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A CLASS OF CONTINUA DEFINED BY MEANS OF CUTTINGS

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As used in the study of dimension, a *cutting* C of a topological space X between two sets A and B is understood to mean a subset of X such that

$$X \setminus C = U \cup V, \ A \subset U, \ B \subset V,$$

where U and V are disjoint open subsets of X. Thus all cuttings are closed subsets. A well-known result concerning the large inductive dimension (see, for example, [1], p. 282) says that, for any finite number of non-empty F_σ-subsets A_i of a metrizable space X ($i = 1,\cdots,k$), there exists a cutting C of X between an arbitrarily chosen pair of disjoint closed subsets such that

$$\operatorname{Ind}(A_i \cap C) \leq \operatorname{Ind} A_i - 1$$

for $i = 1,\cdots,k$. We can apply this result to one-dimensional separable metric spaces X, $k = 2$, and $A_1 = X$. As a consequence, we see that a separable metric space X is at most 1-dimensional if and only if, for each 0-dimensional F_σ-subset A of X, there exists a cutting C of X between an arbitrarily chosen disjoint pair of a singleton and a closed set such that

(*) $\dim C \leq 0, \ A \cap C = \emptyset.$

Removing the requirement that A be an F_σ-subset creates a much stronger condition which distinguishes an interesting class of 1-dimensional spaces and, in particular, of 1-dimensional continua. We say that a continuum X belongs to *Class 0* provided, for each 0-dimensional subset A of X, there exists a cutting C of X between an arbitrarily chosen disjoint pair of a singleton and a closed set such that (*) holds.

The purpose of this paper[1] is to announce initial results and ideas which arose in the investigation of Class O. More complete discussion and the proofs will be published elsewhere. Some less familiar notions and terminology can be found in [3].

Lemma 1. *If* X *is a regular continuum,* $\varepsilon > 0$ *and* $\mathcal{C}$ *is an infinite collection of subcontinua of* X *with* diam $C > \varepsilon$ *for* $C \in \mathcal{C}$, *then there exists a point* $p \in X$ *and an infinite subcollection* $\mathcal{C}' \subset \mathcal{C}$ *with*

$$p \in \bigcap_{C \in \mathcal{C}'} C.$$

Lemma 2. *If* X *is a dendrite and* C *is a cutting of* X *between closed sets* $A, B \subset X$, *then there exists a finite set* $F \subset C$ *such that* F *is a cutting of* X *between* A *and* B.

Theorem. If X *is a local dendrite, then* X *is in Class 0.*

There exists a Suslinian continuum which is not in Class O. This continuum can be taken to be a certain example of a dendroid constructed earlier to illustrate

[1] A former version of the paper was presented to the 21st Spring Topology Conference at the University of Alabama in Birmingham, on March 19, 1987, and to the mathematics faculty and students of Stephen F. Austin State University, Nacogdoches, Texas, on April 10, 1987.

another phenomenon (see [3], pp. 135-137). It contains a dense 0-dimensional G_δ-subset A, namely, the common part of the unions of the interiors of triangles appearing in the construction, such that no cutting C between two vertices of the first triangle satisfies (*). That dendroid is not a rational continuum[2]. It would be worthwhile to decide how Class O fits in other classifications of continua. For instance, consider the following well-known implications:

$$\begin{pmatrix}\text{local}\\ \text{dendrite}\end{pmatrix} \Rightarrow (\text{regular}) \Rightarrow (\text{rational}) \Rightarrow (\text{Suslinian})$$

Problem 1. *Does there exist a continuum in Class O which is not Suslinian?*

Problem 2. *Does there exist a regular continuum which is not in Class O?*

Note that Lemma 2 does not hold for regular continua. To see this, take $T = T_1 \cup T_2$, where T_1 and T_2 are two congruent Sierpiński triangular curves (see [2], p. 276) such that $T_1 \cap T_2 = S$ and S is a side of the largest triangle in both T_1 and T_2. Then T is a regular continuum, S is a cutting of T between two opposite vertices of these triangles, and no proper subset of S is such a cutting.

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[2]Modifying its definition one can, however, obtain an example of another dendroid which is rational and is not in Class O (a construction due to J. Krasinkiewicz, unpublished).

References

[1] R. Engelking, *Dimension theory*, North-Holland (1978).

[2] K. Kuratowski, *Topology*, Vol. II, Academic Press (1968).

[3] A. Lelek, *On the topology of curves II*, Fund. Math. 70 (1971), 131-138.

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