PRIME WALLMAN COMPACTIFICATION

by

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0. Introduction

It has been shown previously, Carlson [7], that there
exists a one-to-one correspondence between the nonempty
closed subsets of the Wallman compactification of a space
and the balanced closed filters on the space. In this
paper it is shown that there exists a one-to-one corre-
spondence between the nonempty closed subsets of the prime
Wallman compactification of the space and the closed filters
on the space. Moreover, it is shown that the absolute of
a space, the hyperabsolute of a space and, if the space is
Hausdorff, the Fomin H-closed extension of a space are all
homeomorphic to subspaces of the prime Wallman compactifica-
tion of that space. A natural representation of the prime
Wallman compactification of a space in terms of its prime
open filters is provided.

1. Preliminaries

By a space we mean a T_1 topological space. For a
space X, let $v(X)$ denote the collection of all prime open
filters on X; $w(X)$ the collection of all prime closed
filters on X. Let $E(X)$ denote the collection of all fixed
open ultrafilters on X; $\Theta(X)$ the collection of all open
ultrafilters on X; and $W(X)$ the collection of all closed
ultrafilters on X.

For F a closed set in X, let F* denote the collection
of all closed ultrafilters on X containing F. The collection

of all such F* forms a base for the closed sets on W(X).
W(X) with this topology is the Wallman compactification of
X. A similar construction yields a topology on w(X),
known as the prime Wallman compactification of X.

   A topology on v(X) is provided by setting O* equal to
the collection of all open prime filters containing the open
set O. The O* then form a base for the open sets for a
topology on v(X). In this paper, v(X) will always be
assumed to have this topology. Moreover, E(X) and Θ(X) with
the subspace topology are known as the absolute of X and
the hyperabsolute of X, respectively. It will be shown that
v(X) and w(X) are homeomorphic.

   1.1 *Definition*. Let X be a topological space. An
open grill is a nonempty collection $\mathcal{G}$ of open sets satis-
fying:

   (A) $\phi \notin \mathcal{G}$

   (B) O $\in \mathcal{G}$, Q open and Q $\supset$ O implies Q $\in \mathcal{G}$.

   (C) For open sets O and Q: O $\cup$ Q $\in \mathcal{G}$ iffi O $\in \mathcal{G}$ or
Q $\in \mathcal{G}$.
Closed grills are defined similarly. Let $\mathcal{G}$ be a closed grill
and $\mathcal{H}$ be an open grill. Set:

   (A) $\mathcal{O}(\mathcal{G})$ = {O: O is open and X-O $\notin \mathcal{G}$}

   (B) $\mathcal{J}(\mathcal{H})$ = {F: F is closed and X-F $\notin \mathcal{H}$}

   Easily prime open (closed) filters are open (closed)
grills and the operators $\mathcal{J}$ and $\mathcal{O}$ will be used on prime open
and closed filters, respectively, most frequently.

1.2 *Definition.* Let O and J be open and closed filters, respectively. Set:

(A) $\mathcal{G}(O)$ = {F: F is closed and the exists O $\in$ O with F $\supset$ O}

(B) $\mathcal{S}(J)$ = {O: O is open and there exists F $\in$ J with O $\supset$ F}

(C) An open (closed) filter is said to be balanced provided it is equal to the intersection of the family of open (closed) ultrafilters that contain it.

(D) An open filter O is said to be a closed generated open filter provided there exists a closed filter J with $O = \mathcal{S}(J)$.

(E) A closed filter J is said to be an open generated closed filter provided there exists an open filter O with $J = \mathcal{G}(O)$.

1.3 *Theorem.* *Let* X *be a topological space and* K *be a prime closed filter and* P *a prime open filter. Then:*

(1) $O(K)$ *is a prime open filter.*

(2) $J(P)$ *is a prime closed filter.*

Let K *be a closed filter and* P *be an open filter. Then:*

(3) $\mathcal{S}(K)$ *is an open filter.*

(4) $\mathcal{G}(P)$ *is a closed filter.*

(5) $O(K)$ *is an open grill.*

(6) $J(P)$ *is a closed grill.*

Let $\mathcal{G}$ *be a closed grill and* H *be an open grill. Then:*

(7) $O(\mathcal{G})$ *is an open filter.*

(8) $J(H)$ *is a closed filter.*

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ (O_1 and O_2) be closed (open) filters.   Then:

   (9) $\mathcal{F}_1 \subset \mathcal{F}_2$ *iffi* $O(\mathcal{F}_2) \subset O(\mathcal{F}_1)$.

   (10) $\mathcal{F}_1 \subset \mathcal{F}_2$ *iffi* $S(\mathcal{F}_1) \subset S(\mathcal{F}_2)$.

   (11) $O_1 \subset O_2$ *iffi* $\mathcal{F}(O_2) \subset \mathcal{F}(O_1)$.

   (12) $O_1 \subset O_2$ *iffi* $\mathcal{G}(O_1) \subset \mathcal{G}(O_2)$.

   (13) $\mathcal{F}_1 = \mathcal{F}_2$ *iffi* $O(\mathcal{F}_1) = O(\mathcal{F}_2)$.

   (14) $O_1 = O_2$ *iffi* $\mathcal{F}(O_1) = \mathcal{F}(O_2)$.

   (15) $O_1 = O(\mathcal{F}(O_1))$.

   (16) $\mathcal{F}_1 = \mathcal{F}(O(\mathcal{F}_1))$.

   1.4 *Corollary.   The mappings* $\mathcal{F}$: v(X) $\to$ w(X) *and*
O: w(X) $\to$ v(X) *are inverse mappings.*

2. Prime Wallman Compactification

   2.1 *Notation.*   Let (X,t) be a T_1 topological space.
Set:

          w(X) = $\{\mathcal{K}$: $\mathcal{K}$ a prime closed filter on X$\}$

          W(X) = $\{\mathcal{N}$: $\mathcal{N}$ a closed ultrafilter on X$\}$

   For F closed in X, set F* = $\{\mathcal{K} \in$ w(X): F $\in \mathcal{K}\}$.   Then
$\{$F*: F closed in X$\}$ forms a base for the closed sets for a
topology on w(X).   When no confusion can result, w(X) will
be denoted by wX.   wX with this topology is called the
prime Wallman compactification of X.

   2.2 *Definition.*   Let O be open in X and F = X-O.   Set:
          O* = $\{\mathcal{K} \in$ wX: O $\in O(\mathcal{K})\}$.

   2.3 *Theorem.* $\{$O*: O open in X$\}$ *is a base for the
open sets in* wX.

2.4 *Corollary.* *Let O be open in X and F = X-O.* *Then*
O* = wX-F*.

Proof. Since {F*: F closed in X} is a base for the
closed sets in wX it follows that {wX-F*: F closed in X}
is a base for the open sets in wX. Let F be closed in X,
it suffices to show that O* = wX-F* where O = X-F. Let
$K \in O*$. Then O $\in O(K)$ which implies that X-O $\not\in K$ and
thus F $\not\in K$ and hence $K \in$ wX-F*.

Suppose $K \in$ wX-F*. Then $K \not\in$ F* and F $\not\in K$ and thus
O = X-F $\in O(K)$. Therefore, $K \in$ O*. Hence O* = wX-F*.

For a T_1 space X, the mapping x → $\mathcal{M}_x$, where $\mathcal{M}_x$ is
the unique closed ultrafilter containing {x} embeds X
into wX. Using this mapping we will identify a subset in
X with the appropriate subset in wX.

2.5 *Theorem.* *Let F be closed in X.* *Then* $cl_{wX}(F) = F*$.

Proof. Let F be closed in X. By the above identifi-
cation, F = {$\mathcal{M}_x$: x $\in$ F} and easily F $\subset$ F* which is closed
in wX. Thus, $cl_{wX}(F) \subset F*$.

Suppose $K \not\in cl_{wX}(F)$. Then there exists a basic open
set O* such that $K \in$ O* and O* $\cap$ {$\mathcal{M}_x$: x $\in$ F} = ϕ. Let
G = X-O. Then, by Corollary 2.4, O* = wX-G*. Now $K \in$ O*
implies G = X-O $\not\in K$. Suppose $K \in$ F*. Then F $\in K$ since
O* $\cap$ {$\mathcal{M}_x$: x $\in$ F} = ϕ it follows that $\mathcal{M}_x \not\in$ O* for each x $\in$ F.
Hence X-O $\in \mathcal{M}_x$ for each x $\in$ F and we have F $\subset$ X-O = G.
If F $\in K$ we must have G $\in K$ but O $\in O(K)$ which implies that
G = X-O $\not\in K$. Therefore, F $\not\in K$ and $K \not\in$ F*. Hence F* $\subset$
$cl_{wX}(F)$.

W(X), with the subspace topology, is the usual
Wallman compactification. It is shown in Carlson [7],
that there exists a one-to-one correspondence between the
nonempty closed subsets of W(X) and the balanced closed
filters. It is the purpose of this section to generalize
this result to the prime Wallman compactification. Addi-
tional information on the prime Wallman compactification
can be found in [12].

The following lemma can be proved using Zorn's lemma.

2.6 *Lemma.* *Let $\mathfrak{I}$ be a closed filter on a space X and*
K *a nonempty closed set not in $\mathfrak{I}$. Then there exists a*
prime closed filter $\mathcal{K}$ such that $\mathfrak{I} \subset \mathcal{K}$ and K $\notin$ $\mathcal{K}$.

Recall that a balanced closed filter is one that is
equal to the intersection of the family of closed ultra-
filters that contain it. It is natural to consider a
corresponding concept for prime closed filters. However,
as the following theorem states, which can be proved using
the above lemma, every closed filter is the intersection
of the family of prime closed filters that contain it. The
author assumes the result is well known but is unaware of
a reference for it.

2.7 *Theorem.* *Every closed filter is the intersection*
of the family of prime closed filters that contain it.

We will now show that there exists a one-to-one
correspondence between the nonempty closed subsets of
wX and the closed filters on X.

2.8 *Theorem.* *Let* $\hat{F} = \{K_\alpha : \alpha \in \Omega\}$ *be a nonempty*
closed subset of wX. *Set* $J = \cap\{K_\alpha : \alpha \in \Omega\}$. *Then* J *is a*
closed filter and $\hat{F} = \{K \in wX : J \subset K\}$.

Proof. Easily J is a closed filter on X. Set G = $\{K \in wX : J \subset K\}$.

Let $K \in$ G. Suppose $K \notin \hat{F}$. Then there exists a
closed set F $\subset$ X such that $\hat{F} \subset F^*$ and $K \notin F^*$. Then F $\notin K$
but F $\in K_\alpha$ for each $\alpha \in \Omega$. Hence F $\in J \subset K$ and we have a
contradiction. Therefore, G $\subset \hat{F}$.

Let $K \in \hat{F}$. Easily $J = \cap\hat{F} \subset K$ which implies $K \in$ G.
Hence $\hat{F} \subset$ G.

2.9 *Theorem.* *Let* $\hat{F} = \{K_\alpha : \alpha \in \Omega\}$ *be a nonempty closed*
subset in wX *and* $J = \cap\{K_\alpha : \alpha \in \Omega\}$. *Then:*

(1) $\hat{F} = \cap\{F^* : F \in J\}$

(2) $\hat{F} = \cap\{cl_{wX}(F) : F \in J\}$

(3) $J = \{F : F$ *is closed in* X *and* $\hat{F} \subset F^*\}$

(4) $J = \{F : F$ *is closed in* X *and* $\hat{F} \subset cl_{wX}(F)\}$

Proof. By Theorem 2.8, $\hat{F} = \{K \in wX : J \subset K\}$. Set
$\hat{G} = \cap\{F^* : F \in J\}$. Let $K \in \hat{G}$. Then $K \in F^*$ for each F $\in J$
and thus F $\in K$ for each F $\in J$. Hence $J \subset K$ and $K \in \hat{F}$.
Conversely, if $K \in \hat{F}$, then $J \subset K$ and F $\in K$, and thus $K \in F^*$
for each F $\in J$. Therefore, $K \in \hat{G}$. Thus (1) holds and
(2) follows by Theorem 2.5.

Let $G = \{F : F$ is closed in X and $\hat{F} \subset F^*\}$. Let F $\in G$.
Then $\hat{F} \subset F^*$, $K_\alpha \in F^*$, or equivalently, F $\in K_\alpha$ for each
$\alpha \in \Omega$. Thus F $\in J$. Now let F $\in J$. Then F $\in K_\alpha$ and
hence $K_\alpha \in F^*$ for each $\alpha \in \Omega$. Thus $\hat{F} \subset F^*$ and (3) holds
and (4) follows by Theorem 2.5.

2.10 *Theorem.* *Let $\mathcal{F}$ be a closed filter on X and*
$\hat{F} = \{K \in wX: \mathcal{F} \subset K\}$. *Then $\hat{F}$ is a nonempty closed subset*
of $\dot{w}X$.

Proof. Let $\hat{G} = \cap\{F^*: F \in \mathcal{F}\}$. Easily $\hat{G}$ is closed in
$w\dot{X}$ and thus it suffices to show that $\hat{G} = \hat{F}$. Let K be a
prime closed filter and $K \in \hat{G}$. Then $K \in F^*$ for each $F \in \mathcal{F}$.
Thus, $F \in K$ for each $F \in \mathcal{F}$ and hence $\mathcal{F} \subset K$. Therefore,
$K \in \hat{F}$ and $\hat{G} \subset \hat{F}$. Let $K \in \hat{F}$. Then $\mathcal{F} \subset K$. Hence, for
each $F \in \mathcal{F}$, it follows that $K \in F^*$ since $F \in K$. Therefore,
$K \in \hat{G}$ and $\hat{F} \subset \hat{G}$.

2.11 *Theorem.* *Let S denote the family of closed*
filters on X and $\mathcal{C}$ denote the family of nonempty closed
subsets of wX. Define h: $S \rightarrow \mathcal{C}$ *by* h(F) $= \{K \in wX: \mathcal{F} \subset K\}$.
Then h *is a one-to-one surjective mapping.*

Proof. Let $\mathcal{F}$ be a closed filter on X. By Theorem 2.10,
$\{K \in wX: \mathcal{F} \subset K\}$ is a nonempty closed subset of wX. Hence h
defines a map from S into $\mathcal{C}$. h is surjective by Theorem
2.8.

h is one-to-one. Suppose $h(\mathcal{F}_1) = h(\mathcal{F}_2)$. Then
$\{K \in wX: \mathcal{F}_1 \subseteq K\} = \{K \in wX: \mathcal{F}_2 \subset K\}$. Since each closed
filter is the intersection of the family of prime closed
filters that contain it, we have
$$\mathcal{F}_1 = \cap\{K \in wX: \mathcal{F}_1 \subset K\} = \cap\{K \in w(X): \mathcal{F}_2 \subset K\} = \mathcal{F}_2.$$
Therefore, h is a one-to-one mapping.

3. Homeomorph of the Prime Wallman Compactification

For O open in a T_1 topological space X, set O* =
$\{\mathcal{P} \in v(X): O \in \mathcal{P}\}$. Let v(X) have the topology generated
by the base $\{O^*: O$ open in $X\}$.

Define f: wX → v(X) by f(K) = $O(K)$.

3.1 *Theorem.* f *is a homeomorphism from the Prime*
Wallman compactification wX *onto the space* v(X).

Proof. By Theorem 1.3 and Corollary 1.4, the mapping
f is a one-to-one mapping from wX onto v(X) and $f^{-1}(P)$ =
$J(P)$ for each $P \in$ v(X).

f is continuous. Let O* be a basic open set in v(X).
Then: $f^{-1}(O^*) = f^{-1}(\{P \in$ v(X): O $\in P\}) = \{J(P)$: O $\in P\}$.
Let F = X-O and F* = $\{K \in$ wX: F $\in K\}$. Now F* is closed
in wX.

Claim: wX-F* = $f^{-1}(O^*)$. Let $K \in$ wX-F*. Then F $\notin K$.
Now f(K) = $O(K)$ = {Q: Q open in X and X-Q $\notin K$}. Hence
O $\in$ f(K) and thus f(K) $\in$ O*. Therefore, $K \in f^{-1}(O^*)$ and
wX-F* $\subset f^{-1}(O^*)$.

Let $K \in f^{-1}(O^*) = \{J(P)$: O $\in P\}$. Thus, there exists
a prime open filter P such that $K = J(P)$ and O $\in P$. Now
$P = O(K)$ = {Q: Q open in X and X-Q $\notin K$}. Since O $\in P$ it
follows that F = X-O $\notin K$. Hence $K \in$ wX-F* and $f^{-1}(O^*) \subset$
wX-F*.

f^{-1} is continuous. It suffices to show that for any
closed set F in X that f(wX-F*) is open in v(X). Let
O = X-F.

Claim: f(wX-F*) = O*. Let $P \in$ O*. Set $K = J(P)$.
Now K = {G: G closed in X and X-G $\notin P$}. Since O $\in P$, it
follows that F = X-O $\notin K = J(P)$ and hence $K \in$ wX-F*. Thus
$P \in$ f(wX-F*).

Let $P \in$ wX-F*. Then there exists $K \in$ wX-F* such that
$P = O(K)$. Now F $\notin K$ which implies that O = X-F $\in O(K) = P$.
Hence $P \in$ O*.

3.3 *Comment.* We have shown that v(X) provides another method for constructing the prime Wallman compactification of a T_1 topological space X. We will show, in section 4, that v(X) contains, in a natural way, both the absolute of X and the hyperabsolute of X, and if X is Hausdorff it contains the Fomin H-closed extension of X.

It follows from Theorem 1.3 that there exists a one-to-one correspondence between the closed filters and open grills on X. By Theorems 1.3, 2.3, 2.4, 2.5, and 3.2 we have the following results.

3.4 *Theorem. Every open grill is the union of the family of prime open filters it contains.*

3.5 *Theorem. Let X be a T_1 topological space. Then:*

(1) *There exists a one-to-one correspondence between the nonempty closed sets in* v(X) *and the open grills on* X.

(2) *Let $\mathcal{G}$ be an open grill on* X. *Then* $\{\mathcal{P} \in$ v(X): $\mathcal{P} \subset \mathcal{G}\}$ *is a nonempty closed subset of* v(X).

(3) *Each nonempty closed set in* v(X) *is of the form* $\{\mathcal{P}: \mathcal{P} \subset \mathcal{G}\}$ *for some open grill $\mathcal{G}$ on* X.

4. Special Subspaces of the Prime Wallman Compactification

4.1 *Notation.* Let X be a T_1 topological space. Set:

v(X) = $\{\mathcal{P}: \mathcal{P}$ a prime open filter on X$\}$

E(X) = $\{\mathcal{M}: \mathcal{M}$ a fixed open ultrafilter on X$\}$

Θ(X) = $\{\mathcal{M}: \mathcal{M}$ an open ultrafilter on X$\}$

M(X) = $\{\mathcal{P}: \mathcal{P}$ a minimal prime open filter on X$\}$

N(X) = $\{\mathcal{P}: \mathcal{P}$ a fixed minimal prime open filter on X$\}$

Let each of these sets be endowed with the subspace topology from $v(X)$. Then $E(X)$ and $\Theta(X)$ are the absolute and hyperabsolute of X, respectively. Thus, we have the following theorem.

4.2 *Theorem.* *The absolute of X and the hyperabsolute of X are embedded in the Prime Wallman Compactification of* X.

If X is T_1, then X is homeomorphic to $S = \{ \mathcal{M}_x \colon \mathcal{M}_x$ the closed ultrafilter converging to x} considered as a sub-space of wX. Easily, S is homeomorphic to $N(X)$ under the mapping f of Theorem 3.2.

Set $F(X) = N(X) \cup (\Theta(X)-E(X))$ with the subspace topology from $v(X)$. We then have the following theorems.

4.3 *Theorem.* *Let* X *be a Hausdorff topological space. Then* $F(X)$ *is homeomorphic to the Fomin* H-*closed extension of* X.

Using the homeomorphism between $w(X)$ and $v(X)$ it is an easy matter to identify these special subspaces in $w(X)$. Specifically:

$w\Theta(X) = \mathcal{J}(\Theta(X)) = \{\mathcal{K} \colon \mathcal{K}$ is a minimal prime closed
     filter}

$wE(X) = \mathcal{J}(E(X)) = \{\mathcal{K} \colon \mathcal{K}$ is a fixed minimal prime
     closed filter}

$wF(X) = \mathcal{J}(F(X)) = \{\mathcal{K} \colon \mathcal{K}$ is a fixed closed ultrafilter
     or $\mathcal{K}$ is a free minimal prime closed filter}.

These subspaces of the prime Wallman compactification
w(X) are homeomorphic to the absolute of X, the hyper-
absolute X, and, if X is a Hausdorff space, the Fomin
H-closed extension of X, respectively.

Making use of the fact that the hyperabsolute of X
is homeomorphic to a subspace of wX, namely $w\Theta(X)$, we will
establish a one-to-one correspondence between the nonempty
closed subsets of the hyperabsolute of X and a special
class of closed filters.

4.4 *Definition.* A closed filter $\mathcal{J}$ is called a minimal
balanced closed filter provided it is equal to the inter-
section of the family of minimal prime closed filters that
contain it.

A closed filter $\mathcal{J}$ is called open generated if there
exists an open filter $\mathcal{O}$ such that $\mathcal{J} = \mathcal{G}(\mathcal{O})$.

4.5 *Theorem.* Carlson [9]. *Let $\mathcal{J}$ be a closed filter
in X. The following statements are equivalent.*

(1) $\mathcal{J}$ *is a minimal balanced closed filter.*

(2) $\mathcal{J}$ *is open generated.*

We are now in a position to characterize the nonempty
closed subsets of the hyperabsolute of X.

4.6 *Theorem. Let X be a T_1 topological space. Let
$\phi \neq F' \subset w\Theta(X)$. F' is closed in $w\Theta(X)$ if and only if
there exists a minimal balanced closed filter $\mathcal{J}$ such that
$F' = \{\mathcal{H} \in w\Theta(X): \mathcal{J} \subset \mathcal{H}\}$.*

Proof.   Suppose F' is a nonempty closed subset of

wΘ(X).   Let $\mathcal{G}$ = ∩{$\mathcal{H}$: $\mathcal{H}$ ∈ F'}.   Since F' ≠ ϕ, $\mathcal{G}$ is a closed

filter.   Let F" = {$\mathcal{H}$ ∈ wΘ(X): $\mathcal{G}$ ⊂ $\mathcal{H}$}.

Since F' is closed, there exists a closed set $\hat{F}$ in wX

and a closed filter $\mathcal{J}$ such that F' = $\hat{F}$ ∩ wΘ(X) and

$\hat{F}$ = {$\mathcal{H}$ ∈ wX: $\mathcal{J}$ ⊂ $\mathcal{H}$}.   Easily, $\mathcal{J}$ ⊂ $\mathcal{G}$.

Now F' ⊂ F" and if $\mathcal{H}$ ∈ F" we have that $\mathcal{J}$ ⊂ $\mathcal{G}$ ⊂ $\mathcal{H}$ and

$\mathcal{H}$ ∈ $\hat{F}$ and thus $\mathcal{H}$ ∈ F'.   Therefore, F' = F" and $\mathcal{G}$ is a

minimal balanced closed filter and F' = {$\mathcal{H}$ ∈ wΘ(X): $\mathcal{G}$ ⊂ $\mathcal{H}$}.

To complete the proof, let $\mathcal{G}$ be a minimal balanced

closed filter and F' = {$\mathcal{H}$ ∈ wΘ(X): $\mathcal{G}$ ⊂ $\mathcal{H}$}.   Now F' ≠ ϕ,

and $\hat{F}$ = {$\mathcal{H}$ ∈ wX: $\mathcal{G}$ ⊂ $\mathcal{H}$} is a closed set in wX by Theorem

2.10.   Moreover, F' = $\hat{F}$ ∩ wΘ(X) and thus F' is closed.

4.7 *Corollary.*   *There exists a one-to-one correspon-*

dence between the nonempty closed subsets of wΘ(X) *and*

the minimal balanced closed filters.

Proof.   The correspondence established by Theorem 4.6

is easily one-to-one.

4.8 *Corollary.*   *There exists a one-to-one correspon-*

dence between the nonempty closed subsets of the hyper-

absolute of X *and the open generated closed filters.*

Proof.   Follows immediately by Corollary 4.7 and

Theorem 4.5.

4.9 *Definition.*   An open grill is called a balanced

open grill provided it is the union of all of the open

ultrafilters it contains.

Using the homeomorphism between wX and v(X) given
by $\mathcal{J} \to \mathcal{O}(\mathcal{J})$ and Corollary 4.7 we have the following theorem.

4.10 *Theorem.*   *There exists a one-to-one correspondence*
between the nonempty closed subsets of the hyperabsolute of
X *and the balanced open grills on* X.

4.11 *Definition.*   A filter is called fixed if it has
a nonempty adherence.   A closed filter is called a fixed
minimal balanced closed filter provided it is equal to the
intersection of the family of fixed minimal prime closed
filters that contain it.   An open grill is called a fixed
balanced open grill provided it is equal to the union of
the family of fixed open ultrafilters it contains.

Using precisely the same techniques that were employed
to prove the above theorems we can establish the following
result.

4.12 *Theorem.*   *Let* X *be a* T_1 *topological space and*
$\phi \neq F' \subset wE(X)$.   F' *is closed if and only if there exists*
a fixed minimal balanced closed filter such that
$F' = \{\mathcal{H} \in wE(X): \mathcal{J} \subset \mathcal{H}\}$.

4.13 *Theorem.*   *There exists a one-to-one correspondence*
between the nonempty closed subsets of the absolute of X
and the fixed minimal prime balanced closed filters on X.

4.14 *Theorem.*   *There exists a one-to-one correspondence*
between the nonempty closed subsets of the absolute of X
and the fixed balanced open grills on X.

5. Special Open and Closed Sets in the Prime Wallman Compactification

5.1 *Notation.*   Let X be a T_1 topological space with

F closed in X and O open in X.   Set:

$$F^* = \{K \in w(X): F \in K\} \text{ and } F' = \{P \in v(X): F \in J(P)\}$$

$$O^* = \{K \in w(X): O \in O(K)\} \text{ and } O' = \{P \in v(X): O \in P\}.$$

Recall: {F*: F closed in X} is a base for the closed sets

in w(X).

{O*: O open in X} is a base for the open sets in

w(X).

{F': F closed in X} is a base for the closed sets

in v(X).

{O': O open in X} is a base for the open sets in

v(X).

Recall that f: w(X) → v(X) given by $f(K) = O(K)$ is a homeo-

morphism and $f^{-1}(P) = J(P)$.   Let x ∈ X.   Set M_x = {F: F

closed and x ∈ F} and O_x = {O: O open and x ∈ O}.   Then

M_x is the unique closed ultrafilter converging to x and

O_x is the open neighborhood filter of x, and since X is T_1,

O_x is a minimal prime open filter.

Let O be open in X.   Set

$$\tilde{O} = \{M_x: x \in O\} \text{ and } \tilde{O}' = \{O_x: x \in O\}$$

Let $\hat{O}$ and $\hat{P}'$ be open sets in wX and v(X), respectively.

Set

$$O = \{x: M_x \in \hat{O}\} \text{ and } P = \{x: O_x \in \hat{O}'\}$$

Now $\tilde{X} = \{M_x: x \in X\}$ and $\tilde{X}' = \{O_x: x \in X\}$.   If $\hat{O}$ is an

open set in wX then $\tilde{X} \cap \hat{O} = \tilde{O}$; if $\hat{O}'$ is an open set in v(X)

then $\tilde{X}' \cap \hat{O}' = \tilde{O}'$.   Moreover, $f(\tilde{O}) = \{O(M_x): M_x \in \hat{O}\} =$

$\{O_x: O_x \in f(\hat{O})\} = \tilde{O}'$, where $f(K) = O(K)$ for $K \in$ wX.   If O

is open in X then $\tilde{O} = \tilde{X} \cap O^*$, and hence $\tilde{O}$ is open in $\tilde{X}$.
Similarly, if $\hat{O}$ is open in wX then $\tilde{O} = \tilde{X} \cap \hat{O}$ is open in $\tilde{X}$.

5.2 *Theorem.*

(1) X, $\tilde{X}$, *and* $\tilde{X}'$ *are homeomorphic.*

(2) $\tilde{X}$ *is dense in* wX.

(3) $\tilde{X}'$ *is dense in* v(X).

Proof. Let h: X $\to$ $\tilde{X}$ be defined by $h(x) = \mathcal{M}_x$. h is
a homeomorphism and X and $\tilde{X}'$ are homeomorphic by what has
been shown previously. To see that $\tilde{X}$ is dense in wX, let
$\hat{O}$ be any nonempty open set in wX. Then there exists
$K \in \hat{O}$ and an open set S in X with $K \in S^* \subset \hat{O}$. Now $K \in S^*$
if and only if $X-S \notin K$. Hence $S \neq \phi$. Let $x \in S$. Then
$X-S \notin \mathcal{M}_x$ and $\mathcal{M}_x \in S^* \subset \hat{O}$. Hence $\hat{O} \cap \tilde{X} \neq \phi$ and we have that
$\tilde{X}$ is dense in wX. Similarly, X and $\tilde{X}'$ are homeomorphic;
a corresponding proof establishes statement (3).

5.3 *Theorem. Let* O *be open and* F *closed in* X. *Then:*

(1) $O \subset F$ *iffi* $O^* \subset F^*$ *iffi* $O' \subset F'$.

(2) *If* F *is closed in* wX, $K \in \hat{F}$ *and* $K \subset L \in$ wX *then*
$L \in \hat{F}$.

(3) *If* $\hat{O}$ *is open in* wX *and* $K \in \hat{O}$ *and* $L \in$ wX *with*
$L \subset K$ *then* $L \in \hat{O}$.

(4) *If* $\hat{F}'$ *is closed in* v(X), $P \in \hat{F}'$ *and* $P' \in$ v(X) *with*
$P' \subset P$ *then* $P' \in \hat{F}'$.

(5) *If* $\hat{O}'$ *is open in* v(X) *and* $P \in \hat{O}'$ *with* $P \subset P' \in$ wX
then $P' \in \hat{O}'$.

(6) *Let* O *and* Q *be open in* X. *Then* $O \cap Q = \phi$ *iffi*
$O^* \cap Q^* = \phi$.

(7) *Let* $\hat{Q}$ *be open in* wX *and* $\hat{F}$ = wX-$\hat{Q}$ = $\cap\{F^*\colon F \in \mathcal{J}\}$

and $\tilde{Q}$ = $\{\mathcal{M}_x\colon \mathcal{M}_x \in \hat{Q}\}$.

   Then: (A) $\tilde{Q} \subset Q^*$

              (B) *If* $\mathcal{M}_x \in \tilde{Q}$ *and* K *is a prime closed filter.*

                   with $K \subset \mathcal{M}_x$ *then* $K \in Q^*$.

              (C) *Let* $F \in \mathcal{J}$ *and* O = X-F. *Then* O $\subset$ Q.

   (8) *Let* $\mathcal{J}$ *be a closed filter with empty adherence.*

Set $\hat{F}$ = $\{K \in$ wX: $\mathcal{J} \subset K\}$ *and* $\hat{Q}$ = wX-$\hat{F}$. *Then* $\tilde{Q}$ = $\hat{Q} \cap \tilde{X}$ = $\tilde{X}$.

Moreover, Q* = wX.

   5.4 *Theorem.* *Let* $\hat{O}$ *be open in* wX *and* $\hat{F}$ = $cl_{wX}(\hat{O})$.

Then:

$$\tilde{O} \subset \hat{O} \subset O^* \subset cl_{wX}(\hat{O}).$$

   Proof. Easily $\tilde{O}$ = $\{\mathcal{M}_x\colon \mathcal{M}_x \in \hat{O}\} \subset \hat{O}$. Suppose $K \in \hat{O}$.

Then there exists an open set S in X with $K \in S^* \subset \hat{O}$.

Now $K \in S^*$ implies X-S $\notin K$ and S $\neq \phi$.

   If $K \notin O^*$ then X-O $\in K$. Now S $\subset$ O and X-S $\supset$ X-O

but X-O $\in K$ and X-S $\notin K$ which is impossible. Therefore,

$K \in O^*$ and we have that $\hat{O} \subset O^*$.

   $O^* \subset cl_{wX}(\hat{O})$. Let $\hat{F}$ = $cl_{wX}(\hat{O})$. Then there exists a

closed filter $\mathcal{J}$ such that $\hat{F}$ = $\cap\{F^*\colon F \in \mathcal{J}\}$. Let x $\in$ O and

F $\in \mathcal{J}$. Then $\mathcal{M}_x \in \hat{O} \subset cl_{wX}(\hat{O}) \subset \hat{F}$ = $\cap\{F^*\colon F \in \mathcal{J}\}$ and

$\mathcal{M}_x \in F^*$. Hence F $\in \mathcal{M}_x$ and x $\in$ F. Thus, for each F $\in \mathcal{J}$ we

have O $\subset$ F, and by Theorem 5.3, O* $\subset$ F*. Therefore, O* $\subset$

$\cap\{F^*\colon F \in \mathcal{J}\}$ = $cl_{wX}(\hat{O})$.

   5.5 *Corollary.* *Let* $\hat{O}'$ *be open in* v(X) *and*

$\hat{F}'$ = $cl_{v(X)}(\hat{O}')$. *Then*

$$\tilde{O}' \subset \hat{O}' \subset O' \subset cl_{v(X)}(\hat{O}') = \hat{F}'.$$

5.6 *Corollary.* *Let* O *be a nonempty open set in* wX
and J = {F: F *is closed and* F $\supset$ O}. *Then* $\text{cl}_{wX}(\hat{O})$ =
{K: $J \subset K$} = $\cap$\{F*: F $\in$ J\}.

5.7 *Corollary.* *Let* O *be a nonempty open set in* X *and*
J = {F: F *is closed and* F $\supset$ O}. *Then* $\text{cl}_{wX}(O*)$ =
{K: $J \subset K$} = $\cap$\{F*: F $\in$ J\}.

Recall that a closed set F is called regularly closed
provided F = cl(int(F)). The following theorem characterizes
the closed filters that give rise to the nonempty regularly
closed subsets in wX.

5.8 *Theorem.* *Let* $\hat{F}$ *be a nonempty closed subset in* wX
and J *the corresponding closed filter on* X. *Then* $\hat{F}$ *is*
regularly closed in wX *if and only if there exists an open*
set O *in* X *such that* J = {F: F *is closed and* F $\supset$ O}.

   Proof. We are given that $\hat{F}$ = {K: $J \subset K$} = $\cap$\{F*: F $\in$ J\}.
Suppose $\hat{F}$ is regularly closed. Let $\hat{O}$ = $\text{int}_{wX}(\hat{F})$. Then
$\hat{F}$ = $\text{cl}_{wX}(\hat{O})$. Let $\tilde{O}$ = {M_x: M_x $\in$ $\hat{O}$} and O = {x: M_x $\in$ $\tilde{O}$}.
By Corollary 5.6, J = {F: F is closed and F $\supset$ O}.

   Conversely, suppose that O is a nonempty open set in
X and J = {F: F is closed and F $\supset$ O} and $\hat{F}$ = {K: $J \subset K$} =
$\cap$\{F*: F $\in$ J\}. Always, $\text{cl}_{wX}(\text{int}_{wX}(\hat{F}))$ $\subset$ $\hat{F}$.

   Since F $\supset$ O we have F* $\supset$ O* for each F $\in$ J by Theorem
5.3. Hence O* $\subset$ $\text{int}_{wX}(\hat{F})$. By Corollary 5.7, $\hat{F}$ = $\cap$\{F*:
F $\in$ J\} = $\text{cl}_{wX}(O*)$ $\subset$ $\text{cl}_{wX}(\text{int}_{wX}(\hat{F}))$ $\subset$ $\hat{F}$. Therefore,
$\hat{F}$ = $\text{cl}_{wX}(\text{int}_{wX}(\hat{F}))$, and $\hat{F}$ is regularly closed.

5.9 *Lemma.* *Let* O_1 *and* O_2 *be open sets with* $\overline{O}_1 = \overline{O}_2 \neq \phi$.
Set $J_i = \{F\colon F$ *is closed and* $F \supset O_i\}$ *for* $i = 1,2$. *Then*
$J_1 = J_2$.

Let O be a nonempty open set in X and $J = \{F\colon F$ is closed and $F \supset O\}$. Then $O(J) = \{Q\colon Q$ is open and $X-Q \not\in J\}$ = $\{Q\colon X-Q \not\supset O\}$ = $\{Q\colon Q \cap O \neq \phi\}$. Thus, we have the following characterization of the nonempty regularly closed subsets of $v(X)$, given by Corollary 5.11.

5.10 *Theorem.* *Let* S *be the collection of equivalence classes of the nonempty open sets with the equivalence relation* $O_1 \sim O_2$ *provided* $\overline{O}_1 = \overline{O}_2$. *Then there exists a one-to-one correspondence between* S *and the nonempty regularly closed subsets of* wX.

5.11 *Corollary.* *Let* F' *be a nonempty closed subset of* $v(X)$. *F' is regularly closed in* $v(X)$ *if and only if there exists a nonempty open set O in X and an open grill* G *defined by* $G = \{Q\colon Q$ *open in X and* $O \cap Q \neq \phi\}$ *such that* $F' = \{P \in v(X)\colon P \subset G\} = \{P \in v(X)\colon P \cap O \neq \phi$ *for each* $P \in P\}$.

5.12 *Theorem.* *Let* $\phi \neq M \neq X$, *with M both open and closed. Let* $J = \{F\colon F$ *is closed and* $F \supset M\}$. *Set* $\hat{F} = \{K \in wX\colon J \subset K\} = \cap\{F^*\colon F \in J\}$. *Then* $\phi \neq \hat{F} \neq wX$ *and* $\hat{F}$ *is both open and closed in* wX.

Proof. Easily $\phi \neq \hat{F} \neq wX$ and $\hat{F}$ is closed. By Corollary 5.7, $\hat{F} = cl_{wX}(M^*)$, where, since M is open, $M^* = \{K \in wX\colon M \in O(K)\} = \{K \in wX\colon X-M \not\in K\}$ and since M

is closed, $M^* = \{K \in wX: M \in K\}$ which is closed in wX.
Hence $M^* = cl_{wX}(M^*)$. Thus, F is both open and closed.

5.13 *Theorem.* *Let* $\hat{F}$ *be both open and closed in* wX *with*
$\phi \neq \hat{F} \neq wX$. *Let* $\tilde{F} = \{M_X: M_X \in \hat{F}\}$ *and* $F = \{x: M_X \in \tilde{F}\}$.
Then $\phi \neq F \neq X$ *and* F *is both open and closed.*

Proof. Since $\hat{F}$ is a nonempty open set it follows
that F is a nonempty open set. Since $\hat{F} \neq wX$, we have that
$F \neq X$. Since $\tilde{X}$ is dense in wX it follows that $F \neq \phi$.
Let $\hat{G} = wX - \hat{F}$. Then G is open in X and $F = X - G$ is closed.

5.14 *Theorem.* X *is connected if and only if* wX *is*
connected.

By Theorem 5.3, we have that if O is open and F is
closed with $O \subset F$ then $O^* \subset F^*$. Using this result, we
are able to characterize the nowhere dense subsets of wX.

5.15 *Theorem.* *Let* $\hat{F}$ *be a nonempty closed subset of*
wX *and* $\mathcal{J}$ *the closed filter on* X *such that* $\hat{F} = \{K \in wX:$
$\mathcal{J} \subset K\} = \cap\{F^*: F \in \mathcal{J}\}$. *If there exists a nonempty open*
set O *in* X *with* $O \subset F$ *for each* $F \in \mathcal{J}$ *then* $O^* \subset \hat{F}$.

5.16 *Definition.* A closed filter $\mathcal{J}$ on X will be
called nowhere dense if for each nonempty open set O there
exists $F \in \mathcal{J}$ with $O \not\subset F$.

5.17 *Theorem.* *Let* $\hat{F}$ *be a nonempty closed subset of*
wX *and* $\mathcal{J}$ *the generating closed filter.* $\hat{F}$ *is nowhere dense*
if and only if $\mathcal{J}$ *is nowhere dense.*

5.18 *Theorem.* wX *is second category if and only if*
given any countable family $\{\mathcal{J}_i: i \in N\}$ *of nowhere dense*
closed filters there exists a prime closed filter K *such*
that $\mathcal{J}_i \not\subset K$ *for each* i ∈ N.

5.19 *Theorem.* wX *is a Baire space if and only if*
for each countable family $\{\mathcal{J}_i: i \in N\}$ *of nowhere dense*
closed filters there exists a closed filter $\mathcal{J}$ *such that*
$\{K \in wX: \mathcal{J} \not\subset K\} \subset \cup\{\mathcal{J}_i: i \in N\}$.

5.20 *Theorem.*   *Let* $\hat{F}$ *be a nonempty closed subset of*
wX *generated by the closed filter* $\mathcal{J}$. F *is a* G_δ *closed set*
if and only if there exists a countable family $\{\mathcal{J}_i: i \in N\}$
of closed filters such that for each $K \in wX$ *we have* $\mathcal{J} \subset K$
if and only if $\mathcal{J}_i \not\subset K$ *for each* i ∈ N.

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