TOPOLOGY PROCEEDINGS

Volume 11, 1986

Pages 415–426

http://topology.auburn.edu/tp/

PROBLEM SECTION

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

PROBLEM SECTION

CONTRIBUTED PROBLEMS

Some of the problems listed here are related to talks that were given at the 1986 Spring Topology Conference in Lafayette, Louisiana, for which this volume of TOPOLOGY PROCEEDINGS is the journal of record. Others were communicated to the Problems Editor at the Baku international conference in October of 1987; these are indicated with a [B].

The Problems Editor invites anyone who has published a paper in TOPOLOGY PROCEEDINGS or attended a Spring Topology Conference to submit problems to this section. They need not be related to any articles, but if they are, please provide a reference. Please define any terms not in a general topology text or in referenced articles.

A. Cardinal Invariants

21. *(Shapirovskii)* Let A be a subset of a space X and let $x \in A'$. [A' denotes the derived set.] Define the accessibility number $a(x,A)$ to be $\min\{|B|: B \subset A, x \in B'\}$. Define $t_c(x,X)$ to be $\sup\{a(x,F): F \text{ is closed and } x \in F'\}$. As usual, define $t(x,X)$ as $\sup\{a(x,A): x \in A'\}$. Can we ever have $t_c(x,X) < t(x,X)$ in a compact Hausdorff space? [B]

See also B30, C51, and C52.

B. Generalized Metric Spaces and Metrization

30. *(M. E. Rudin)* A Collins space is one in which each point x has a special countable open base W_x with the

property that, if U is a neighborhood of a point y, there is a neighborhood V of y such that, for all $x \in V$ there is a $W \in W_x$ with $y \in V \subset U$. [Recall that a Collins space is metrizable precisely if W_x can be made a nested decreasing sequence for each x.] It is easy to see that every space with a point countable base is a Collins space. Is the converse true?

Comment by M. E. Rudin: "The conjecture [that the converse is true] has become doubly interesting to me since I now know that I do *not* know how to construct a counterexample."

See also C51.

C. Compactness and Generalizations

51. *(Malyhin)* Recall that a space is *weakly first countable* if to each point x one can assign a countable filterbase F_x of sets containing x such that a set U is open iff for each $x \in U$ there is $F \in F_x$ such that $F \subset U$. Is there a weakly first countable compact space which is not first countable? One that is of cardinality $> c$? [Yes if CH] [B]

52. *(Shapirovskii)* Is it true that every infinite compact Hausdorff space contains either $\beta\omega$, or a point with countable π-character, or a nontrivial convergent sequence? [B]

53. *(Uspensky)* Is every Eberlein compact space of nonmeasurable cardinal bisequential? [B]

54. *(Nyikos)* Is there a compact sequential space of nonmeasurable cardinal that is not hereditarily α-realcompact?

(A space is called α-realcompact if every maximal family of closed sets with the c.i.p. has nonempty intersection.) [Yes if $\clubsuit$.]

See also A21, D37, F29, P27, and P28.

D. Paracompactness and Generalizations

37. *(Nyikos)* Is there a "real" example of a locally compact, realcompact, first countable space of cardinality $\aleph_1$ that is not normal?

See also C53, C54, and U2.

F. Continua Theory

29. *(Nyikos)* Is there a (preferably first countable, or, better yet, perfectly normal) locally connected continuum without a base of open subsets with locally connected closures? [Yes to the general question if CH.]

H. Homogeneity and Mappings of General Spaces

13. *(H. Kato)* Do refinable maps preserve countable dimension?

14. *(A. Koyama)* Do c-refinable maps between normal spaces preserve Property C?

See also O10, O11, and P26.

L. Topological Algebra

7. *(Arkhangel'skii)* Let F(X) denote the free topological group on the space X. If dim $\beta X = 0$, does dim $\beta F(X) = 0$ follow? [B]

O. Theory of Retracts; Extension of Continuous Functions

10. *(A. Koyama)* Let $r: X \to Y$ be a refinable map and let K be a class of ANR's. If Y is extendible with respect to K, then is X also extendible with respect to K?

11. *(H. Kato)* Do refinable maps preserve FANR's?

P. Products, Hyperspaces, Remainders, and Similar Constructions

26. *(Ibula Ntantu)* Let X be a Tychonoff space and $K(X)$ the hyperspace of its nonempty compact subsets. Recall that a continuous $f: Z \to Y$ is called compact-covering if each compact subset of Y is the image of some compact subset of Z. If $K(X)$ with the Vietoris (i.e. finite) topology is a continuous image of ω^ω, must X be a compact-covering image of ω^ω? [The converse is true.]

27. *(Malyhin)* A point $x \in X$ is called a *butterfly point* if there exist disjoint sets A and B of X such that $\overline{A} \cap \overline{B} = \{x\}$. Is it consistent that there is a non-butterfly point in $\omega^* = \beta\omega - \omega$? Is it consistent with MA? [B]

[*Editor's note.* If PFA, non-butterfly points of ω^* would be exactly those points p for which $\omega^* - \{p\}$ would be normal. Malyhin has withdrawn the claim (which would have implied the nonexistence of such points under PFA) made in his 1973 paper with Shapirovskii, that MA implies $\omega^* - \{p\}$ is always non-normal. It is still an unsolved problem whether there is a model in which $\omega^* - \{p\}$ is normal for some $p \in \omega^*$.]

28. *(Malyhin)* Is there a model in which $\beta(\omega^* - \{p\}) = \omega^*$ for some points of ω^* but not for others? [B]

[*Editor's note.* For background information, see the sub-section on solutions to earlier problems, solution to P20 (vol. 7).]

See also L7.

U. Uniform Spaces

2. *(Steve Carlson)* If a proximity space admits a compatible complete uniformity, is it rich?

PROBLEMS FROM OTHER SOURCES

At the end of his paper in Uspehii. Mat. Nauk. 33:6 (1978), 29-84 (=Russian Math. Surveys 33:6 (1978), 33-96), A. V. Arkhangel'skii posed a list of 26 open problems with the comment, "[translation] The solution of many of them seems to me to require original ideas, and methods." Here we list the ones which have either been solved or have had some consistent answer given to them. In future issues, solutions will be incorporated in the sub-section, "Information on Earlier Problems."

1. Does there exist "naïvely" a non-sequential compact space of countable tightness? *Solution.* No, PFA implies the nonexistence of such a compact space *(Balogh, AMS Proceedings, to appear)*, and one does not need large cardinals for this answer *(Dow)*. It was known, of course, that such spaces do exist under $\diamondsuit$ *(Ostaszewski, Fedorchuk)* and more recently this was shown compatible with MA + $\neg$CH *(Nyikos)*.

2. Is it true "naïvely" that each non-empty sequential compact space is first countable at some point? *Solution.* No. In the model produced by adding one Cohen real to a model of $p = c > \omega_1$, there is a Frechet-Urysohn compact space in which every point is of character ω_1 *(Malyhin).* It was known, of course, that CH implies an affirmative answer *(Mrówka).*

5. Is there a non-metrizable homogeneous Eberlein compact space? *Solution.* Yes *(J. van Mill).*

7. Does there exist "naïvely" a compact Fréchet-Urysohn space whose square is not Fréchet-Urysohn? *Solution.* Yes *(Petr Simon, Comment. Math. Univ. Carolinae 21 (1980), 749-753).*

8. Does there exist "naïvely" a regular space for which $hl(X^n) \leq \tau$ for all $n \in N$ and $d(X) > \tau$? *Solution.* Yes, for any τ that is the successor of a regular cardinal except possibly $\tau = \omega_2$ *(Shelah, communicated by I. Juhász).*

11. Is every left-separated regular space zero-dimensional? *Solution.* No *(R. C. Solomon).* There is even a connected, completely regular, Moore counterexample *(P. de Caux).*

13. Does it follow from MA + $\neg$CH that every regular, first countable, hereditarily separable space is Lindelöf? *Solution.* No *(Abraham and Todorčević, Handbook of Set-Theoretic Topology)*, but it does follow from PFA *(Baumgartner and Todorčević)* and large cardinals are not needed *(Todorčević).*

14. Are the following "naïvely" equivalent: a) there is an L-space; b) there is an S-space? *Solution.* No *(Todorčević).*

16. Let X be a regular ccc space with a G_δ-diagonal. Is it true that $|X| \leq c$? *Solution.* There is no upper bound on the cardinality of X. *(D. B. Shakhmatov, Comment. Math. Univ. Carol.* 25 (1984), 731-746*).*

23. Is there a Lindelöf symmetrizable regular space which is not separable (necessarily an L-space)? *Consistency result.* There is a forcing construction of such an example. *(D. B. Shakhmatov).* The example has a coarser topology which is separable, metrizable, and zero-dimensional, which is relevant to Problem B14 [see next subsection].

24. Is $t(X \times X) = t(X)$ for each countably compact Tychonoff space? *Consistency result.* In the model adding one Cohen real to a model of MA + $\neg$CH, there is a counter-example which is even a hereditarily separable Fréchet-Urysohn topological group *(Malyhin and Shakhmatov).*

25. Does there exist "naïvely" a compact space X such that $c(X \times X) > c(X)$? *Solution.* Yes *(Todorčević).*

INFORMATION ON EARLIER PROBLEMS

From *Lectures on Set-Theoretic Topology*

C7. Is the density $\leq$ the smallest cardinal greater than the spread for compact spaces? *Solution.* Yes. *(Shapirovskii, Amer. Math. Soc. Transl.* (2) *vol.* 134, 1987, *pp.* 93-118. = *Seminar on General Topology, P. S. Alexandroff ed., Moskov. Gos. Univ.,* 1981, *pp.* 162-187.*)* [*Caution.* This paper uses s(X) for the density of X and $\bar{c}(X)$ for the spread.]

From Topology Proceedings

Volume 1.

D6. *(Alster and Zenor)* Is every perfectly normal, locally Euclidean space collectionwise normal? *Remarks.* M. E. Rudin has withdrawn her claim of a counterexample from $\diamondsuit^+$. The status of this problem is that it is consistent for the answer to be affirmative: $MA(\omega_1)$ implies perfectly normal manifolds are metrizable. But no consistency result in the opposite direction is known.

Volume 2.

Classic problem VI. Is there a compact space of countable tightness which is not sequential? *Solution.* See the solution to Problem 1 of the preceding sub-section. Fedorchuk's Φ $(=\diamondsuit)$ example [referenced in vol. 2] also combines with the Balogh-Dow result to solve related problem D: Is there a compact space of countable tightness which is not sequentially compact?

Classic problem VI, related problem E. Is every separable, countably compact space of countable tightness compact? *Partial Solution.* *(Nyikos)* For Hausdorff spaces the answer is negative. For regular spaces there are only "consistent" counterexamples. For normal spaces, it is independent: No if $\diamondsuit$, yes if PFA, and (Dow) large cardinals are not needed for "yes."

Volume 3.

B14. *(Arkhangel'skii)* Let X be regular, Lindelöf, and symmetrizable. Is X separable? Does X have a G_δ-diagonal? *Consistency result.* It is consistent that the answer to the first is negative, but the construction does have a G_δ-diagonal (see preceding sub-section, Problem 23) *(Shakhmatov)*.

C16. *(J. Hagler)* Does there exist a compact space K with a countable dense subset D such that every sequence in D has a convergent subsequence, but K is not sequentially compact? (We may assume without loss of generality that K is a compactification of ω, i.e. that the points of D are isolated.) *Consistency result.* Yes if $\mathfrak{s} = \mathfrak{c}$. In fact, $\mathfrak{s} = \mathfrak{c}$ implies that $2^{\mathfrak{c}}$ itself is an example of such a K *(Nyikos)*.

C18. *(Nyikos)* Is there a compact non-metrizable space X such that X^2 is hereditarily normal? *Consistency result.* Yes if $\mathfrak{p} > \omega_1$ *(Nyikos)* and yes if CH *(Gruenhage)*.

Volume 4.

C29. *(Nyikos)* Does there exist a first countable, countably compact, noncompact regular space which does not contain a copy of ω_1? [Yes if $\clubsuit$; also yes in any model which is obtained from a model of $\clubsuit$ by iterated ccc forcing, so that "yes" is compatible with MA + $\neg$CH.] *Solution.* No is also consistent. It follows from PFA that no such space exists *(Balogh)* and a negative answer is also equiconsistent with ZFC *(Dow)*.

Volume 5.

C32. *(Nyikos)* Is every separable, first countable, normal, countably compact space compact? [No if $\mathfrak{p} = \omega_1$] *Solution.* Yes if PFA *(Fremlin)*, and an affirmative answer is equiconsistent with ZFC *(Dow)*. But also, a negative answer is consistent with MA + $\neg$CH and with PFA$^-$ *(Nyikos)*.

E8. *(Watson)* Is there a locally compact, normal, non-collectionwise normal space? [Yes if $\mathfrak{p} > \omega_1$ or "Shelah's principle"] *Consistency result.* If κ is supercompact and κ Cohen or random reals are added, the answer is negative in the resulting model *(Balogh)*. It is not yet known whether a negative answer is equiconsistent with ZFC.

M4. *(Nyikos)* Is every normal manifold collectionwise normal? [Yes if cMEA] *Consistency result.* No if $\diamondsuit^+$ *(M. E. Rudin)*. It is still not known whether an affirmative answer is possible without large cardinals.

Volume 6.

L4. *(van Douwen)* Must every locally compact Hausdorff topological group contain a dyadic neighborhood of the identity? *Solution.* Yes *(Uspensky)*.

Volume 7.

P19. *(Nyikos)* Is there a chain of clopen subsets of ω^* of uncountable cofinality whose union is regular open? [Yes if $\mathfrak{p} > \omega_1$ or $\mathfrak{b} = \mathfrak{d}$ or in any model obtained by adding uncountably many Cohen reals.] *Solution.* No is also consistent, by a modification of Miller's rational forcing *(Dow and Steprans)*.

P20. *(van Douwen)* Can one find in ZFC a point p of ω^* such that $\beta(\omega^*-\{p\}) \neq \omega^*$ or, better yet, $\zeta(\omega^*-\{p\}) \neq \omega^*$? It is known that p is as required if it has a local base of cardinality ω_1, and that it is consistent for there to be $p \in \omega^*$ with $\beta(\omega^*-\{p\}) = \omega^*$. *Solution.* No. If PFA, $\beta(\omega^*-\{p\}) = \omega^*$ for all $p \in \omega^*$ *(van Douwen, Kunen, and van Mill)* and this is also true in any model where at least as many Cohen reals are added as there are reals in the ground model *(Malyhin)*.

[*Editor's note.* Eric van Douwen apparently had PFA in mind when he wrote ". . . it is consistent for there to be $p \in \omega^*$ . . ." in posing the problem; at that time it was not yet known that PFA implied every point of the remainder had the stated property. This leaves unsolved Malyhin's problem of whether it is consistent for some points to have the property and others not.]

Volume 8.

C45. *(van Douwen)* Is there a compact Fréchet-Urysohn space with a pseudocompact noncompact subspace? *Solution.* Yes, there is even a Talagrand compact space X with a point p such that $X = \beta(X-\{p\})$ *(Reznichenko)*.

Volume 9.

A19. *(van Douwen)* Is a first countable T_1 space normal if every two disjoint closed sets of size $\leq c$ can be put into disjoint open sets? *Solution.* There exists a counterexample unless there is a proper class of measurable cardinals in an inner model, but if the consistency of a supercompact cardinal is assumed, then an affirmative answer is consistent *(I. Juhász)*.