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CONVERGENCE IN THE BOX PRODUCT OF COUNTABLY MANY METRIC SPACES[1]

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0. Introduction and Theorems

Notation. Suppose that for each $i \in \omega$, X_i is a metric space, and let X be the box product $\square_{i\in\omega} X_i$ (the point-set of X is $\Pi_{i\in\omega} X_i$ and a typical base element is $\Pi_{i\in\omega} U_i$ where each U_i is a proper open set in X_i). Is X normal or paracompact? This problem was originally posed by A. H. Stone over twenty years ago and remains in large part unsolved (see the survey articles of E. K. van Douwen, [vD], and S. Williams, [W]). The first positive consistency result was obtained by M. E. Rudin in [R]: the Continuum Hypothesis implies that if each X_i is locally compact and σ-compact, then X is paracompact. In [K], K. Kunen generalized both the factor spaces and the method of proof. At the heart of the Rudin-Kunen strategy is the following decomposition of X.

Define two points in X to be equivalent if they disagree at most a finite number of times, and for each p, let $E(p)$ be the equivalence class to which p belongs. For each $p \in X$ and each $i \in \omega$, let $F_i(p) = \{q \in X: (\forall j > i)(q_j = p_j)\}$. Then $E(p) = \cup_{i\in\omega} F_i(p)$. Let $\nabla_{i\in\omega} X_i$ be the quotient space on X induced by E (sometimes called the

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nabla product), and let σ be the quotient map, so $\sigma(p)$ denotes $E(p)$ as a point in the quotient space.

The purpose of this paper is to show how convergence in X, across and inside the fibers of the Rudin-Kunen decomposition, depends upon local compactness in the factor spaces.

Proposition. Suppose $p \in X$ and $Y \subseteq X$. Then (1) p is a limit point of a countable subset of Y iff p is a limit point of $Y \cap E(p)$; and (2) there is a sequence in $Y \setminus \{p\}$ converging to p iff there exists $i \in \omega$ such that p is a limit point of $Y \cap F_i(p)$.

Theorem 1. *Suppose* X_i *is locally compact for each* $i \in \omega$, *and* C *is a closed subset of* X. *Then for every limit point* p *of* C, *there is a sequence in* $C \setminus \{p\}$ *converging to* p.

So in light of the Proposition, if C is closed and $C \cap E(p) = \{p\}$, then p is an isolated point of C.

Theorem 2. *Suppose that for each* $i \in \omega$, p_i *is a point in* X_i *that does not have a compact neighborhood. Then there is a closed set* $C \subseteq X$ *containing* p *such that:*

(1) $C \cap E(p) = \{p\}$;

(2) p *is a limit point of* C;

(3) p *is the only limit point of* C.

So in light of the Proposition, there is a closed set C where p is isolated from each countable subset but is nevertheless a limit point.

1. Lemmas

Notes. Lemma 1 below is in [K], where it is attributed to Rudin. We include a proof for the convenience of the reader. Lemma 2 is a generalization of the fact that σ is closed if each X_i is compact, which is in [K] and implicitly in [R].

Lemma 1. *The quotient space,* $\vee_{i\in\omega}X_i$, *is* ω_1*-open (i.e., the intersection of every countable collection of open sets is open).*

Proof. First observe that σ is an open map (so $\{\sigma(U)\colon U$ is open in $X\}$ is a base for the quotient space), and for all $U = \Pi_{i\in\omega}U_i$ and $V = \Pi_{i\in\omega}V_i$, $\sigma(U) \subseteq \sigma(V)$ iff $(\exists j \in \omega)(\forall i > j)(U_i \subseteq V_i)$. Suppose $p \in X$, and for each $n \in \omega$, $U(n)$ is a basic open set in X containing p. Define $V = \Pi_{i\in\omega}V_i$ by $V_i = \cap\{U(n)_i\colon n \le i\}$. Then $p \in V$ and for each n, $\sigma(V) \subseteq \sigma(U(n))$.

Lemma 2. *Suppose that for each* $i \in \omega$, $K_i \subseteq X_i$ *is compact, and let* $K = \Pi_{i\in\omega}K_i$. *Then for each closed set* $A \subseteq K$, $\sigma(A)$ *is closed in the quotient space. We will prove and subsequently use the following limit point version of the statement that* $\sigma|K$ *is a closed map. (Restricting* σ *to* K *is not to be confused with changing the points of the quotient space by intersecting the equivalence classes with* K*; the quotient map and space remain intact.) Suppose* $p \in K$ *and* $A \subseteq K$ *where* $A \cap E(p) = \emptyset$ *and* $\sigma(p)$ *is a limit point of* $\sigma(A)$ *in the quotient space. Then there exists* $q \in K \cap E(p)$ *where* q *is a limit point of* A *in* X.

Proof. Assume our conclusion does not follow and let S be a basic open cover of $K \cap E(p)$ where $\cup S \cap A = \emptyset$. By the compactness of each K_i, we can choose a sequence T of finite subcollections of S where for each n, $K \cap F_n(p) \subseteq \cup T(n)$. We also assume that each set in $T(n)$ intersects $F_n(p)$; equivalently, for each $U \in T(n)$ and each $i > n$, $p_i \in U_i$. Define $V = \Pi_{i\in\omega} V_i$ by $V_0 = X_0$ and for $i > 0$ by $V_i = \cap\{U_i\colon (\exists n < i)(U \in T(n))\}$. Then $p \in V$ and $\cup\{K \cap E(q)\colon q \in V\}$ is contained in $\cup S$ and is therefore disjoint from A. Since $A \subseteq K$, $\cup\{E(q)\colon q \in V\} \cap A = \emptyset$, so $\sigma(V) \cap \sigma(A) = \emptyset$ in the quotient space. This result (and the fact that σ is an open map) contradicts the hypothesis.

2. Proofs of the Proposition and the Theorems

Proof of the Proposition. Part (1)--Sufficiency. For each $i \in \omega$, let $\{U_i(n)\colon n \in \omega\}$ be a local base at p_i. For each $j \in \omega$ and each $s\colon \omega \to \omega$, let $G_j(s) = \Pi_{i\in\omega} U_i(s_i) \cap F_j(p)$. Let Z be a countable subset of Y such that for each j and s, if $Y \cap G_j(s) \neq \emptyset$, then $Z \cap G_j(s) \neq \emptyset$. We can take Z to be countable since $G_j(s) = G_j(t)$ if s and t agree on $[0,j]$.

We claim that p is a limit point of Z. Let $s\colon \omega \to \omega$. By hypothesis, there exists $q \in \Pi_{i\in\omega} U_i(s_i) \cap Y \cap E(p)$. Let $j \in \omega$ such that $q \in F_j(p)$. Then $q \in G_j(s)$, so $Z \cap G_j(s) \neq \emptyset$.

Part (1)--Necessity. Suppose $q\colon \omega \to Y \setminus E(p)$. Then for each $n \in \omega$, there exists $k(n) \geq n$ such that $q(n)_{k(n)} \neq p_{k(n)}$. Since the map k is finite to one, we can choose an open set U about p so that for each $i \in \mathcal{R}m\, k$

(the image of k) and each $n \in k^{-1}(i)$, $q(n)_i \notin U_i$. So p is not a limit point of $\mathcal{I}m$ q.

Part (2)--Sufficiency is immediate. For Part (2)--Necessity, suppose $q: \omega \to Y$ where there does not exist $i \in \omega$ with $\mathcal{I}m\ q \subseteq F_i(p)$. Then we can choose a subsequence $q \circ \ell$ where for each $n \in \omega$, there exists $k(n) \geq n$ such that $q(\ell(n))_{k(n)} \neq p_{k(n)}$. As in the proof of Part (1)--Necessity, the finite to one property of k implies the existence of an open set U about p that excludes each term of the subsequence. So q does not converge to p.

Proof of Theorem 1. Suppose $C \subseteq X$ is closed and $p \in C$ where p is a limit point of C.

Claim 1. We first show that p is a limit point of $C \cap E(p)$. Assume otherwise and by the regularity of X and the local compactness of the factors, choose a basic open set $U = \Pi_{i\in\omega}U_i$ where $p \in U$, $\overline{U} \cap C \cap E(p) = \{p\}$, and for each $i \in \omega$, $\overline{U}_i$ is compact (as usual the horizontal bar denotes the closure operator in the appropriate space). For each $(i,j) \in \omega \times \omega$, where j is nonzero, let $A(i,j) = \{q \in U \cap C$: the distance between q_i and p_i is at least $\frac{1}{j}\}$. Note that by the choice of U, each A(i,j) is disjoint from E(p). Since p is a limit point of $B = \cup\{A(i,j): (i,j) \in \omega \times \omega, j \neq 0\}$, $\sigma(p)$ is a limit point of $\sigma(B)$ in the quotient space. By Lemma 1, there exist i_0 and j_0 such that $\sigma(p)$ is a limit point of $\sigma(A(i_0,j_0))$. By Lemma 2, there is a point in $\overline{U} \cap C \cap E(p)$ that is a limit point of $A(i_0,j_0)$ in X. By the definition of $A(i_0,j_0)$, this point is necessarily different from p. This result contradicts the choice of U.

Claim 2. There exists $i \in \omega$ such that p is a limit point of $C \cap F_i(p)$. We assume otherwise and by an inductive process choose the projections of a basic open set $U = \Pi_{i\in\omega}U_i$ where $p \in U$ and $U \cap C \cap E(p) = \{p\}$. The existence of such an open set contradicts Claim 1.

Since p is not a limit point of $C \cap F_0(p)$, we can choose an open set $U_0 \subseteq X_0$ containing p_0 such that $\bar{U}_0$ is compact and $(\bar{U}_0 \times \Pi_{i>0}X_i) \cap C \cap F_0(p) = \{p\}$. We can now choose an open set $U_1 \subseteq X_1$ containing p_1 such that $\bar{U}_1$ is compact and $(\bar{U}_0 \times \bar{U}_1 \times \Pi_{i>1}X_i) \cap C \cap F_1(p) = \{p\}$. Otherwise, there is a sequence $(q(n))$ in $(\bar{U}_0 \times \Pi_{i>0}X_i) \cap C \cap F_1(p)$ where $(q(n)_1)$ is a sequence in $X_1 \setminus \{p_1\}$ converging to p_1; but then $(q(n))$ has a convergent subsequence, and this contradicts either the assumption (if the subsequence converges to p) or the choice of U_0. Continue this process until for each $i \in \omega$, U_i has been chosen using the compactness of $\bar{U}_0 \times \cdots \times \bar{U}_{i-1}$.

Proof of Theorem 2. For each $i \in \omega$, let $\{U_i(n): n \in \omega\}$ be a nested local base about p_i, and let $\psi_i: \omega \times \omega \to X_i$ be a 1-1 function, where for each $m \in \omega$, $\{\psi_i(m,n): n \in \omega\}$ is closed and discrete in X_i and is a subset of $U_i(m)\setminus\overline{U_i(m+1)}$.

We first define a closed set C' that satisfies (1) and (2) in the conclusion of the theorem. We then define a closed set C where $C \subseteq C'$ and C satisfies all three properties. Let $\phi: {}^{\omega}\omega \to X$ where $\phi(s)_i = \psi_i(s_i, s_{i+1})$, and let $C' = \overline{\mathcal{I}m\ \phi}$ (the closure of the image of ϕ). Note that p is a limit point of C'.

We claim that $C' \cap E(p) = \{p\}$. Let ℓ be a limit point of $\mathcal{I}m\ \phi$ distinct from p. Let $i \in \omega$ be the least index with $\ell_i \neq p_i$. Since p_i is the only limit point in X_i of the i-th projection of C', there are integers s_i and s_{i+1} with $\ell_i = \psi_i(s_i,s_{i+1})$. Then ℓ is a limit point in X of $\{q \in \mathcal{I}m\ \phi: q_i = \psi_i(s_i,s_{i+1})\}$. For each q in this collection, $q_{i+1} \in \{\psi_{i+1}(s_{i+1},j): j \in \omega\}$. This set of values does not have any limit points in X_{i+1}, so there is an integer s_{i+2} with $\ell_{i+1} = \psi_{i+1}(s_{i+1},s_{i+2})$. We now have that ℓ is a limit point in X of $\{q \in \mathcal{I}m\ \phi: q_i = \psi_i(s_i,s_{i+1})$ & $q_{i+1} = \psi_{i+1}(s_{i+1},s_{i+2})\}$. Continuing this process generates the tail end of a function $s \in {}^{\omega}\omega$ where ℓ agrees with $\phi(s)$ on indices $j \geq i$. So $\ell \notin E(p)$. Moreover, we have a characterization of the limit set of $\mathcal{I}m\ \phi$: $\{\ell \in X: \ell = p$, or, $(\exists s \in {}^{\omega}\omega)\ (\exists i \in \omega)\ (\forall j \in \omega)\ ((j \leq i \rightarrow \ell_j = p_j)$ & $(j > i \rightarrow \ell_j = \psi_j(s_j,s_{j+1})))\}$. This characterization indicates the adjustment that must be made in the domain of ϕ to eliminate all limit points except p.

Let $D \subseteq {}^{\omega}\omega$ such that (1) D is strictly dominant (i.e., $(\forall s \in {}^{\omega}\omega)\ (\exists t \in D)\ (\forall i \in \omega)\ (s_i \leq t_i)$), and (2) each pair of distinct functions in D disagree on an infinite number of indices. Let $C = \overline{\mathcal{I}m(\phi|D)}$.

Remark. The referee pointed out the following variation on the construction used to prove Theorem 2. Let C be the closure of $\{q \in E(p): (\exists m,n \in \omega$ with $m \neq 0)\ (q_0 = \psi_0(m,n)$ & $q_m = \psi_m(n,0)$ & $(\forall i \neq 0,m)\ (q_i = p_i))\}$. Then C is countable and p is the unique limit point of C, but p is not the limit of a sequence in $C\setminus\{p\}$. To see this, first note that each point in $C \setminus \{p\}$ is different from p in the first coordinate. Suppose $q: \omega \to C \setminus \{p\}$ converges to p. Then there exists $i \in \omega$ such that $\mathcal{I}m\ q \subseteq F_i(p)$. This implies that $\{m \in \omega: (\exists k,n \in \omega)\ (q(k)_0 = \psi_0(m,n))\}$ is bounded above by i, so p_0 is isolated from $\{q(k)_0: k \in \omega\}$, in contradiction to the choice of q. This proves the converse of Theorem 1 since the construction of C in this example requires the failure of local compactness in only one coordinate.

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