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0. Introduction

Recently there has been some interest in fractal functions, i.e. functions whose graph is a fractal set, especially in the ones which are generated by iterating a given class of continuous mappings. These mappings are defined via a set of interpolation or data points and the graph of the so-generated continuous but in general nowhere differentiable function passes through this set of interpolation points. Two-dimensional fractal functions of this type (by this we mean fractal functions whose graph is a subset of $\mathbf{R}^2$) were first introduced in [Bl] and are used to model natural objects which exhibit some kind of geometric self-similarity, such as mountain ranges, rivers and clouds.

A calculus of 2-dimensional fractal functions was developed in [BHa] and a formula for the (fractal) dimension for the graphs of a special class of fractal functions was derived.

The question of the connection to dynamical systems and in particular to the Lyapunov dimension commenced in [HM]. A more general dimension formula was also presented. An extended formula containing all the previous cases is derived in [BEHM] and its generalization to n-dimensional fractal functions is given in [M2].

The investigation of connections between fractal functions generated by a deterministic algorithm using methods from the theory of iterated function systems and the associated dynamical system led to the discovery of a new class of fractal functions, the so-called "hidden-variable fractal functions." This new class arises from a relation between an attractor for an iterated function system and its associated code space. This relation, provided certain conditions on the attractor and its defining maps hold, defines the graph of a continuous fractal function having the same dimension as the attractor. The projections of this function onto R^2 yields then the hidden-variable fractal functions, objects that depend continuously on all the "hidden" variables. Formulas for the dimension of these new fractal functions, a relation to the dimension of the embedding space of the original attractor and connections to the associated dynamical system, although only briefly, were considered in [BEHM] and in more detail in [M1].

We felt the need for combining all these results and for showing their common origin as representations of an associated dynamical system. The former has partially been done in [BEHM] but without reference to the underlying dynamical system. Barnsley considered parts of the latter in [B2] but the dynamics of his system is different from ours.

We also will show that for our fractal functions the Lyapunov dimension of the associated dynamical system equals one plus the fractal dimension of the graph of the fractal function.

The structure of this paper is as follows. In section 1 we introduce iterated function systems, define the associated dynamical system and some of its characteristics, and give an example and some illustrations. Section 2 is then devoted to the presentation of the results.

At this point I would like to mention Michael F. Barnsley, Jeff Geronimo and Douglas Hardin. The collaboration with them in the past has proved to be very fruitful and I am thankful for their advice and their helpful suggestions.

1. Definitions and Preliminaries

Let X be a compact metric space and $\mathbf{w}:=\{w_i: i = 1,\ldots,n\}$, $n \in \mathbf{N}$, a collection of Borel measurable functions $w_i: X \to X$. Let $\mathbf{p}:=\{p_i: i = 1,\ldots,n\}$ be a set of non-zero probabilities, i.e. $p_i \in (0,1)$ and $\Sigma p_i = 1$.

Definition 1. The pair $(X,\mathbf{w})$ is called an *iterated function system* (IFS) if $\exists \mathbf{p} = \{p_i: i = 1,\ldots,n\}$ such that the operator T defined by

$$(Tf)(x) := \sum_i p_i (f \cdot w_i)(x), \quad \forall f \in C^0(X)$$

maps $C^0(X)$ into itself.
Note that if $w_i \in C^0(X)$, $\forall i = 1,\ldots,n$, then $(X,\mathbf{w})$ is an IFS for any set of probabilities.

Convention. From now on we assume that all $w_i \in C^0(X)$.

$(X,\mathbf{w})$ is called a hyperbolic IFS (HIFS) if $\exists s \in [0,1)$ such that

$$\frac{d(w_i(x_1),w_i(x_2))}{d(x_1,x_2)} \leq s \quad \forall i, \quad \forall x_1,x_2 \in X$$

(here d denotes the metric on X).

Associated with every IFS is an invariant measure, called the **p**-balanced measure μ, satisfying

$$\mu E = \sum_i p_i \mu(w_i^{-1}E) \qquad \forall E \in B(X)$$

or
$$\int_X f d\mu = \sum_i p_i \int_X f \cdot w_i d\mu \qquad \forall f \in C^0(X)$$

(B(X) denotes the Borel sets of X).

If $\exists A \in P(X)$ so that

$$A = \cup_{i=1}^n w_i A$$

then A is called an attractor for the IFS (X,**w**).

We note that $A \in K(X)$ and that $A = \operatorname{supp} \mu$. If furthermore (X,**w**) is a HIFS then the attractor A is unique. It can be shown that A can be obtained as follows: Let $x_0 \in X$, define $x_m := \mathbf{w}(x_{m-1})$, $m \in \mathbf{N}$, where **w** is interpreted as a set-valued map $\mathbf{w}: H(X) \to H(X)$, $\mathbf{w}(S) := \cup w_i(S)$ $\forall S \in H(X)$. Then

$$A = \lim_{m\to\infty} \mathbf{w}^m(x_0)$$

and A is independent of x_0.

It follows from the above characterization that A can be generated by iterating a starting point x_0 using the map w_i with probability p_i to obtain $x_1 = w_i(x_0)$. In general we have then after m iterations

$$x_m = w_{\omega_1 \ldots \omega_m}(x_0)$$

where
$$w_{\omega_1 \ldots \omega_m} := w_{\omega_m} \cdot \ldots \cdot w_{\omega_1} \text{ and } \omega_j \in \{1,\ldots,n\} \ \forall j$$

If we set $\Omega := \{1,\ldots,n\}^{\mathbf{N}}$ then $x_m = w_\omega(x_0)$ for some $\omega \in \Omega$.

We call Ω the code space associated with the IFS (X,**w**).

With the metric $|\cdot,\cdot|: \Omega^2 \to R_0^+$ defined by

$$|\omega,\tilde{\omega}|: = \sum_{j\geq 1} \frac{|\omega_j - \tilde{\omega}_j|}{(n+1)^j}$$

$(\Omega, |\cdot,\cdot|)$ is a compact metric space homeomorphic to the classical Cantor set. There exists also a surjection $S \in C^0(\Omega, A)$ such that

$$S(\omega) = \lim_{m\to\infty} w_{\omega_1 \cdots \omega_m}(x_0) \quad \text{where } (\omega_1, \ldots, \omega_m) \in \Omega$$

and this limit is uniformly independent of $x_0 \in X$ (for more details and proofs we refer the reader to [BD]).

To better understand the dynamics of the maps w_i we associate a dynamical system with the IFS $(X,\mathbf{w})$ as follows (see also [P]): Let $I: = [0,1] \subset \mathbb{R}$ and denote by m uniform Lebesgue measure on I. Define $M: = X \times I$ and a map $F: M \to M$ by

$$F(x,t): = (w_i(x), h_i(t)) \quad \text{if } (x,t) \in X \times I_i$$

where $I_i: = [p_1 + \ldots + p_{i-1}, p_1 + \ldots + p_i)$, $i = 1,\ldots,n-1$, $I_n: = [p_1 + \ldots + p_{n-1}, 1]$, and where $h_i \in C^0(I)$,

$$h_i(t): = \frac{t - (p_1 + \ldots + p_{i-1})}{p_i} \quad \forall i = 1,\ldots,n.$$

Note that F is piece-wise C^0. We could make F continuous by connecting the components of graph (F) by appropriate C^∞-functions having support on $[p_1 + \ldots + p_i - \varepsilon, p_1 + \ldots + p_i + \varepsilon]$, $\varepsilon > 0$.

F possesses a (strange) attractor A(M) and its associated invariant measure ν is given by $\nu = \mu \times m$. Furthermore

$$\mathrm{proj}_X A(M) = A(X), \text{ the attractor of } (X,\mathbf{w})$$

$$\mathrm{proj}_X \nu = \mu$$

Notice that if $F \in C^0(M)$ then its invariant measure $\tilde{\nu}$ is "close" to ν in the weak*-topology.

The triple $\mathbb{D} = (M,F,\nu)$ is called the dynamical system associated with the IFS (X,w). As an example let us consider $X: = [0,1] \subset \mathbf{R}$ and $\mathbf{w}: = \{w_1,w_2\}$ where $w_i: X \to X$ is defined by

$$w_1(x): = \frac{1}{3}\,x, \quad w_2(x): = \frac{1}{3}\,x + \frac{2}{3}$$

$A(X)$ is then the classical middle-thirds Cantor set. If we choose the probabilities $p_1 = p_2 = 1/2$ then

$$h_1(t) = 2t, \quad h_2(t) = 2t - 1$$

and

$$F(x,t) = \begin{cases} (\frac{1}{3}\,x, 2t) & (x,t) \in X \times [0,\frac{1}{2}] \\ (\frac{1}{3}\,x + \frac{2}{3}, 2t - 1) & (x,t) \in X \times [\frac{1}{2},1] \end{cases}$$

The action of F on $\mathbb{D}$ is depicted in Figure 1. We are interested in the (fractal) dimension or as it is sometimes called the capacity of $A(M)$.

Recall that for a bounded set $S \subset \mathbf{R}^k$ the fractal dimension is defined by

$$\dim(S): = \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{\log \varepsilon^{-1}}$$

where $N(\varepsilon)$ denotes the minimum number of k-dimensional ε-balls needed to cover S. J. Yorke et al. (see for instance [FKYY]) defined another notion of dimension, called the Lyapunov dimension, to further characterize dynamical systems and their attractors. They conjectured that this dimension agrees with the Hausdorff-Besicovitch dimension for "typical" attractors. We showed that this conjecture is false for a wide class of "typical" attractors (see [M1]). For our dynamical systems we will see that there exists a certain set of probabilities $\mathbf{p}^*$ for which the Lyapunov dimension attains a maximum value and this maximum value agrees with the *fractal* dimension.

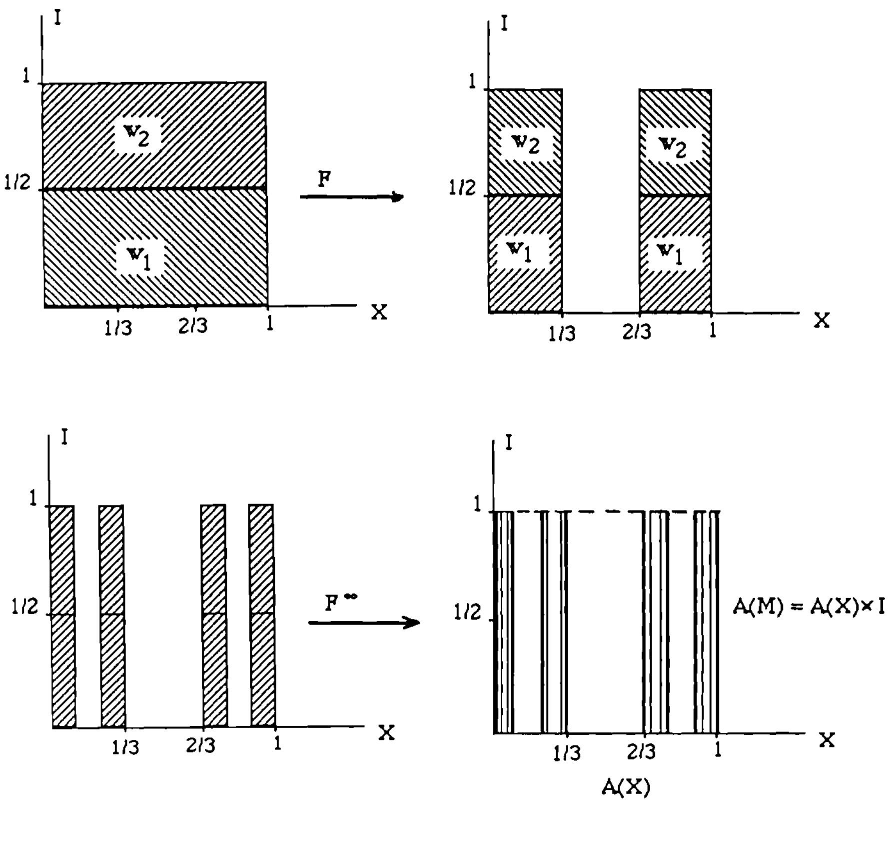

Figure 1. The action of F on M = X × I

Let us state the definition of Lyapunov dimension.

Definition 2. Let $\mathbb{D} = (M,F,\nu)$, M compact k-dimensional manifold, be a dynamical system and let $\lambda_1 \geq \ldots \geq \lambda_m$ be the Lyapunov exponents of F. Let $q: = \max\{j \in \{1,\ldots,m\}: \lambda_1 + \ldots + \lambda_j > 0\}$. If no such q exists then the *Lyapunov dimension* $\Lambda(\nu)$ of ν is defined to be zero. If $1 \leq q < k$ then

$$\Lambda(\nu): = q + \frac{\lambda_1 + \ldots + \lambda_q}{|\lambda_{q+1}|}$$

If $q = k$ then $\Lambda(\nu): = k$.

Remark. For 2-dimensional dynamical systems $\Lambda(\nu)$ = Hausdorff-Besicovitch dimension = fractal dimension. This was shown by L. S. Young (see [Y]).

2. Dynamical Systems and Fractal Functions

We are interested in a special class of IFS's, namely the ones for which $A(X) = \text{graph}(f)$ for some $f \in C^0$.

Continuous functions defined this way will be referred to as *fractal functions* since their graph is in general a fractal set. We will define two classes of IFS's which generate fractal functions.

2.1 Fractal Interpolation Functions

Let $X \in K(\mathbf{R}^k)$, $k \geq 2$, and suppose that $T: = \{(\tau_j,\xi_j) \in \mathbf{R} \times \mathbf{R}^{k-1}: \tau_0 < \ldots < \tau_n, j = 0,1,\ldots,n, n \in \mathbf{N}\}$ is a given set of interpolation points in X. Set $J: = [\tau_0,\tau_n]$. Define maps $w_i: X \to X$ by

$$w_i(\tau,\xi): = (\phi_i(\tau),\psi_i(\tau,\xi)) \quad \forall i = i,\ldots,n$$

where $\phi_i: J \to [\tau_{i-1}, \tau_i]$ is a linear homeomorphism with

$$\phi_i(\tau_0) := \tau_{i-1} \text{ and } \phi_i(\tau_n) := \tau_i \quad \forall i = 1,\ldots,n$$

and $\psi_i: J \times \mathbf{R}^{k-1} \to \mathbf{R}^{k-1}$ is a linear C^0-map with

$$\psi_i(\tau,\cdot) \text{ contractive}$$

$$\psi_i(\cdot,\xi) \in \mathrm{Lip}(\mathbf{R}^{k-1}) \quad \forall i, \forall \xi$$

$$\psi_i(\tau_0,\xi_0) := \xi_{i-1} \text{ and } \psi_i(\tau_n,\xi_n) := \xi_i$$

The maps w_i are then uniquely determined by T together with parameters $0 \le |e_{m,i}| < 1$, $i = 1,\ldots,n$ and $m = 1,\ldots,k-1$ (we refer to the $e_{m,i}$ as the ξ-component scaling factors). If α denotes the constant of contractivity of the ϕ_i and γ the Lipschitz constant of the ψ_i we can define a new metric $\Theta: X^2 \to R_0+$ by setting

$$\Theta(x,\tilde{x}) := d_1(\tau,\tilde{\tau}) + \frac{1-\alpha}{(n+1)\gamma} \sum_m d_2(\xi_m,\tilde{\xi}_m)$$

$$\text{with } \xi = (\xi_m)_{1\le m\le k-1} \text{ and } \tilde{\xi} = (\tilde{\xi}_m)_{1\le m\le k-1}$$

where d_1 and d_2 denote metrics on J and $\mathbf{R}$, respectively.

It is straight-forward to show that (X,Θ) is a compact metric space and that in this new metric $(X,\mathbf{w})$ is a HIFS. Hence $(X,\mathbf{w})$ has a unique attractor $A(X)$.

Proposition 1. $A(X) = \mathrm{graph}(f)$ *where* $f \in C^0(J,\mathbf{R}^{k-1})$, $f(\tau_j) = \xi_j$ $\forall j = 0,1,\ldots,n$ *and* $f(\phi_i(\tau)) = \psi_i(\phi_i^{-1}(\tau), f(\phi_i^{-1}(\tau))$ $\forall i = 1,\ldots,n$.

Proof. Let $\mathbf{F} := \{g \in C^0(J,\mathbf{R}^{k-1}): g(\tau_0) = \xi_0$ and $g(\tau_n) = \xi_n\}$ and let $d(g,h) := \max\{|g(\tau) - h(\tau)|: \tau \in J\}$, $g,h \in \mathbf{F}$. Then $(\mathbf{F},d)$ is a complete metric space.

If we define an operator T on $\mathbf{F}$ by

$$(Tg)(\tau) := \psi_i(\phi_i^{-1}(\tau), g(\phi_i^{-1}(\tau)) \quad \forall \tau \in \phi_i J, \ \forall g \in \mathbf{F}$$

then it is easy to show that T is well-defined, maps $\mathbf{F}$ into itself and is a contraction with the same constant of contractivity as the ψ_i. Hence T has a unique fixed point $f \in \mathbf{F}$. Let G: = graph(f). G is then an attractor for the IFS (X,**w**) and by uniqueness $G \equiv A(X)$. The remaining statements of the theorem follow immediately from the definition of T.

Note. We refer to f as a (k-1)-dimensional fractal interpolation function since graph(f) $\subset \mathbf{R}^{k-1}$ and since it interpolates the points in $\mathcal{T}$.

The associated dynamical system $\mathfrak{D}$ is then given by $M = X \times I \subset \mathbf{R}^{k+1}$ and F is as above. Figure 2 shows the action of F on M in the case k = 2.

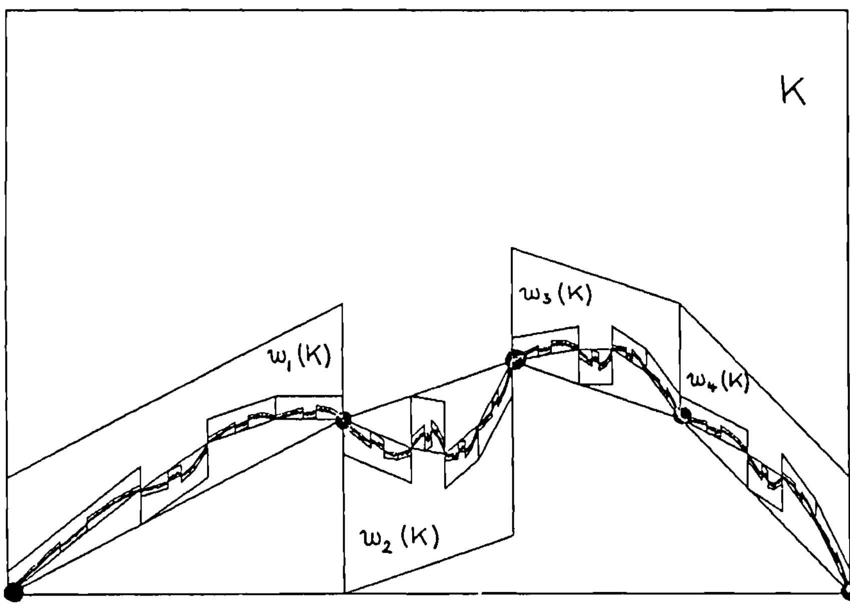

Figure 2. The action of $F|_X$ on M, i.e. the generation of A(X) for n = 4. The interpolation points are indicated by •.

Another attractor $A(X) \subset R^2$ is shown in Figure 3.

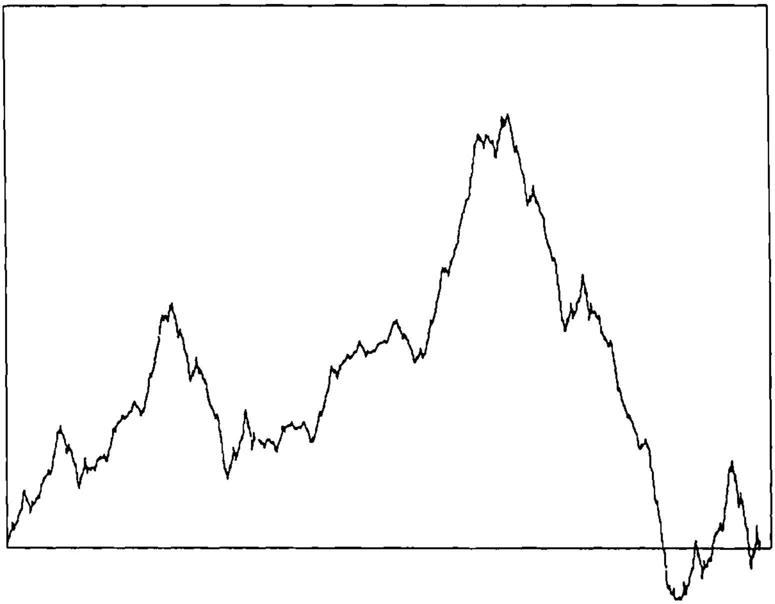

Figure 3. An attractor A(X)

The following theorem gives a formula for the (fractal) dimension of A(X). We won't give the rather lengthy and involved proof here (see [M2]).

Theorem 1. *Let* (K,**w**) *be the* IFS *defined above. Let* A(X) *be the graph of the fractal interpolation function* **f** *generated by* (K,**w**). *Let* $E_i := \Pi_m|e_{m,i}|$. *Suppose that* T *is not co-planar (i.e.* T *is not contained in any hyperplane of* $\mathbf{R}^k$*).*

Let $b_i := \tau_i - \tau_{i-1}$, $\forall i$. *Then if*

a) $\Sigma_i E_i > 1$, $\dim A(X) = d$ *where* d *is the unique positive solution of*

$$\sum_i E_i b_i^{d-k+1} = 1$$

b) $\Sigma_i |e_{m,i}| \leq 1$, $\forall m = 1,\ldots,k-1$, $\dim A(X) = 1$

c) $\Sigma_i E_i \leq 1$ *and*

$\Sigma_i |e_{m,i}| > 1$ *for* $m = 1,\ldots,h$

$\Sigma_i |e_{m,i}| \leq 1$ *for* $m = h+1,\ldots,k-1$ $\quad h = 1,\ldots,k-2$

let $E_i^{(p)} := \Pi_{j\in\pi} |e_{m,i}|$ *with* π *denoting a* π*-tupel of elements of* $\{1,\ldots,h\}$. *Let* $\mathbf{q} := \max\{\pi : \pi \subset \{1,\ldots,h\}\}$ *such that* $\Sigma_i E_i^{(q)} > 1$. *Then* $\dim A(X) = \max\{d^{(q)} : \mathbf{q} \subset \{1,\ldots,h\}\}$ *where* $d^{(q)}$ *is the unique positive solution of*

$$\sum_i E_i b_i^{d-q+1} = 1$$

If $T \subset H^r$ *where* H^r *is a hyperplane of co-dimension* r *of* R^k, $1 \leq r \leq k-1$, *then conclusions* a)-c) *hold with* $k+1$ *replaced by* $k+1-r$. *If* $T \subset H^k$ *then* $\dim A(X) = 1$.

Let us now show that the Lyapunov dimension $\Lambda(v)$ of $\mathfrak{D}$ equals $\dim A(M) = 1 + \dim A(X)$.

First notice that the Lyapunov exponents of F are given by

$$\lambda_1 = -\Sigma p_i \log(p_i) > 0$$

$$\lambda_m = \Sigma p_i \log(|e_{m,i}|) < 0 \qquad \forall m = 1,\ldots,k-1$$

The Lyapunov dimension $\Lambda(v)$ equals then

$$\Lambda(v) = (q+1) - \frac{\sum_i p_i \log\left(\frac{E_i^{(q)}}{p_i}\right)}{\sum_i p_i \log(E_i^{(q+1)})}$$

where $q := \max\{j = 1,\ldots,k : \lambda_1 + \ldots + \lambda_j > 0\}$ and $E_i^{(q)}$ is as in the statement of Theorem 1.

Using methods from calculus it can be shown that there exists a set of probabilities $\mathbf{p}^*$ which maximizes $\Lambda(v)$ and this maximum value Λ^* satisfies

$$\sum_i E_i^{(q)} b_i^{\Lambda^*-1-(k-1)+q} = 1$$

for $\mathcal{T}$ not contained in any hyperplane of dimension $< q$. But this implies that $\dim A(X) = \Lambda^* - 1$.

2.2 Hidden Variable Fractal Functions

Let again $X \in K(\mathbf{R}^k)$, $k \geq 2$. Suppose that $\{x_j\}_{0 \leq j \leq n}$, $n \in \mathbf{N}$, is a collection of distinct points in X with $d(x_j, x_{j+1}) < d(x_0, x_n)$, $\forall j$, and that the polygon $\Pi(x_0, \ldots, x_n) \cong [0,1]$ (here d denotes the metric in X).

Recall that a map $S \in C^0(X)$ is a *similitude* or *similarity map* if it is given by

$$S(x) = sR(x) + t$$

where $s \in [0,1)$, $R(x) \in SO(k)$ and $t \in X$.

Let $\mathbf{w}: = \{w_i: i = 1, \ldots, n\}$ be a collection of similitudes $w_i: X \to X$ satisfying

C1) $x_0 = w_1(x_0)$, $x_n = w_n(x_n)$, $w_{i+1}(x_0) = w_i(x_0) = x_i$ $\forall i = 1, \ldots, n$

C2) Open Set Condition (Hutchinson): $\exists$ open set $G \subset X$ so that

$$\cup w_i G \subset G \quad \text{and} \quad w_i G \cap w_j G = \emptyset \quad \text{for } i \neq j$$

$(X, \mathbf{w})$ is a HIFS with unique attractor $A(X)$ and associated code space Ω.

Proposition 2. *Let* $z \in I = [0,1] \subset \mathbf{R}$. *Let* $z = (z_1 z_2 \ldots z_r \ldots)$, $z_m \in \{1, \ldots, n\}$, *denote the* n-*ary expansion of* z. *Let* $P: I \to \Omega$ *be defined by*

$$P(z = (z_1 \ldots z_r \ldots)): = \sigma_+(z_1 \ldots z_r \ldots)$$

where σ_+ *is the right-shift operator. Then* P *is a homeomorphism.*

The proof is straight-forward.

Now define a map $f: I \to A(X) \subset X$ by

$$f(z) := SP(z)$$

where S is the continuous surjection from Ω onto A(X). Since $S \in C^0(\Omega, A(X))$ f is a continuous function. Furthermore, f passes through $\{(z_j, x_j) \in I \times X: z_j := j/n, j = 0,1,\ldots,n\}$.

Let $A(I \times X) :=$ graph(f). Then $A(I \times X)$ is the unique attractor of the HIFS $(I \times X, \tilde{w})$ where $\tilde{\mathbf{w}} := \{\tilde{w}_i: i = 1,\ldots,n\}$ and

$$\tilde{w}_i: I \times X \to I \times X$$

$$\tilde{w}_i(z,x) := \begin{bmatrix} \frac{1}{n}(z + i - 1) \\ w_i(x) \end{bmatrix} \quad \forall i$$

(for more details see [M1]).

Note that $A(X) = \text{proj}_X \text{graph}(f)$. We then have the following result.

Theorem 2. dim A(X) = dim graph(f).

Remark. It is well known that under the above conditions on $(X,\mathbf{w})$ the Hausdorff-Besicovich dimension of A(X) is the unique positive solution of $\Sigma s_i^d = 1$ where $s_i := \text{Lip}(w_i)$, $\forall i = 1,\ldots,n$ (see for instance [Hu], [BD], [M1]). We furthermore have that d = dim A(X) agrees with fractal dimension of A(X), and the over the probabilities maximized Lyapunov dimension $\Lambda(v)$ of the associated dynamical system equals 1 + d (the set $\mathbf{p}^*$ of probabilities is given by $p_i^* = s_i/(\Sigma s_i)$; see [M1] for more details).

Now let us project $A(I \times X)$ onto $I \times \mathbf{R}$. We obviously obtain the graph of a continuous fractal function $f^*: I \to \mathbf{R}$. Since f^* still depends continuously on all the "hidden variables" graph(f^*) is in general not self-affine, i.e. graph(f^*) $\neq \cup w_i$graph(f^*). The projections f^* are thus called *hidden variable fractal functions*.

Figure 4 shows an attractor A(X) and the projections of $A(I \times X)$ onto $\mathbf{R}^2$ and Figure 5 the projections of an attractor $A(I \times X)$ onto $\mathbf{R}^2$.

The following theorem gives a formula for the (fractal) dimension of graph(f^*).

Theorem 3. $\dim \text{graph}(f^*) = 1 + \log_n(\Sigma s_i)$. (2.1)

The proof can be found in [M1].

There exists an interesting relation between dim graph(f^*) and the dimension of the embedding space of A(X). To derive this relationship notice that we have

$$\Sigma s_i \geq 1 \quad \text{and} \quad \Sigma s_i^k \leq 1 \qquad (2.2)$$

(these inequalities follow immediately from C1) and C2): the first reflects the fact that A(X) is connected and the latter the fact that $A(X) \subset \mathbf{R}^k$ and thus $\dim A(X) \leq k$).

Applying the Chauchy-Schwartz inequality to (2.1) and (2.2) yields

Theorem 4. $1 \leq \dim \text{graph}(f^*) \leq 2 - k^{-1}$.

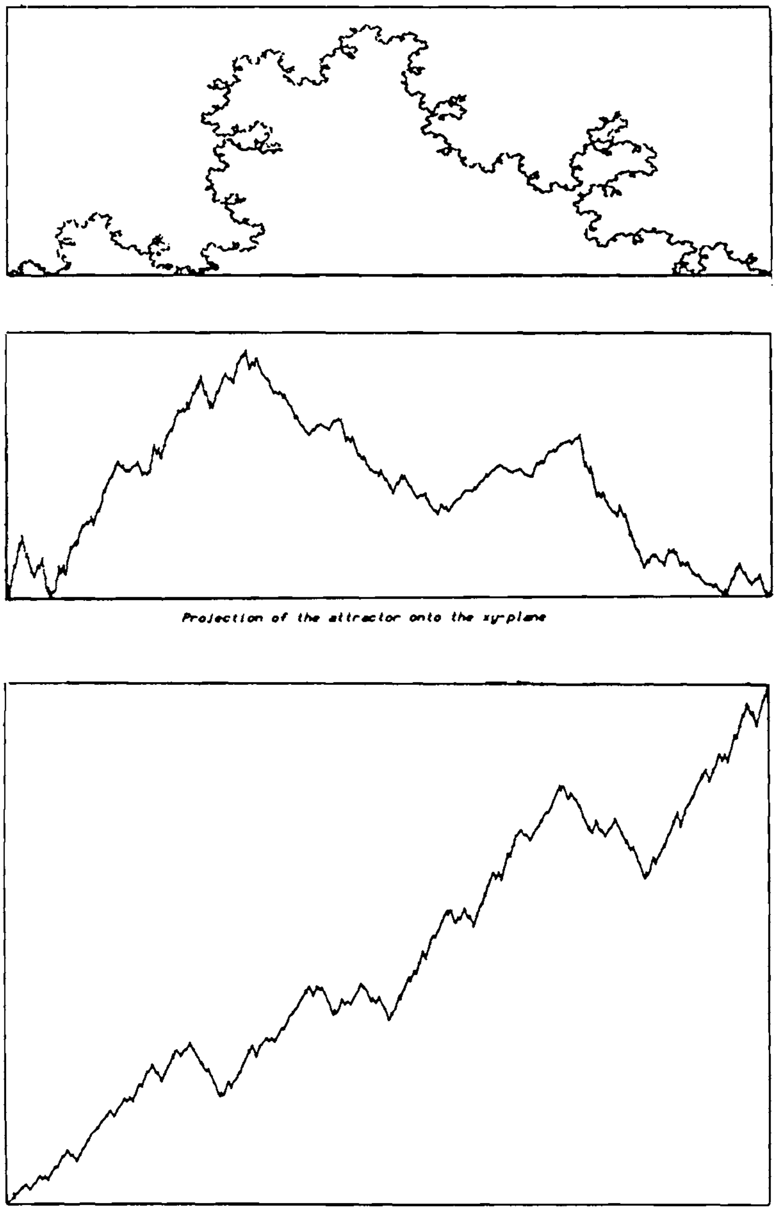

Figure 4. The attractor A(X) in X and the projections of A(I × X) onto I × R

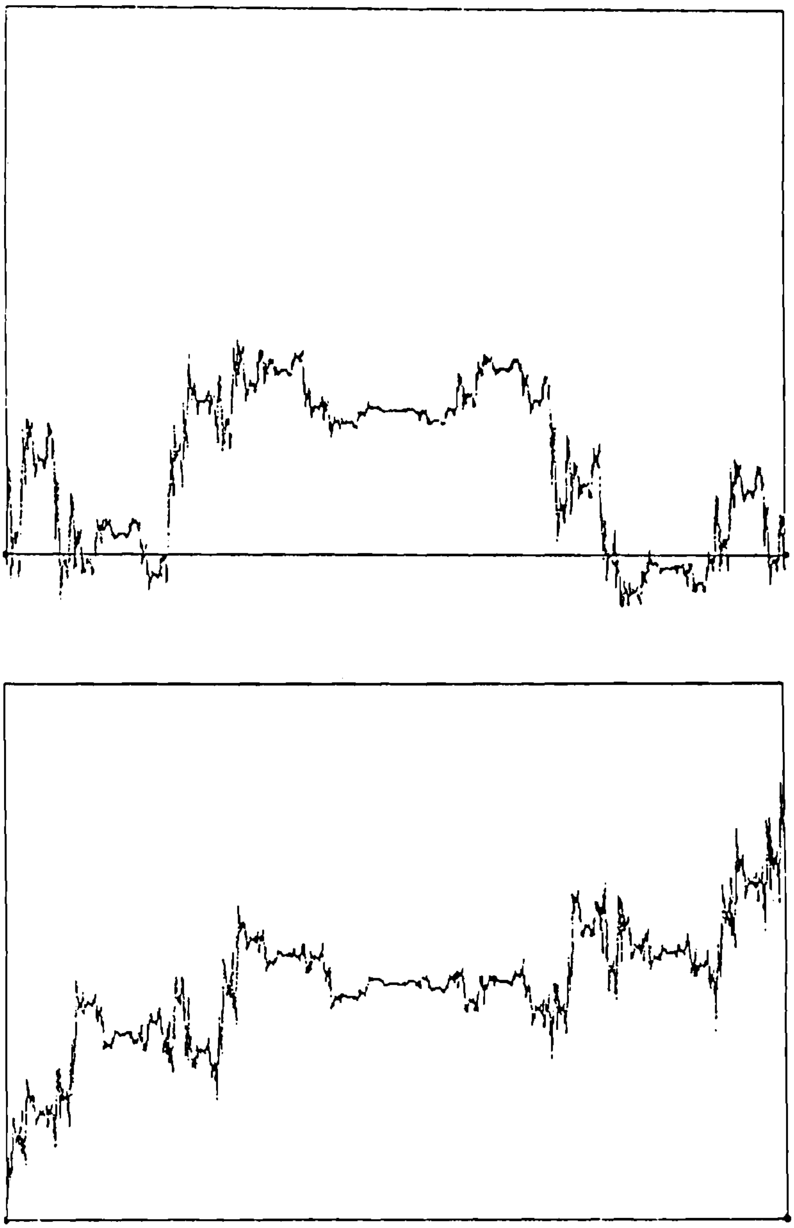

Figure 5. The projections of an attractor $A(I \times X)$ onto $I \times R \subset R^2$

Bibliography

[B1] M. F. Barnsley, *Fractal functions and interpolation*, Constructive Approximation 2 (1986), 303-329.

[B2] ________, *Making chaotic dynamical systems to order*, in Chaotic Dynamics and Fractals, ed. M. F. Barnsley, Academic Press, New York (1986).

[BD] ________ and S. G. Demko, *Iterated function systems and the global construction of fractals*, Proc. Roy. Soc. Lond., Ser. A, 399 (1985), 243-275.

[BEHM] M. F. Barnsley, J. H. Elton, D. P. Hardin and P. R. Massopust, *Hidden variable fractal interpolation functions*, Georgia Tech (1986) (preprint).

[BHa] M. F. Barnsley and A. Harrington, *The calculus of fractal interpolation functions*, Georgia Tech (1985) (preprint).

[FKYY] P. Frederickson, J. L. Kaplan, E. D. Yorke and J. A. Yorke, *The Lyapunov dimension of strange attractors*, J. Diff. Eq. 49 (1983), 185-207.

[H] D. P. Hardin, Ph.D. Thesis, Georgia Institute of Technology (1985).

[HM] ________ and P. R. Massopust, *The capacity for a class of fractal functions*, Commun. Math. Phys. 105 (1986), 455-460.

[Hu] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. J. Math. 30 (1981), 731-747.

[M1] P. R. Massopust, Ph.D. Thesis, Georgia Institute of Technology (1986).

[M2] ________, N-*dimensional fractal interpolation functions and their dimension*, LaGrange College (1987) (preprint).

[P] S. Pelikan, *Invariant densities for random maps on the interval*, Trans. Am. Math. Soc. 281 (2) (1984), 813-815.

[Y] L. S. Young, *Dimension, entropy and Lyapunov exponents*, Ergod. Th. and Dynam. Sys. 2 (1982), 109-124.

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