TOPOLOGY PROCEEDINGS

Volume 12, 1987

Pages 75–83

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

In functional analysis, C(X) is a beautiful space, where X = [0,1]. Riesz gave a well-known conclusion--The dual of C(X) is the space of all finite signed measures on X with the norm defined by $||\nu|| = |\nu|(X)$. If X is a compact and Hausdorff space, the same result can be obtained. Thompson [1] first introduced the concept of S-closed spaces. References [2-4] studied a series of topological properties of S-closed spaces. In this paper, a measure on S-closed spaces with certain properties is constructed. Some S-closed spaces are neither compact nor Hausdorff, but some interesting results can still be obtained. For example, if X is a S-closed space, then to each bounded linear functional F on C(X), the set of all continuous real-valued functions on X, there corresponds exactly one finite signed F-S measure ν on X such that $F(f) = \int f d\nu$, for each $f \in C(X)$ and $||F|| = |\nu|(X)$.

Let X be a topological space; a set $P \subset X$ is called a regular closed set of X, if $P = P^{o-}$, where o and - are the interior and the closure operations on X; a set $Q \subset X$ is called a regular open set of X, if $Q = Q^{-o}$. A topological space X is said to be S-closed if every cover for X, consisting of regular closed sets, has a finite subcover.

Example 1. Let $S = \{x: 0 < x < 1\}$ be the open unit interval. $\tau = \{\phi\} \cup \{X \setminus A: A \subset X \text{ and } |A| \leq \omega_0\}$. Then $X = (S,\tau)$ is a topological space.

Let $A \subset X$. If A is countable, then $A^o = \phi$; and if A is uncountable, then $A^- = X$. Whence there are only two regular closed sets in X. It is not hard to see that X is an S-closed T_1 space, but not a Hausdorff space; therefore, not a compact space either.

Let X be a topological space; $A \subset X$ is said to be an S-closed set of X if every cover of regular closed sets in X for A has a finite subcover.

Proposition 1. *The finite union of* S-*closed sets of a topological space is* S-*closed.*

The proof is straightforward and is omitted.

Proposition 2. *If* P *is a regular closed set of an* S-*closed space* X, *then* P *is* S-*closed.*

Proof. Let $\{U_t : t \in T\}$ be a family of regular closed sets of X, which covers P. That is

$$\cap\{X - U_t : t \in T\} \subset X - P \subset (X - P)^-.$$

It follows from Theorem 4 in [3] that there exists a finite subfamily $\{X - U_{t_1},\cdots,X - U_{t_n}\}$ such that

$$\cap_{t=1}^{n}(X - U_{t_i}) \subset (X - P)^-.$$

From that P is a regular closed set it follows that $(X - P)^{-o} = X - P$ and that

$$[\cap_{i=1}^{n}(X - U_{t_i})]^o = \cap_{i=1}^{n}(X - U_{t_i}) \subset (X - P)^{-o} = X - P.$$

This implies that $P \subset \cup_{i=1}^{n} U_{t_i}$.

Proposition 3. If g: X → Y *is a continuous mapping from an* S-*closed space* X *into a metric space* Y, *then* g(X) *is a bounded set of* Y.

Proof. For every $x \in X$, choose a unit open ball $V_x = B(g(x),1)$ of $g(x)$. The continuity of g implies that $[g^{-1}(V_x)]^-$ is regular closed in X, and

$$\cup\{[g^{-1}(V_x)]^- : x \in X\} \supset X.$$

Since X is an S-closed space, there exists a finite family $\{[g^{-1}(V_{x_i})]^-: i = 1,2,\cdots,n\}$ such that

$$\cup_{i=1}^{n}[g^{-1}(V_{x_i})]^- \supset X.$$

It follows from the continuity of g that

$$\cup_{i=1}^{n}V_{x_i}^- = [\cup_{i=1}^{n} g \circ g^{-1}(V_{x_i})]^- \supset g(X).$$

This implies that g(X) is bounded.

A topological space X is called a locally S-closed space if for every $x \in X$, there exists a neighborhood U_x of the point x such that U_x^- is contained in an S-closed set of X.

Proposition 4. Every S-*closed set of a* T_1^* *space* X *is closed.*

Proof. Let A be an S-closed set of X and let p be a point of $X \setminus A$. For every $x \in X\setminus\{p\}$, there exists a regular open neighborhood U_x of the point p such that $x \notin U_x$ and that $\cap\{U_x: x \in X\setminus\{p\}\} = \{p\}$. Hence

$$X - \{p\} = \cup\{X\setminus U_x: x \in X\setminus\{p\}\} \supset A.$$

As A is an S-closed set of X, there exists a finite family $\{X\setminus U_{x_1}, X\setminus U_{x_2}, \cdots, X\setminus U_{x_k}\}$ such that

$$\cup_{j=1}^{k}(X\setminus U_{x_j}) \supset A.$$

Take $U(p) = \cap_{j=1}^{k} U_{x_j}$. Hence $U(p) \cap A = \phi$. That is $U(p) \subset X\setminus A$.

Corollary. Every S-*closed set of a Hausdorff space is closed.*

Proposition 5. *Let* A *be an* S-*closed set of a topological space* X. *If* $G \subset A$ *and* G *is regular open in* X, *then* G *is* S-*closed in* X.

Proof. Let $\{U_s^- : s \in S\}$ be a family of regular closed sets of X which covers G. Then $\{U_s^- : s \in S\} \cup \{X\setminus G\}$ is a cover of A of regular closed sets. Since A is S-closed in X, there exists a finite subcover $\{U_{s_1}^-, U_{s_2}^-, \cdots, U_{s_n}^-\} \cup \{X\setminus G\}$ for the set A. Hence $\{U_{s_1}^-, U_{s_2}^-, \cdots, U_{s_n}^-\}$ is a finite subcover for G.

2. The Measure on S-Closed Spaces

Lemma 1. *Let* X *be a locally* S-*closed* T_1^* *space. Then for any* S-*closed set* $A \subsetneqq X$ *there exists a both closed and open set* $U \subsetneqq X$ *such that* $A \subset U$ *and* U *is contained in an* S-*closed set of* X.

Proof. For every $x \in A$, choose an open neighborhood V_x of x and an S-closed set W_x of X such that $V_x^- \subset W_x$. Pick a point $p \in X\setminus A$ and a regular open neighborhood U_x of x such that $p \notin U_x$. Let $Y_x = (V_x \cap U_x)^{-o}$. Then Y_x is regular open in X with $p \notin Y_x \subset Y_x^- \subset W_x$. So by Proposition 5, Y_x is S-closed in X. By Proposition 4, Y_x is closed in

X. Thus $\{Y_x: x \in A\}$ is a family of regular closed sets which covers A. From that A is S-closed in X it follows that there exists a finite subcover $\{Y_{x_i}: i = 1,2,\cdots,n\}$ for A. Then $U = \cup_{i=1}^{n} Y_{x_i} \supset A$ is closed, open and S-closed in X.

Let X be a topological space. Take C(X) to indicate the family of all real-valued continuous functions on X. And define

$$C_0(X) = \{f \in C(X): \text{there exists an S-closed set } A \text{ of } X \text{ such that } f(x) \neq 0 \text{ implies } x \in A\}.$$

The class of F - S sets is defined to be the smallest σ-algebra B of subsets of X such that functions in $C_0(X)$ are measurable with respect to B. A measure μ is called an F - S measure on X, if its domain of definition is the σ-algebra B of F - S sets, and $\mu(A) < \infty$ for each S-closed set A in B.

Lemma 2. *If* X *is a topological space, then* $C_0(X)$ *is a vector lattice.*

Proof. It suffices to show that $\alpha f + \beta g$, $f \vee g$ and $f \wedge g$ belong to $C_0(X)$, whenever $f,g \in C_0(X)$ and $\alpha,\beta \in R$, the set of all real numbers. Since $\{x \in X: (\alpha f + \beta g)(x) \neq 0\} \subset \{x \in X: f(x) \neq 0\} \cup \{x \in X: g(x) \neq 0\}$. It follows from Proposition 1 that $\alpha f + \beta g \in C_0(X)$. For $f \wedge g = f + g - (f \vee g)$ and $f \vee g = (f - g) \vee 0 + g$, we only need to prove that if $f \in C_0(X)$ then $f \vee 0 \in C_0(X)$. Indeed, $f \vee 0$ is continuous and $\{x: f(x) \neq 0\} \supset \{x: (f \vee 0)(x) \neq 0\}$. Hence, $f \in C_0(X)$ implies $f \vee 0 \in C_0(X)$.

Theorem 1. *Let* X *be a locally* S-*closed* T_1^* *space,* I *a positive linear functional on the set* $C_0(X)$. *Then there is an* F - S *measure* μ *such that for each* $f \in C_0(X)$, $I(f) = \int f d\mu$.

Proof. The set $C_0(X)$ is a vector lattice by Lemma 2. Now we show that I is a Daniell integral on $C_0(X)$ (see [5]). To see this end, let $\zeta \in C_0(X)$ and $\langle \zeta_n \rangle$ be an increasing sequence of functions in $C_0(X)$ such that $\zeta \le \lim \zeta_n$. We may assume that ζ and each ζ_n are non-negative. Take $K = \{x \in X: \zeta(x) \neq 0\}$. Then K^- is S-closed in X. In fact, since $\zeta \in C_0(X)$, there exists an S-closed set G of X such that $G \supset K^-$. Proposition 5 implies that the regular open set K^{-o} is S-closed in X. So, Proposition 4 implies that K^{-o} is closed. That is $K^{-o} = K^-$. So K^- is S-closed in X.

Take a non-negative $g \in C_0(X)$ such that $g(x) = 1$, for each $x \in K^-$. By Lemma 1, this can be done.

For any given $\varepsilon > 0$, the set K^- is covered by regular sets $\{O_n^-: n = 1,2,\cdots\}$, where $O_n = \{x \in X: \zeta(x) - \varepsilon g(x) < \zeta_n(x)\}$. Since K^- is S-closed in X, and O_n^-'s are increasing, there must be an N such that $K^- \subset O_N$. Hence $\zeta - \varepsilon g < \zeta_N$ on K. Since $\zeta \equiv 0$ outside K, $\zeta - \varepsilon g \le \zeta_N$ holds everywhere. So

$$I(\zeta) - \varepsilon I(g) \le I(\zeta_N) \le \lim I(\zeta_n).$$

Since ε was arbitrary and $I(g) < \infty$, it must be that

$$I(\zeta) \le \lim I(\zeta_n).$$

Thus I is a Daniell integral.

It follows from [5] Stone Theorem that there is a measure μ defined on the class B of F - S sets such that for each f in $C_0(X)$,

$$I(f) = \int f d\mu.$$

It remains only to show that if K is an S-closed set in B, then $\mu(K) < \infty$. In fact, from Lemma 1 there exists $h \in C_0(X)$ such that $h(x) = 1$, for each $x \in K$, then $\mu(K) \leq \int h d\mu = I(h) < \infty$.

Theorem 2. *If* X *is an* S-*closed space and* I *a positive linear functional on* C(X), *then there is a unique* F - S *measure* μ *on* X *such that* $I(f) = \int f d\mu$, *for each* $f \in C(X)$.

Proof. It follows from the proof of Theorem 1 that it suffices to show that μ is unique. Because $1 \in C(X)$, Theorem 20 in [5] implies the uniqueness.

Theorem 3. *Let* X *be an* S-*closed space. Then to each bounded linear functional* F *on* C(X), *there corresponds a unique finite signed* F - S *measure* ν *on* X *such that*

$$F(f) = \int f d\nu,$$

for each $f \in C(X)$. *Moreover*, $||F|| = |\nu|(X)$.

Proof. By Proposition 3, C(X) is a normed linear space with the norm $||\cdot||$ defined by $||f|| = \sup|f(x)|$, for each $f \in C(X)$.

Let $F = F_+ - F_-$ be defined as in [5] Proposition 23. Then by Theorem 2, there are finite F - S measures μ_1 and μ_2 such that

$$F_+(f) = \int f d\mu_1 \text{ and } F_-(f) = \int f d\mu_2,$$

for each $f \in C(X)$.

Set $\nu = \mu_1 - \mu_2$; then ν is a finite signed F - S measure, and $F(f) = \int f d\nu$, for each $f \in C(X)$. Now, for each $f \in C(X)$, $|F(f)| \leq \int |f| d|\nu| \leq ||f|| \, |\nu|(X)$. Hence, $||F|| \leq |\nu|(X)$. But

$$|\nu|(X) \leq \mu_1(X) + \mu_2(X)$$
$$= F_+(1) + F_-(1) = ||F||.$$

Thus, $||F|| = |\nu|(X)$.

To show the uniqueness of ν, let ν_1 and ν_2 be two finite signed F - S measures on X such that

$$\int f d\mu_i = F(f), \ i = 1,2.$$

Then $\lambda = \nu_1 - \nu_2$ would be a finite signed F - S measure on X such that $\int f d\lambda = 0$, for each $f \in C(X)$. Let $\lambda = \lambda^+ - \lambda^-$ be the Jordan decomposition of λ. Then the integration with respect to λ^+ gives the same positive linear functional on C(X) as that given by λ^-; and by Theorem 2, it must be $\lambda^+ = \lambda^-$. Hence $\lambda = 0$ and $\nu_1 = \nu_2$.

Theorem 4. *Let* X *be an* S-*closed space. Then to each bounded functional* F *on* C(X) *and* $0 < p < +\infty$, *there corresponds one finite* F - S *measure* U *on* X *such that for each* $f \in C(X)$, $F(f) = (\int |f|^p dU)^{1/p}$ *if and only if there exists a unique positive linear functional* I *on* C(X) *such that* $F^p(f) = I(|f|^p)$, *for each* $f \in C(X)$. *Moreover,* $U(X) = F^p(1)$.

The proof is straightforward and is omitted.

We conclude this paper with a problem: Let X be an S-closed T_1 space; then the dual of C(X) is (isometrically isomorphic to) the space of all finite signed F - S measures on X with the norm defined by $||\nu|| = |\nu|(X)$.

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