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1. Introduction

Let X be a compact, connected ANR. Let $E(X)$ and $H(X)$ be the space of self-homotopy equivalences and the group of homeomorphisms of X, respectively. Let $\gamma: E(X) \to X$ and $\gamma': H(X) \to X$ be the evaluation maps at $x \in X$. Then γ and γ' induce $\gamma_{\#}: \pi_1(E(X),id) \to \pi_1(X,x)$ and $\gamma'_{\#}: \pi_1(H(X),id) \to \pi_1(X,x)$ such that if $i: H(X) \to E(X)$ denotes the inclusion map, then we have the following commutative diagram

$$\begin{tikzcd} \pi_1(H(X),id) \arrow[rr, "i_{\#}"] \arrow[dr, "\gamma'_{\#}"'] & & \pi_1(E(X),id) \arrow[dl, "\gamma_{\#}"] \\ & \pi_1(X,x) & \end{tikzcd}$$

McCarty [6] has shown that $\gamma'_{\#}(\pi_1(H(X),id))$ lies in the center $Z(\pi_1(X,x))$ and each element $\alpha \in \gamma'_{\#}(\pi_1(H(X),id))$ acts trivially on $\pi_k(X,x)$ for all $k \geq 1$ if X is an admissible space, i.e., X is at least locally compact, locally connected and Hausdorff. Therefore, the natural question is whether $\alpha \in \gamma'_{\#}(H(X),id)$ if any $\alpha \in \pi_1(X,x)$ acts trivially on $\pi_k(X,x)$, for all $k \geq 1$. This question arose while I was studying the Nielsen fixed point theorems, which heavily depend on the structure of the fundamental group [1].

All those elements $\alpha \in \pi_1(X,x)$ which act trivially on $\pi_k(X,x)$ are called k-simple elements of $\pi_1(X,x)$ [4]. Then our question can be rephrased as: if $\alpha \in \pi_1(X,x)$ k-simple for all $k \geq 1$, then is $\alpha \in \gamma'_{\#}\pi_1(H(X),id))$?

Let $P(X,x) = \{\alpha \in \pi_1(X,x) | \alpha$ is k-simple for all $k \geq 1\}$. Then $P(X,x) \subset Z(\pi_1(X,x))$, the center of $\pi_1(X,x)$. In 1965, Gottlief [3] showed that if X is aspherical, then $\gamma_\#(\pi_1(E(X),id)) \simeq P(X,x) \simeq Z(\pi_1(X,x))$. Now our question can be rephrased as follows: if X is aspherical then under what conditions $\gamma'_\#(\pi_1(H(X),id)) \simeq P(X,x) \simeq Z(\pi_1(X,x))$, that is, whether $\gamma'_\#$ hits the center.

Let $E(X,x)$ and $H(X,x)$ be the based self-homotopy equivalences and the based homeomorphisms at $x \in X$. For any homeomorphism $g \in H(X,x)$, we have an induced automorphism $g_\#: \pi_k(X,x) \to \pi_k(X,x)$. If g is based homotopic to g' then the induced automorphisms agree, i.e., $g_\# = g'_\#$, and yield a representation

$$\psi: \pi_0(H(X,x),id) \to \mathrm{Aut}(\pi_k(X,x)).$$

We answer the above question in the following form. Let X = M be a closed, connected aspherical manifold. Then $\gamma'_\#$ hits the center if and only if the representation $\psi: \pi_0(H(M,x),id) \to \mathrm{Aut}(\pi_1(M,x))$ is faithful. This implies that if $\pi_1(M,x)$ is centerless then the representation ψ is faithful. At the end we will give some examples satisfying our hypothesis.

Finally I would like to thank Gottlieb for his comments made on the original version of this paper.

2. On the Homeomorphism Groups and Representations

Let X = M be a closed, connected aspherical manifold. A manifold M is called aspherical if its universal covering space $\tilde{M}$ is contractible, i.e., M is a $K(\pi,1)$-space. As before, let $E(M)$ and $H(M)$ be the self-homotopy equivalences and the based homeomorphisms at $x \in M$ respectively. The evaluation maps $\gamma\colon E(M) \to M$, $\gamma'\colon H(M) \to M$ defined by $\gamma(h) = h(x)$ and $\gamma'(g) = g(x)$ at $x \in M$ are fiberings [2], [6], and we have the following fiber-homotopy commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(H(M),id) \arrow[r, "\gamma'_\#"] \arrow[d] & \pi_1(M,x) \arrow[r, "d'_\#"] \arrow[d] & \pi_0(H(M,x),id) \arrow[r] \arrow[d] & \pi_0(H(M),id) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \pi_1(E(M),id) \arrow[r, "\gamma_\#"] & \pi_1(M,x) \arrow[r, "d_\#"] & \pi_0(E(M,x),id) \arrow[r] & \pi_0(E(M),id) \arrow[r] & 0
\end{tikzcd}$$

Lemma 1. *Let* M *be a closed, connected aspherical manifold. If the induced homomorphism* ${}_0i_\#\colon \pi_0(H(M,x),id) \to \pi_0(E(M,x),id)$ *is a monomorphism then* ${}_1i_\#\colon \pi_1(H(M),id) \to \pi_1(E(M),id)$ *is an epimorphism.*

Proof. From the Gottlieb theorem [3], we know $\pi_1(H(M,x),id) = 0$ and $\pi_1(E(M,x),id) = 0$, and we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(H(M),id) \arrow[r] \arrow[d, "{}_1i_\#"'] & \pi_1(M,x) \arrow[r] \arrow[d, equal] & \pi_0(H(M,x),id) \arrow[r] \arrow[d, "{}_0i_\#"'] & \pi_1(H(M),id) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(E(M),id) \arrow[r] & \pi_1(M,x) \arrow[r] & \pi_0(E(M,x),id) \arrow[r] & \pi_0(E(M),id) \arrow[r] & 0
\end{tikzcd}$$

We can see that ${}_1i_\#$ is a monomorphism, since it factors through

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(H(M),id) \arrow[r, "r'_\#"] \arrow[d, "{}_1i_\#"'] & \pi_1(M,x) \\
0 \arrow[r] & \pi_1(E(M),id) \arrow[ur, "r_\#"'] &
\end{tikzcd}$$

Now by diagram chasing, i.e., by the Weak Four Lemma [7], we know that ${}_1i_\#$ onto if ${}_0i_\#$ is a monomorphism.

Corollary 2. $\gamma'_\#(\pi_1(H(M),id)) \simeq Z(\pi_1(M,x))$ *if* ${}_0i_\#$ *is a monomorphism.*

Proof. Since $\gamma_{\#}(E(M),id) \simeq Z(\pi_1(M,x))$ [3] and ${}_1i_{\#}$ is onto from the lemma 1, we have the result.

Lemma 3. *Let* M *be a closed, connected aspherical manifold such that* $\pi_0(H(M),id) \to \pi_0(E(M),id)$ *is a monomorphism. If* ${}_1i_{\#}$ *is an epimorphism then* ${}_0i_{\#}$: $\pi_0(H(M,x),id) \to \pi_0(E(M,x),id)$ *is a monomorphism.*

Proof. This lemma again follows from diagram chasing. This time we apply The Five Lemma [7]. Note that ${}_0i_{\#}$ is an epimorphism if $\pi_0(H(M),id) \to \pi_0(E(M),id)$ is onto.

Corollary 4. *With the hypothesis of lemma* 3, *we have a representation* ψ: $\pi_0(H(M,x),id) \to Aut(\pi_1(M,x))$, *which is faithful.*

Proof. Let $\psi = {}_0i_{\#}\ \phi$, where ϕ: $\pi_0(E(M,x),id) \to Aut(\pi_1(M,x))$ is an isomorphic representation [2]. Since ${}_0i_{\#}$ becomes monic, the result follows.

Combining these two lemmas, we have

Theorem 5. *Let* M *be a closed, connected aspherical manifold such that* $\pi_0(H(M),id) \simeq \pi_0(E(M),id)$. *Then* ${}_1i_{\#}$ *is an isomorphism if and only if* ${}_0i_{\#}$ *is an isomorphism i.e., there is an isomorphic representation* ψ: $\pi_0(H(M,x),id) \to Aut(\pi_1(M,x))$.

Corollary 6. *Let* M *be a closed, connected aspherical manifold. If* $\pi_1(M,x)$ *has no non-trivial center, then there is a faithful representation.* ψ: $\pi_0(H(M,x),id) \to Aut(\pi_1(M,x))$.

Remark. Let M be an arbitrary manifold. If $[\alpha] \in \pi_1(M,x)$ then we can lift the loop α to α^* in $H(M)$ such that α^* is a path from the identity homeomorphism to $g \in H(M,x)$. McCarty [6] has shown that the induced automorphism $g_\#: \pi_k(M,x) \to \pi_k(M,x)$ is the same as the standard action of $[\alpha]$ on higher homotopy groups for all k [4]. Thus if $[\alpha] \in P(M,x)$, then $g_\#$ becomes the identity automorphism for all $k \geq 1$. If this implies that g is isotopic to the identity homeomorphism relative to x on M then g belongs to the identity path component in $\pi_0(H(M,x))$. Let β be a path from g to identity map in $H(M,x)$. Then $\beta \circ \alpha^*$ is a loop in $H(X)$ such that $\gamma'(\beta \circ \alpha^*) = \alpha$. This implies $\gamma'_\#(\pi_1(H(M),id) = P(M,x)$. We ask the following question. If $g_\#: \pi_k(M,x) \to \pi_k(M,x)$ is the identity automorphism for all $k \geq 1$, what conditions are necessary to ensure that g belongs to the path component of the identity homeomorphism in $H(M,x)$.

Example 1. Let M be a closed, connected 3-manifold which is irreducible and sufficiently large [8]. These manifolds are aspherical and Waldhausen has shown $\pi_0(E(M)) \approx \pi_0(H(M))$. On the other hand Laudenbach [5], pushing further Waldhausen's result, has shown $\pi_1(E(M),id) \approx \pi_1(H(M),id)$ if in addition M is P^2-irreducible. Thus these manifolds satisfy our hypothesis, and there is an isomorphic representation

$$\psi: \pi_0(H(M,x),id) \to \text{Aut } \pi_1(M,x).$$

Example 2. For the higher dimensional examples, we show "The model aspherical manifolds" from [2]. Let (W,N) be a properly discontinuous action of a discrete group N on a contractible manifold W so that W/N is compact. Then for each torsion free extension $1 \to Z^k \to \pi \to N \to 1$ the space $M = (T^k \times W)/N$ is an aspherical manifold and the map $M \to W/N$ is a Seifert fibering. These manifolds M satisfy our hypothesis; $\pi_0(E(M)) \simeq \pi_0(H(M))$ and $\pi_1(E(M),id) \simeq \pi_1(H(M),id)$. Thus we have an isomorphic representation $\psi\colon \pi_0(H(M,x),id) \to \operatorname{Aut}(\pi_1(M,x))$.

References

1. R. F. Brown, *The Lefschetz fixed point theorem*, Scott Foresman and Co., Glenview, Ill., 1971.
2. P. E. Conner and F. Raymond, *Deforming homotopy equivalences to homeomorphisms in aspherical manifolds*, Bulletin of A.M.S. 83 (1977), 36-85.
3. D. H. Gottlieb, *A certain subgroup fo the fundamental group*, Amer. J. Math. 87 (1965), 840-856.
4. S. T. Hu, *Homotopy theory*, Academic Press, New York, 1959.
5. F. Laudenbach, *Topologie de la dimension trois: homotopie et isotopie*, Astérisque, Soc. Math. de Franc 12 (1974).
6. G. S. McCarty, Jr., *Homeotopy groups*, Trans. of A.M.S. 106 (1963), 293-304.
7. S. Maclane, *Homology*, Academic Press, New York, 1963.
8. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. 18 (1968), 56-88.

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