
TOPOLOGY PROCEEDINGS



Volume 12, 1987

Pages 193–196

<http://topology.auburn.edu/tp/>

Research Announcement:
NON-HOMOGENEITY OF
INTERMEDIATE UNIVERSAL CONTINUA

by
WAYNE LEWIS

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

NON-HOMOGENEITY OF INTERMEDIATE UNIVERSAL CONTINUA

Wayne Lewis*

For each pair of non-negative integers n and k , $k > n$, Menger [6] has described an n -dimensional continuum M_n^k in E^k which is universal with respect to containing homeomorphic copies of every n -dimensional continuum which can be embedded in E^k . For any $k > 0$, M_0^k is the standard Cantor set. M_1^2 is the Sierpiński universal plane curve [8], [9], and M_1^k ($k \geq 3$) is the Menger universal curve.

The Cantor set is clearly homogeneous. It is a classical result that the Sierpiński curve is not homogeneous. Anderson [1] proved in 1958 that the Menger universal curve is homogeneous and gave a characterization of the continuum.

In early 1983 the author announced informally [5] that M_n^k is never homogeneous for $n > 0$, $k < 2n + 1$, and indicated an argument for the result. The result was not written up for publication at the time since the question of homogeneity for the case $k \geq 2n + 1$, when no embedding restrictions apply to the universality of M_n^k , remained open. Since then, Bestvina [2] has given a very nice proof of the homogeneity of M_n^k for $k \geq 2n + 1$ and a complete characterization of such spaces. Several persons have expressed interest in the result announced earlier for $k < 2n + 1$, or uncertainty about the status of this problem. It is the purpose of this note to clarify this.

*This work was supported in part by N.S.F. grant DMS 83-00826.

Alternative constructions of universal continua have been given by Lefschetz [4] and Pasynkov [7]. The space M_n^k in Lefschetz' construction is the intersection of a nested sequence of k -manifolds with boundary. Let $M_{(0,n)}^k$ be a k -simplex. Inductively, let $M_{(i+1,n)}^k$ be the star of the n -skeleton of $M_{(i,n)}^k$ in the second barycentric subdivision of $M_{(i,n)}^k$. Set $M_n^k = \bigcap_{i=0}^{\infty} M_{(i,n)}^k$. It remains an open problem to characterize M_n^k for $k < 2n + 1$ as well as to determine whether the constructions of Lefschetz, Menger, and Pasynkov yield homeomorphic continua. (Some results are known for the case $k = n + 1$ [3],[9].) Our argument is described in terms of Lefschetz' construction, but can be appropriately modified for the other two.

Theorem. M_n^k is not homogeneous for $n > 0$, $k < 2n + 1$.

Sketch of proof. The case $k = n + 1$ is already known from arguments similar to that for the Sierpiński curve, i.e. local separation vs. its lack. However, this is also covered by the general case, which uses a generalization of these arguments.

Let $B = \{x \in M_n^k \mid x \text{ is in the boundary of some } M_{(i,n)}^k \text{ in the defining sequence for } M_n^k\}$, and
 let $I = \{x \in M_n^k \mid x \text{ is in the interior of every } M_{(i,n)}^k \text{ in the defining sequence for } M_n^k\}$.

There are two alternative but closely related methods to topologically distinguish between points in B and points in I .

The first involves local linking properties. If $x \in B$, there is an n -sphere Σ in M_n^k containing x (and a subset of

the boundary of some $M_{(i,n)}^k$) such that every $(k - n - 1)$ -sphere in $M_n^k - \Sigma$ is null-homotopic in $M_n^k - \Sigma$. No point in I has this property. If $y \in I$ and S is any n -sphere in M_n^k containing y , there is a $(k - n - 1)$ -sphere T in $M_n^k - S$ such that every null homotopy of T in M_n^k intersects S . (One can if desired restrict one's attention to S being as nice as desired in terms of ulc properties, etc., since it is to be compared to the Σ containing x , which can be chosen to have any such desired properties.)

Alternatively, one can use a variation on the disjoint disk property. If $x \in B$, there is an embedding $f: C^n \rightarrow M_n^k$ with $f(0) = x$, where C^n is an n -simplex and 0 its barycenter, such that for any map $g: C^{k-n} \rightarrow M_n^k$ and any $\epsilon > 0$ there are maps $\tilde{f}: C^n \rightarrow M_n^k$ and $\tilde{g}: C^{k-n} \rightarrow M_n^k$, with $\text{dist}(f, \tilde{f}) < \epsilon$ and $\text{dist}(g, \tilde{g}) < \epsilon$ such that $\tilde{f}(C^n) \cap \tilde{g}(C^{k-n}) = \phi$.

No point of I has this property. If $y \in I$ and $h: C^n \rightarrow M_n^k$ is any embedding with $h(0) = y$, when there exist $j: C^{k-n} \rightarrow M_n^k$ and $\epsilon > 0$, such that if $\tilde{h}: C^n \rightarrow M_n^k$ and $\tilde{j}: C^{k-n} \rightarrow M_n^k$ are two maps with $\text{dist}(h, \tilde{h}) < \epsilon$ and $\text{dist}(j, \tilde{j}) < \epsilon$ then $\tilde{h}(C^n) \cap \tilde{j}(C^{k-n}) \neq \phi$.

For the first argument, the non-existence of null-homotopies of n spheres in any M_n^k prevents it from applying to M_n^{2n+1} . For the second argument, the fact that M_n^{2n+1} has the disjoint n -cell property, and $k - n > n$ in this case, prevents the argument from applying for $k > 2n$. It is the fact that M_n^{2n+1} satisfies the disjoint n -cell property which is central in the characterization of M_n^{2n+1} and the proof of its homogeneity.

Since the above arguments distinguish points based on local properties, they also show that no Sierpiński manifold (i.e. continuum locally homeomorphic to M_n^k for some $n > 0$, $n < k < 2n + 1$) is homogeneous. Left open are the questions of how many orbits M_n^k has under the action of its homeomorphism group and whether either of the sets B or I constitutes an orbit, as well as homogeneity properties of M_n^k under classes of maps more general than homeomorphisms.

References

- [1] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. (2) 67 (1958), 313-324.
- [2] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Bull. Amer. Math. Soc. 11 (1984), 369-370.
- [3] J. W. Cannon, *A positional characterization of the (n-1)-dimensional Sierpiński curve in S^n ($n \neq 4$)*, Fund. Math. LXXIX (1973), 107-112.
- [4] S. Lefschetz, *On compact spaces*, Ann. of Math. (2) 32 (1931), 521-538.
- [5] W. Lewis, *Continuum theory problems*, preliminary draft (also presented at Topology Conference, Univ. of Houston, March 1983).
- [6] K. Menger, *Kurventheorie*, Teubner, Leipzig (1932).
- [7] B. A. Pasynkov, *Partial topological products*, Trans. Moscow Math. Soc. 13 (1965), 153-271.
- [8] W. Sierpiński, *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée*, C. R. Paris 162, 629-632.
- [9] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. 45 (1948), 320-324.

Texas Tech University

Lubbock, Texas 79409-1042