A(u,s^1) < y < A(u,s^2)$ *and*

$$L(u) - O \subset L(u) - (\overline{P(u,s^1)} \cup \overline{Q(u,s^2)}).$$

Proof. Let $y \in A_u$, so $y = A(u,s)$ for some $s = \{s_n\}_{n=1}^{\infty} \in \Omega$. Let U be a region containing y and lying in O. Since $y \in U$ then $J = \{n | (s_n, n) \in U\} \in u$ and $\{(s_n, n) | n \in J\} \in y$. For each $n \in J$ let U_n be the component of $U \cap I_n$ which contains (s_n, n). Let s^1 and s^2 be elements of Ω so that

$$s_n^1 < s_n < s_n^2 \quad \text{and} \quad s_n^1, s_n^2 \in U_n$$

for all $n \in J$. It is easy to verify that s^1 and s^2 are the required elements of Ω.

Corollary M9. *Let* $u \in N^*$ *and let* s^1 *and* s^2 *be elements of* Ω. *Then* $\{y \in L(u) \mid A(u,s^1) \underset{u}{\leq} y \underset{u}{\leq} A(u,s^2)\}$ *is connected and is* $P(u,s^2) \cap Q(u,s^1)$.

Proof. The fact that $P(u,s^2) \cap Q(u,s^1) = \{y \in L(u) \mid A(u,s^1) \underset{u}{\leq} y \underset{u}{\leq} A(u,s^2)\}$ follows from the definitions of P and Q. By corollary M6 and corollary M7 it follows that each two points of $P(u,s^2) \cap Q(u,s^1)$ lie in a proper subcontinuum of that set, hence $P(u,s^2) \cap Q(u,s^1)$ is connected.

Let us introduce some notation. If $u \in N^*$, and s_1 and s_2 are elements of Ω then let $J^u(s^1,s^2)$ denote $P(u,s^2) \cap Q(u,s^1)$ and $\overline{J^u(s^1,s^2)}$ denote $\overline{P(u,s^2)} \cap \overline{Q(u,s^1)}$. Note that $\overline{J^u(s^1,s^2)} = Cl_{\beta X} J^u(s^1,s^2) = J^u(s^1,s^2) \cup \{A(u,s^1), A(u,s^2)\}$.

Corollary M10. *Let* $u \in N^*$, *then* $L(u)$ *is locally connected at each point of* A_u.

Proof. Let O be an open set containing $y \in A_u$. Then let s^1 and s^2 be elements of Ω so that as in corollary M8, $A(u,s^1) < y < A(u,s^2)$ and

$$L(u) - O \subset L(u) - (\overline{P(u,s^1)} \cup \overline{Q(u,s^2)}).$$

Since $\overline{P(u,s^1)}$ and $\overline{Q(u,s^2)}$ are compact then $L(u) - (\overline{P(u,s^1)} \cup \overline{Q(u,s^2)})$ is open in $L(u)$ and contains y. Furthermore, $L(u) - (\overline{P(u,s^1)} \cup \overline{Q(u,s^2)}) = J^u(s^1,s^2)$ which is connected by corollary M9.

Theorem M11. *Let* $u \in N^*$, $\Omega' \subset \Omega$, $H = \bigcap_{s \in \Omega'} \overline{P(u,s)}$ *and* $E = H \cap Cl_{\beta X}(L(u) - H)$. *Then* H *is irreducible from* O_u *to each point of* E *and* E *is the end of* H *from* O_u.

Corollary M12 . *Let* $u \in N^*$, $\Omega' \subset \Omega$, $H = \cap_{s\in\Omega'}\overline{P(u,s)}$ *and* $K = \cap\{\overline{Q(u,t)} \mid t < s$ *for all* $s \in \Omega'\}$. *Then* $\mathrm{End}(H,0_u) = H \cap K = \mathrm{End}(K,1_u) = \cap\{\cap\{\overline{P(u,s)} \cap \overline{Q(u,t)} \mid t < s$ *for all* $s \in \Omega'\} \mid s \in \Omega'\}$.

Corollary M13. *Let* $u \in N^*$, *suppose* $x \in L(u)$ *is a cut point of* $L(u)$, *and* $y \in L(u) - \{x\}$. *Then either* $y \underset{u}{<} x$ *or* $x \underset{u}{<} y$.

Proof. Let u, x and y be as in the hypothesis and suppose $y \not< x$ and $x \not< y$. Since x is a cut point of L(u) then $L(u) - \{x\}$ is the union of two mutually separated point sets P and Q containing 0_u and 1_u respectively. Assume $y \in P$. Let 0 be a region in βX so that $O \cap L(u) \subset P$, $y \in O$ and $Cl_{\beta X}O \cap Q = \emptyset$. Let $M = \cup\{P(u,s) \mid A(u,s) \in O\}$.

Claim 6.2.1. $x \notin M$.

Suppose $x \in M$. Then $x \in P(u,s)$ for some $s \in \Omega$, so $x \notin \overline{Q(u,s)}$, so $\overline{Q(u,s)} \subset Q$, and hence $A(u,s) \in Q$ which is a contradiction.

Claim 6.2.2. $y \notin M$.

Suppose $y \in M$. Then $y \in P(u,s)$ for some $s \in \Omega$ and by the previous claim $x \notin \overline{P(u,s)}$. So $x \in Q(u,s)$ and hence

$$y < A(u,s) < x$$

which contradicts our original assumption.

Therefore since $y \notin M$ and $Cl_{\beta X}O \cap L(u)$ is a subset of $Cl_{\beta X}M$ it follows that $y \in Cl_{\beta X}M$. Therefore since $y \notin M$ and y is a limit point of M it follows that $y \in \cap_{s\in\Omega'}\overline{Q(u,s)}$ where $\Omega' = \{s \in \Omega \mid A(u,s) \in O\}$. Let $H = \cap_{s\in\Omega'}\overline{Q(u,s)}$. It is easy to verify that $M = L(u) - H$. Let $E = H \cap Cl_{\beta X}M$. Then by corollary M12 H is irreducible from 1_u to each point of E and $Cl_{\beta X}M$ is irreducible from 0_u to E.

Claim 6.2.3. $x \in Cl_{\beta X}M$.

Suppose $x \notin Cl_{\beta X}M$. Then by theorem 6 there exists $A(u,s)$ such that $x \in \overline{Q(u,s)}$ and $\overline{Q(u,s)} \cap M = \emptyset$. But then $y \in P(u,s)$ so $y < A(u,s) \leq x$ which contradicts our original assumption.

Therefore $x \in Cl_{\beta X}M - M$ and $y \in Cl_{\beta X}M - M$ by theorem M11 x and y are elements of E. By claim 6.2.1 $x \notin M$ and $0_u \in M$ and M is connected so $M \subset P$. $Cl_{\beta X}M - \{x\}$ is connected so $Cl_{\beta X}M - \{x\} \subset P$. By theorem M11 O intersects $L_u - Cl_{\beta X}M$ so there exists $A(u,r) \in O \cap (L_u - Cl_{\beta X}M)$. Thus $A(u,r) > z$ for all $z \in Cl_{\beta X}M$. So $x \notin \overline{Q(u,r)}$, for otherwise $y < A(u,s) \leq x$. But then $Q(u,s) \cup (Cl_{\beta X}M - \{x\})$ is connected and contains 0_u and 1_u which contradicts the fact that x is a cut point of L(u).

Corollary M14. *Let* $u \in N^*$ *and let* G *be the collection to which* E *belongs if and only if there is a subset* Ω' *of* Ω *so that* E *is an end of* $\cap_{s\in\Omega'}\overline{P(u,s)}$. *Then* G *is an upper semi-continuous decomposition of* L(u) *and the decomposition space* L(u)/G *is a Hausdorff arc.*

Proof. The fact that G is upper semi-continuous can be seen by noticing that $\{J^u(s,r) \mid s,r \in \Omega\}$ forms a basis for the decomposition space (with appropriate modifications made for the endpoints 0_u and 1_u.) If E_1 and E_2 are elements of G then define $E_1 < E_2$ if and only if there exists

$x_1 \in E_1$ and $x_2 \in E_2$ so that $x_1 \underset{u}{<} x_2$. It is not difficult to verify that "<" is a linear order on G. Furthermore, since L(u) is a continuum it follows that L(u)/G is a continuum. Since the sets $\{\{E | A(u,r) < E < A(u,s)\} | r \in \Omega$ and $s \in \Omega\}$ forms a basis for L(u)/G (again with appropriate modifications for the endpoints) then L(u)/G is a linearly ordered continuum so it is a Hausdorff arc.

Theorem M15. *Let* $u \in N^*$ *and* $\{w^i\}_{i=1}^{\infty}$ *be a sequence of elements of* Ω *so that*

$$A(u,w^n) < A(u,w^{n+1}).$$

Let $E = (\cap_{n=1}^{\infty} \overline{Q(u,w^n)}) \cap (\cap\{\overline{P(u,w)} | A(u,w^n) < A(u,w^{n+1})$ for all $n \in N\} = \mathrm{End}(\cap_{n=1}^{\infty} \overline{Q(u,w^n)}, 1_u)$. Then E is a nondegenerate indecomposable continuum.

Definition. If X is a space and $x \in X$ then x is a P-point of X means that if $\{O_n\}_{n=1}^{\infty}$ is a countable collection of open sets each containing x then x lies in the interior of $\cap_{n=1}^{\infty} O_n$.

Theorem 1. *Let* $u \in N^*$. *Then every point of* A_u *is a P-point of* L(u).

Proof. Let $y = A(u,s) \in A_u$ and $s = \{s_n\}_{n=1}^{\infty} \in \Omega$. Suppose $\{O_n\}_{n=1}^{\infty}$ is a countable sequence of open sets each containing y so that $y \notin \mathrm{Int}_{L(u)}(\cap_{n=1}^{\infty} O_n)$. Then y is a limit point of $\cup_{i=1}^{\infty}(L(u) - O_i)$. So y is a limit point of either $P(u,s) \cap (\cup_{i=1}^{\infty}(L(u) - O_i))$ or of $Q(u,s) \cap (\cup_{i=1}^{\infty}(L(u) - O_i))$. Let us assume without loss of generality that y is a limit point of $P(u,s) \cap (\cup_{i=1}^{\infty}(L(u) - O_i))$. By corollaries M7-M14 we can construct a sequence of points $A(u,s^n)$ with $s^n \in \Omega$ so that

$$P(u,s) - \cap_{i=1}^{n} O_i \subset P(u,s^n)$$

and $A(u,s^n) < A(u,s^{n+1})$. Suppose z is a limit point of $\{A(u,s^n)\}_{n=1}^{\infty}$. Then $A(u,s^n) < z$ for all $n \in N$. Also since $z \in P(u,s)$ we have $z < y$. Let $r \in \Omega$ be such that $z < A(u,r) < y$, then $Q(u,r)$ is an open set in $L(u)$ which contains $A(u,s)$. Furthermore since $A(u,s^n) < A(u,r)$ for all $n \in N$ we have $Q(u,r) \cap P(u,s^n) = \emptyset$ so $Q(u,r) \cap (P(u,s) - \cap_{i=1}^{n} O_i) = \emptyset$. Therefore $A(u,s)$ is not a limit point of $P(u,s) \cap (\cup_{i=1}^{\infty}(L(u) - O_i))$ which is a contradiction. So the only limit point of $\{A(u,s^n)\}_{n=1}^{\infty}$ is y but this contradicts the fact that X* does not contain a convergent sequence. Therefore y lies in the interior of $\cap_{i=1}^{\infty} O_i$ and hence y is a P-point of $L(u)$.

Lemma 2.1. *Let* $u \in N^*$ *and let* $\{x_i\}_{i=1}^{\infty}$ *and* $\{y_i\}_{i=1}^{\infty}$ *be sequences of elements of* $L(u)$ *so that*

$$x_i < x_{i+1} < y_{j+1} < y_j \text{ for all } i,j \in N.$$

Then there exist $s,r \in \Omega$ *so that*

$$x_i < A(u,r) < A(u,s) < y_j \text{ for all } i,j \in N.$$

Proof. Let u, $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be as in the hypothesis. From the definition of "$\underset{u}{<}$" it follows that there exist sequences $\{r^n\}_{n=1}^{\infty}$ and $\{s^n\}_{n=1}^{\infty}$ of elements of Ω so that

$$x_i < A(u,r^i) < x_{i+1} \text{ for all } i \in N$$

and $$y_{j+1} < A(u,s^j) < y_j \text{ for all } j \in N.$$

By induction construct a sequence of sets $\{H^n\}_{n=1}^{\infty}$ of u so that

a) $r_i^n < r_i^{n+1} < s_i^{n+1} < s_i^n$ for all $i \in H^n$,

b) $H^{n+1} \subset H^n$ and

c) $\cap_{n=1}^{\infty} H^n = \emptyset$.

Let r and s be elements of Ω so that for each $k \in H^n - H^{n+1}$ we have

$$r_k^n < r_k < s_k < s_k^n.$$

Claim 2.1.1. $A(u,r_k) < A(u,s_k)$.

Let $H = H_1$ then if $k \in H$ there exists $m \in N$ so that $k \in H^m - H^{m+1}$. Then $r_k < s_k$. Therefore $r_k < s_k$ for all $k \in H$. So $A(u,r) < A(u,s)$.

Claim 2.1.2. $A(u,r^n) < A(u,r)$ for all $n \in N$.

By condition c if $k \in H^n$ there exists $m > n$ so that $k \in H^m - H^{m+1}$. Now $k \in H^m$ so $r_k^m < r_k$,

$k \in H^m \subset H^{m-1}$ so $r_k^{m-1} < r_k^m$,

$k \in H^m \subset H^{m-2}$ so $r_k^{m-2} < r_k^{m-1}$,

$\vdots$

$k \in H^m \subset H^n$ so finally we have

$$r_k^n < r_k^{n+1} < \cdots < r_k^{m-1} < r_k^m < r_k.$$

Therefore $r_k^n < r_k$. Thus $r_k^n < r_k$ for all $k \in H^n$. Therefore $A(u,r_k^n) < A(u,r)$.

Claim 2.1.3. $A(u,s) < A(u,s^n)$ for all $n \in N$.

The proof is similar to the proof of claim 2.1.2.

From claims 2.1.2 and 2.1.3 and the definitions of $\{s^n\}_{n=1}^{\infty}$ and $\{r^n\}_{n=1}^{\infty}$ we have

$$x_i < A(u,r^i) < A(u,r) < A(u,s) < A(u,s^j) < y_j$$

for all $i,j \in N$.

Lemma 2.2. *Let* $u \in N^*$ *and* $\{w^n\}_{n=1}^{\infty}$ *be a sequence of elements of* Ω *so that* $A(u,w^n) < A(u,w^{n+1})$ *for all* $n \in N$. *Let* $H = \cap_{n=1}^{\infty}\overline{Q(u,w^n)}$ *and* $E = \mathrm{End}(H,l_u)$ *then no point of* E *is a limit point of a countable set of cut points of* H.

Proof. From above $E = H \cap (\cap\{\overline{P(u,w)} \mid A(u,w^n) < A(u,w)$ for all $n \in N\})$. Since E is an end of H no point of E is a cut point of H. Also every cut point of H is a cut point of $L(u)$. Suppose that $\{z_i\}_{i=1}^{\infty}$ is a countable sequence of cut points of H which have a limit point y in E. Since no point of $\{z_i\}_{i=1}^{\infty}$ is in E we can construct by induction a sequence $\{(A(u,r^n))\}_{n=1}^{\infty}$ so that $A(u,r^n) \in H$, $A(u,r^n) < z_i$ for all $i \in N$ with $i \le n$, and $y < A(u,r^n)$ for all $n \in N$. Let $x_i = A(u,w^i)$ and $y_i = A(u,r^i)$ for all $i \in N$. Then the sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ satisfy the hypothesis of lemma 2. So there exist $r \in \Omega$ so that

$$x_i < A(u,r) < y_i \text{ for all } i \in N.$$

But since $x_i = A(u,w^i) < A(u,r)$ and E contains no cut point of $L(u)$ (theorem M15) then $P(u,r)$ is open in $L(u)$ contains y and no point of $\{y_i\}_{i=1}^{\infty}$ and hence no point of $\{z_i\}_{i=1}^{\infty}$. So E contains no limit points of $\{z_i\}_{i=1}^{\infty}$.

Theorem 2. *Let* $u \in N^*$. *Then there are at least eight non-homeomorphic subcontinua of* $L(u)$.

Proof. The eight continua are as follows:

Let C_1 denote a degenerate subset of $L(u)$.

Let $\{w^n\}_{n=1}^{\infty}$ be a seuqence of elements of Ω so that $A(u,w^n) < A(u,w^{n+1})$ let $H = \cap_{n=1}^{\infty}\overline{Q(u,w^n)}$ and let $E = \mathrm{End}(H,l_u)$.

Then by Theorem M15 E is a non-degenerate indecomposable continuum. Let $C_2 = E$.

It has been shown that there are three different types of ends; an end which is a singleton points set (corollary M7), an end which is non-degenerate and contains a limit point of a countable collection of cut points (theorem M11) and an end which is non-degenerate and contains no limit point of a countable collection of cut points (lemma 2.2).

Let $C_3 = L(u)$,

$C_4 = H = \cap_{n=1}^{\infty} \overline{Q(u,w^n)}$, and

$C_5 = Cl_{\beta X}(\cup_{n=1}^{\infty} P(u,w^n))$.

Let $\{r^n\}_{n=1}^{\infty}$ be a sequence of points of Ω so that

$$A(u,r^{n+1}) < A(u,r^n) < A(u,w^1)$$

and let $\hat{H} = Cl_{\beta X}(\cup_{n=1}^{\infty} \overline{Q(u,r^n)})$.

Let $C_6 = \hat{H} \cap C_5$.

Let $\{s^n\}_{n=0}^{\infty}$ be a sequence of elements of Ω so that $A(u,s^0) \in C_4 = H$ and $A(u,s^0) < A(u,s^{n+1}) < A(u,s^n)$ for all $n \in N$.

Let $C_7 = H \cap (\cap_{n=1}^{\infty} \overline{P(u,s^n)})$.

Let $C_8 = \hat{H} \cap (\cap_{n=1}^{\infty} \overline{P(u,s^n)})$.

The continua $\{C_i\}_{i=1}^{8}$ are pairwise non-homeomorphic because C_1 and C_2 are non-homeomorphic indecomposable continua and C_i is decomposable if $i \in \{3, 4, 5, 6, 7, 8\}$. The continua $\{C_i\}_{i=3}^{8}$ are pairwise non-homeomprphic because their end structures are different.

Definition. Suppose that X is a continuum and $p \in X$. Then the composant of X at p is the set to which x belongs if and only if there is a proper subcontinuum of X containing x and p. A composant of X is a composant of X at some point of X. It is well known [M,K] that the composants of an indecomposable continuum are disjoint, and that a non-degenerate indecomposable metric continuum has c composants.

Remarks. There are some set theoretic axioms which will give additional continua different from the ones constructed above. For example the axiom of near coherence of filters NCF yields that A^* has only one composant [Bl,Mi]. Bellamy [Be] showed that A^* can be mapped onto every metric continuum. Let C be a non-degenerate indecomposable metric continuum and let $f: A^* \to C$ be an onto map. Then there is a subcontinuum D of A^* so that D is mapped by f onto C but no proper subcontinuum of D is mapped onto D. Then D is indecomposable and must have at least c composants. So under NCF, A^* and D are non-homeomorphic non-degenerate indecomposable continua.

It follows from the above remarks that it is consistent that there exist non-degenerate indecomposable subcontinua of A^* with different numbers of composants. We conjecture that the continuum hypothesis CH implies that every non-degenerate indecomposable subcontinuum of A^* has 2^c composants. Rudin [R] has shown that A^* has 2^c composants.

We will now examine the structure of some of the indecomposable layers of the continua $L(u)$ for $u \in N^*$. If $f: N \to N$ is a mapping then βf denotes the unique extension $\beta f: \beta N \to \beta N$. We will use the symbol CH to indicate the continuum hypothesis. The near coherence of filters, which we will denote by NCF, is equivalent to the following statement [Bℓ], *the near coherence of filters axiom* (NCF):

If u and v are two ultrafilters in N^* then there exists a non-decreasing finite to one map $f: N \to N$ so that $\beta f(u) = \beta f(v)$. The assumption of NCF has been shown to be independent of the axioms of set theory by Blass and Shelah [BS].

By a map is meant a continuous function. We will assume that all spaces are completely regular.

The following theorem follows from a theorem of Bellamy and Rubin [BR].

Theorem A. *Suppose* $\{I_n\}_{n=1}^{\infty}$ *is a sequence of Hausdorff continua so that for each* $n \in N$, *with* $n > 1$, I_n *is irreducible from* I_{n-1} *to* I_{n+1}, $I_n \cap I_k \neq \emptyset$ *if and only if* $|n - k| \leq 1$, *and* $W = \cup_{n=1}^{\infty} I_n$ *is locally compact and not compact. Then* W^* *is an indecomposable continuum.*

The following theorem was proven in [S2].

Theorem B. *Suppose that* $\{I_n\}_{n=1}^{\infty}$ *is a sequence of Hausdorff continua so that for each* $n \in N$, *with* $n > 1$, I_n *is irreducible from* I_{n-1} *to* I_{n+1}, $I_n \cap I_k \neq \emptyset$ *if and only if* $|n - k| \leq 1$ *and* $W = \cup_{n=1}^{\infty} I_n$ *is locally compact and not compact. Then* CH *implies that* W^* *has* 2^c *composants.*

Lemma 3.1. *Let* X *be a locally compact Hausdorff space,* $x \in X^*$, $H \in x$, *and let* $\{M_t | t \in \Gamma\}$ *be a collection of subcontinua of* X *so that* $H \subset \cup_{t \in \Gamma} M_t$. *Furthermore if* $J \subset X$ *let* M_j *denote* $\cup\{M_t | M_t \cap J \neq \emptyset\}$. *Then* $M = \cap\{Cl_{\beta X} M_J | J \in x\}$ *is a continuum which contains* x.

Proof. Suppose $J \in x$ and $J \subset H$. Then since each point of J lies in some element of $\{M_t | t \in \Gamma\}$ we have

$J \subset M_J$. So $Cl_{\beta X}J \subset Cl_{\beta X}M_J$. But $x \in Cl_{\beta X}J$. Therefore $x \in Cl_{\beta X}M_J$. Therefore $x \in Cl_{\beta X}M_J$ for all $J \in x$ so $x \in M$.

Suppose that M is not a continuum. Then M is the union of two disjoint compact sets A and B. Assume $x \in A$. Let U_A and U_B be disjoint open sets in βX with disjoint closures containing A and B respectively.

For each $y \in \beta X - U_A \cup U_B$, $y \notin M$. So there exists an element $H_y \in x$ and an open set R_y so that $y \notin Cl_{\beta X}M_{H_y}$, $y \in R_y$, and $(Cl_{\beta X}M_{H_y}) \cap R_y = \emptyset$. By compactness some finite subcollection $\{R_{y_i}\}_{i=1}^n$ of $\{R_y | y \in \beta X - U_A \cup U_B\}$ covers $X - U_A \cup U_B$. Let $J = \cap_{i=1}^n H_{y_i}$, let $\hat{J}$ be such that $\hat{J} \in x$ and $\hat{J} \subset U_A$, and let $J' = J \cap \hat{J}$. Then $J' \in x$. So $Cl_{\beta X}M_{J'}$ intersects both U_A and U_B, because $A \cup B \subset Cl_{\beta X}M_{J'}$. So $M_{J'}$ intersects both U_A and U_B. Let $z \in M_{J'} \cap U_B$. So $z \in M_t$ for some $t \in \Gamma$ such that $M_t \cap J' \neq \emptyset$. So M_t intersects both U_A and U_B. But since $M_t \subset M_{H_{y_i}}$ for all $i \in \{1,2,\cdots,n\}$ we have $M_t \cap R_{y_i} = \emptyset$ for all $i \in \{1,2,\cdots,n\}$. So $M_t \subset U_A \cup U_B$. So M_t is not connected. This contradicts the original hypothesis. So the lemma is established.

Theorem 3. *Suppose* $\{I_n\}_{n=1}^\infty$ *is a sequence of irreducible continua so that for each positive integer* $n > 1$, I_n *is irreducible from* I_{n-1} *to* I_{n+1}, $I_n \cap I_k \neq \emptyset$ *if and only if* $|n - k| \leq 1$, *and* $X = \cup_{n=1}^\infty I_n$ *is locally compact and not compact. Then* NCF *implies that* X^* *has only one composant.*

Proof. For each $n \in N$ let W_n be a zero set in X such that $I_n \subset W_n$ and $W_n \cap W_k \neq \emptyset$ if and only if $|n - k| \leq 1$ and let Z_n be a zero set in X such that $Z_n \subset I_n - I_{n-1} \cup I_{n+1}$ and $W_n \cap Z_k \neq \emptyset$ if and only if $n = k$. Let p and q be two points of X*. We will construct a proper subcontinuum of X* containing p and q. Since p and q are ultrafilters of zero sets there exist i, j $\in \{0,1,\cdots,6\}$ so that $\cup_{n=1}^{\infty} W_{7n+1} \in p$ and $\cup_{n=1}^{\infty} W_{7n+j} \in q$. There exists an integer $\ell \in \{0,1,2,\cdots,6\}$ such that $|i - \ell| \geq 2$ and $|j - \ell| \geq 2$. Let $J_n = I_{7n+\ell+1} \cup I_{7n+\ell+2} \cup \cdots \cup I_{7n+\ell+6}$. Then J_n is a continuum and for each $n \in N$ the continuum $I_{7n+\ell}$ intersects neither $\cup_{k=1}^{\infty} W_{7k+i}$ nor $\cup_{k=1}^{\infty} W_{7k+j}$. Therefore $\{J_n\}_{n=1}^{\infty}$ is a sequence of disjoint subcontinua of X and p and q are both points of $Cl_{\beta X}(\cup_{n=1}^{\infty} J_n)$. Without loss of generality let us assume that $\ell = 0$. Note that $W_{7n+i} \cup W_{7n+j} \subset J_n$. If $L \subset X$ then define $H_L = \{n | L \cap J_n \neq \emptyset\}$, and if $x \in X^*$ define $u_x = \{H_L \subset N | L \in x\}$.

Claim 1. If $x \in X^*$ and $\cup_{n=1}^{\infty}(W_{7n+i} \cup W_{7n+j}) \in x$ then u_x is an ultrafilter in N*.

Proof. Suppose $H \in u_x$ and $K \subset N$ is such that $H \subset K$. Then there exists $L \in x$ such that $H_L = H$. Let $\hat{L} = L \cup \{Z_n | n \in K - H\}$. Since L and $\cup\{Z_n | n \in K - H\}$ are both zero sets it follows that $\hat{L}$ is a zero set and since $\hat{L}$ is a zero set and $L \subset \hat{L}$ we have $\hat{L} \in x$. Therefore $H_{\hat{L}} \in u_x$ and $H_{\hat{L}} = H_L \cup \{n | n \in K - H\} = K$ so $K \in u_x$.

Suppose H_1 and H_2 are elements of u_x. Then there exist L_1 and L_2 in x so that $H_{L_1} = H_1$ and $H_{L_2} = H_2$. Then $L = L_1 \cap L_2 \in x$. Consider H_L. If $n \in H_L$ then there exists $z \in L_1 \cap L_2 \cap J_n$ and we have $n \in H_{L_1} \cap H_{L_2}$. Therefore $H_{L_1 \cap L_2} \subset H_{L_1} \cap H_{L_2}$, but $H_{L_1 \cap L_2} \in x$ so by the previous argument $H_{L_1} \cap H_{L_2} \in u_x$. Therefore we have established that u_x is a filter.

Suppose now that u_x is not an ultrafilter. Then there exists a set $K \subset N$ such that $K \notin u_x$ but $K \cap H \neq \emptyset$ for all $H \in u_x$. Let $\hat{W}_n = W_{7n+i} \cup W_{7n+j}$ and let $M = \cup\{\hat{W}_n | n \in K\}$. By hypothesis $W = \cup_{n=1}^{\infty} \hat{W}_n \in x$ so if $L \in x$ then $\tilde{L} = L \cap W$ is an element of x and $H_{\tilde{L}} \in u_x$. Then $H_{\tilde{L}} \cap K \neq \emptyset$ so there exists m so that $m \in H_{\tilde{L}} \cap K$, so $\tilde{L} \cap J_m \neq \emptyset$ but $\tilde{L} \subset W$ so $\tilde{L} \cap \hat{W}_m \neq \emptyset$. But $W_m \subset M$ so $\tilde{L} \cap M \neq \emptyset$. Therefore M is a zero set which intersects every element of x so $M \in x$ and since $H_M = K$ we have $K \in u_x$. Therefore u_x is an ultrafilter and the claim is established.

Therefore u_p and u_q are both ultrafilters in N^*. By NCF there exists a function $f: N \to N$ so that f is non-decreasing and finite to one and so that $\beta f(u_p) = \beta f(u_q) = w$. For each integer n let

$$r_n = \text{Min}\{k | f(k) = n\}$$

$$s_n = \text{Max}\{k | f(k) = n\}$$

and let $K_n = \cup\{I_k | 7r_n + 1 \leq k \leq 7s_n + 6\}$. Then since f is finite to one K_n is a subcontinuum of X.

Let $M = \cap\{\text{Cl}_{\beta X}(\cup\{K_n | n \in E\} | E \in w\})$. By lemma 3.1 M is a subcontinuum of βX and since $w \in N^*$ we have $M \subset X^*$. Note that $\cup\{J_k | k \in f^{-1}(n)\} \subset K_n$.

Claim 2. $p \in M$ and $q \in M$.

Proof. We will prove that $p \in M$, the proof that $q \in M$ is similar. Suppose $E \in w$. Thus $f^{-1}(E) \in u_p$ (since $\beta f(u_p) = w$). Therefore there exists $L \in p$ such that

$H_L = f^{-1}(E)$ and $L \subset W = \cup_{n=1}^{\infty}\hat{W}_n$. Suppose $t \in L$, then $t \in J_n$ for some integer n and $n \in f^{-1}(E)$. So $f(n) \in E$. Let $m = f(n)$; so

$$r_m \leq n \leq s_m$$

$$J_n = I_{7n+1} \cup \cdots \cup I_{7n+6}, \text{ and}$$

$$7r_m + 1 \leq 7n + 1 \leq 7n + 6 \leq 7s_m + 6, \text{ so}$$

$J_n \subset K_m$ and $t \in K_m$. Therefore $L \subset \cup\{K_m | m \in E\}$, and so

$$p \in \mathrm{Cl}_{\beta X}L \subset \mathrm{Cl}_{\beta X}(\cup\{K_m | m \in E\}).$$

So $p \in \mathrm{Cl}_{\beta X}(\cup\{K_m | m \in E\})$ for all $E \in w$ and hence $p \in M$. This establishes the claim.

Claim 3. M is a proper subcontinuum of X^*.

Proof. Let $H \subset N$ be such that $H \notin w$ and let $G = f^{-1}(H)$. There exists $\hat{H} \in w$ so that $\hat{H} \cap H = \emptyset$. Let $\hat{G} = f^{-1}(\hat{H})$. Then $\hat{G} \cap G = \emptyset$. For each $k \in H$ let $t_k \in f^{-1}(k)$ and let $x_k \in \hat{W}_{t_k}$. Let

$$a_k = \mathrm{Max}\{j \in \hat{G} | j \leq t_k\} \text{ and}$$

$$b_k = \mathrm{Min}\{j \in \hat{G} | t_k \leq j\}.$$

So $a_k < t_k < b_k$. Since f is nondecreasing we have $f(a_k) < f(t_k) < f(b_k)$. So by definition of K_k and since f is nondecreasing $\hat{W}_{t_k} \cap (K_{f(a_k)} \cup K_{f(b_k)}) = \emptyset$ and so $\hat{W}_{t_k} \cap K_n = \emptyset$ for all $n \in \hat{H}$. Therefore $\{x_k | k \in H\} \cap (\cup\{K_n | n \in \hat{H}\}) = \emptyset$ and $\mathrm{Cl}_{\beta X}(\{x_k | k \in H\}) \cap \mathrm{Cl}_{\beta X}(\cup\{K_n | n \in \hat{H}\}) = \emptyset$ and hence $\mathrm{Cl}_{\beta X}(\{x_k | k \in H\}) \cap M = \emptyset$. This establishes claim 3.

Therefore M is a proper subcontinuum of X^* that contains p and q.

Definition. The subspace W of the space Y is said to be C^*-*embedded* in Y if and only if whenever $f\colon W \to (-\infty,\infty)$

is a bounded continuous function then f extends to a bounded continuous function $\hat{f}: Y \to (-\infty,\infty)$.

If Y is a space and H and K are two subsets of Y then H and K are said to be *completely separated* in Y if and only if there exists a map $f: Y \to [0,1]$ such that $f(H) = 0$ and $f(K) = 1$.

The following two theorems which relate these concepts will be needed.

Theorem C. (Urysohm's extension theorem, see [GJ].) *If* Y *is a space then the subspace* K *of* Y *is* C* *embedded in* Y *if and only if any two completely separated sets in* K *are completely separated in* Y.

Theorem D. (See [GJ].) *A subspace* S *of the space* X *is* C* *embedded in* X *if and only if* βS *is homeomorphic to* $Cl_{\beta X}S$.

Theorem 4. *Suppose* X *is a locally compact separable metric space and* K *is a* σ*-compact subset of* X*. *Then* K *is* C* *embedded in* X *and hence* $Cl_{\beta X}K$ *is homeomorphic to* βK.

Proof. We will use theorem C to establish that K is C* embedded in βX. So let $\hat{L}$ and $\hat{M}$ be arbitrary sets which are completely separated in K. Let $f: K \to [0,1]$ be a map such that $f(\hat{L}) = 0$ and $f(\hat{M}) = 1$ and let $L = f^{-1}(0)$ and $M = f^{-1}(1)$. Then L and M are closed in K. Let $\{J_i\}_{i=1}^{\infty}$ be a sequence of compact sets such that $K = \cup_{i=1}^{\infty} J_i$ and $J_n \subset J_{n+1}$ for all $n \in N$.

Since X is a locally compact separable metric space we can find a sequence $\{U_i\}_{i=1}^{\infty}$ of open sets in X so that for each n, $X - U_n$ is compact and $Cl_{\beta X}U_{n+1} \subset U_n$, and $X^* = \cap_{n=1}^{\infty} U_n$. By induction construct sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of sets in βX so that for each positive integer n,

a) $A_n \subset U_n$ and $B_n \subset U_n$,

b) $M \cap J_n \subset A_n$ and $L \cap J_n \subset B_n$,

c) $Cl_{\beta X}A_n \cap Cl_{\beta X}L = \emptyset$ and $Cl_{\beta X}B_n \cap Cl_{\beta X}M = \emptyset$, and

d) $Cl_{\beta X}A_n \cap (Cl_{\beta X} \cup_{k=1}^{n-1} B_n) = \emptyset$ and
$Cl_{\beta X}B_n \cap (Cl_{\beta X} \cup_{k=1}^{n} A_n) = \emptyset$.

Condition c can be guaranteed because $M \cap J_n$ is compact and contains no limit points of L and because $L \cap J_n$ is compact and contains no limit points of M. Let $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$.

Claim. $Cl_X(A \cap X) \cap Cl_X(B \cap X) = \emptyset$.

Proof. Suppose $x \in Cl_X(A \cap X) \cap Cl_X(B \cap X)$. Then $x \in U_{n-1} - U_n$ for some integer n. So $x \notin Cl_{\beta X}U_{n+1}$ and hence $x \notin Cl_{\beta X}A_k$ and $x \notin Cl_{\beta X}B_k$ for all $k \geq n+1$. So $x \in Cl_X(A_i \cap X)$ and $x \in Cl_X(B_j \cap X)$ for some $i \leq n$ and some $j \leq n$. But this contradicts condition d. So the claim is established.

Therefore $Cl_{\beta X}(A \cap X)$ and $Cl_{\beta X}(B \cap X)$ are also disjoint. Furthermore $Cl_X(A \cap X)$ and $Cl_X(B \cap X)$ are closed in X so there is a map $h: X \to [0,1]$ so that $h(Cl_X(A \cap X)) = 1$ and $h(Cl_X(B \cap X)) = 0$. So h has a unique extension $\beta h: X \to [0,1]$. Since $h(Cl_X(A \cap X)) = 1$ and $h(Cl_X(B \cap X)) = 0$ we have $\beta h(Cl_{\beta X}(A \cap X)) = 1$ and $\beta h(Cl_{\beta X}(B \cap X)) = 0$. But $M \subset \cup_{i=1}^{\infty} A_i \subset Cl_{\beta X} \, {}_{i=1}^{\infty} A_i = Cl_{\beta X}(A \cap X)$, so $\beta h(M) = 1$. Similarly $\beta h(L) = 0$. Therefore $\hat{L}$ and $\hat{M}$ are completely separated in βX. The remainder of the conclusion follows from theorem D.

Let $X = [0,1] \times N$. By theorem M2 the set of components of X^* is in a 1-1 correspondence with the points of N^* and $L: N^* \to$ components of X^* is this correspondence. Recall the order "$\underset{u}{\leq}$" on $L(u)$ which was defined above. The following theorem was first proven by Miodsuszewski [Mi]. It also follows from theorems A and 4.

Theorem 5. (Theorem 19 [Mi].) *Suppose* $u \in N^*$, $\{p_i\}_{i=1}^{\infty}$ *is a sequence of a cut point of* $L(u)$ *with* $p_n \underset{u}{\leq} p_{n+1}$ *and* J_n *the subcontinuum of* X^* *irreducible from* p_n *to* p_{n+1} *for each* $n \in N$. *Then* $J = \mathrm{Cl}_{\beta X}(\cup_{n=1}^{\infty} J_n) - \cup_{n=1}^{\infty} J_n$ *is an indecomposable continuum.*

Proof. If $n > 1$ is a positive integer then p_n is a cut point of $L(u)$ which separates p_{n+1} from p_{n-1} in $L(u)$. So $J_n \cap J_k \neq \emptyset$ if and only if $|k - n| \leq 1$. Then by theorem 4 J is homeomorphic to $\beta(\cup_{n=1}^{\infty} J_n) - \cup_{n=1}^{\infty} J_n$ and so by theorem A J is indecomposable.

We can now apply theorem A and theorem 4 with theorem 4 of [S1] and theorem A and theorem 4 with theorem 3 to obtain the following:

Theorem 6. *If* J *is as in the hypothesis of theorem* 5 *then* CH *implies that* J *has* 2^c *composants and* NCF *implies that* J *has only one composant.*

Corollary 6.1. *If* $A = [0,\infty)$ *and* U *is an open set in* A^* *then* CH *implies that* U *contains an indecomposable* *continuum with* 2^c *composants and* NCF *implies that* U *contains an indecomposable continuum with only one composant.*

Remarks. Let $A = [0,\infty)$. If we assume NCF then A^* is a one composant indecomposable continuum; also every nondegenerate subcontinuum of A^* contains a one composant indecomposable continuum (by theorem 6, and the constructions in [S1] which show that every subcontinuum of A^* contains a copy of some L(u).) We also know that every nondegenerate subcontinuum of A^* maps onto every metric continuum [Be]. Let Z be a nondegenerate metric indecomposable continuum and let Y be a nondegenerate subcontinuum of A^*. Then Y maps onto Z and hence some subcontinuum K of Y maps irreducibly onto Z (K maps onto Z but no proper subcontinuum of K maps onto Z). Any such continuum K must be indecomposable and have at least c composants. Therefore NCF implies that A^* contains indecomposable continua which also have at least c composants. Such continua are clearly not homeomorphic to A^* and by theorem 6 they cannot be positioned in A^* like the continuum J in theorem 5. We conjecture (even under ZFC) that A^* contains indecomposable continua which are not homeomorphic to A^* and which are positioned in A^* differently than the continuum J described in theorem 5. However we have not determined if this is the case even under CH.

The following problems arise naturally from our discussion.

Q1. Does A^* have 2^c different subcontinua? It would be of interest if A^* can be shown to have at least infinitely many different subcontinua.

Q2. If u and v are different types [F1,F2] of points of N^* then are $L(u)$ and $L(v)$ non-homeomorphic? It is easy to see that if u and v are points of the same types then $L(u)$ is homeomorphic to $L(v)$.

Q3. Does CH imply that every non-degenerate indecomposable subcontinuum of A^* have 2^c composants?

Q4. Does A^* have infinitely many nonhomeomorphic indecomposable subcontinua? (Even under any extra set theoretic axioms.)

Q5. If under some set theoretic axiom A^* has K composants, then does it follow that every continuum positioned in A^* like the continuum J in theorem 5 also have K composants?

Remark. E. van Douwen has asked in conversation whether every cut point of $L(u)$ is in the form $A(u,s)$ for some $s \in \Omega$. We intend to show in a future paper that this is not the case under some set theoretic assumptions.

If one is willing to assume some extra set theoretic assumptions then some partial answers to these questions can be made. It was pointed out to the author by Stewart Baldwin that Martin's Axiom (MA) implies that if $u \in N^*$ there is a subset of A_u which is well ordered by $\leq_u$ and has cardinality c. Furthermore, MA + "c is weakly inaccessible" is consistent relative to the consistency of the existence of an inaccessible cardinal (see [J], [vD2]). Therefore we can assume that there are c non-homeomorphic subarcs of L(u)/G hence MA + c is weakly inaccessible implies there are at least c different subcontinua of L(u).

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