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by

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HOMEOMORPHISMS OF COMPOSANTS IN KNASTER CONTINUA

W. Debski and E. D. Tymchatyn[1]

1. Introduction

H. Cook classified the solenoids in [2]. He showed that there exists a family $\mathcal{S}$ if solenoids such that $\mathcal{S}$ has cardinaltiy c and no two distinct members of $\mathcal{S}$ are homeomorphic. Recently Debski [4] (see also Watkins [9]) gave a similar classification of the simplest Knaster indecomposable continua. However, realtively little is known about individual composants of Knaster continua and of solenoids.

Two composants M and L of an indecomposable continuum K are said to be in the *same position* if there exists a homeomorphism $g: K \to K$ such that $g(M) = L$. It is obvious for example that every pair of composants of a homogeneous indecomposable continuum are in the same position. Hence, every pair of composants of the pseudo arc are in the same position.

Bellamy [1] described a homeomorphism $h: K_{\underline{2}} \to K_{\underline{2}}$ of Knaster's dyadic indecomposable continuum which fixes exactly two composants of $K_{\underline{2}}$. Debski [5] showed that two composants L and M of $K_{\underline{2}}$ are in the same position if and only if there exists an integer n such that $h^n(L) = M$. He obtained analogous results for the other simplest Knaster indecomposable continua. In particular, for each Knaster

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indecomposable continuum K and each composant L of K there exist at most countably many composants of K with the same position as L.

It is our purpose in this paper to make a preliminary investigation of composants of solenoids and Knaster continua and to compile a short list of problems.

2. Preliminaries

All spaces considered in this paper are separable and metric. A *continuum* is a compact, connected, metric space. A *continuum* is *indecomposable* if it is not the union of two proper subcontinua. If $p \in X$ and X is a continuum then the *composant* of p in X is the union of all proper subcontinua of X which contain p. If X is an indecomposable metric continuum the composants of X are pairwise disjoint and dense in X and X has c composants [8].

Let R be the topological group of real numbers with addition. Let Z be the subgroup of integers in R. Let $\pi: R \to R/Z$ be the natural homomorphism of R onto the quotient group R/Z. Then R/Z is topologically isomorphic to the unit circle in the complex plane.

Let $\underline{n} = \{n_i\}_{i=1}^{\infty}$ be a sequence of integers greater than 1. For each i let $R_i = R$, $Z_i = Z$, $R/Z_i = R/Z$, $\pi = \pi_i$: $R_i \to R/Z_i$ and let $n_i: R_{i+1} \to R_i$ be the homeomorphism given by $n_i(x) = n_i x$. We have the cummutative diagram

$$\begin{array}{ccccccccc}
Z_1 & \overset{n_1}{\leftarrow} & Z_2 & \overset{n_2}{\leftarrow} & \cdots & Z_i & \leftarrow & \cdots & \\
\cap & & \cap & & & \cap & & \phi_i & \\
R_1 & \overset{n_1}{\leftarrow} & R_2 & \overset{n_2}{\leftarrow} & \cdots & R_i & \leftarrow & & L_{\underline{n}} \\
\pi_1\downarrow & & \pi_2\downarrow & & & \pi_i\downarrow & & \Psi_i & \pi_\infty\downarrow \\
R/Z_1 & \overset{n_1}{\leftarrow} & R/Z_2 & \overset{n_2}{\leftarrow} & \cdots & R/Z_i & \leftarrow & \cdots & S_{\underline{n}} \supset C_0
\end{array}$$

Let $L_{\underline{n}} = \varprojlim(R_i, n_i)$ and $S_{\underline{n}} = \varprojlim(R/Z_i, n_i)$ be the inverse limits [6]. Let $\phi_i\colon L_{\underline{n}} \to R_i$ and $\Psi_i\colon S_{\underline{n}} \to R/Z_i$ be the natural projections of the inverse limit space to the coordinate space. Since each $n_i\colon R/Z_i \to R/Z_i$ is a topological group homomorphism of the compact abelian group R/Z_i we have $S_{\underline{n}}$ is a compact abelian topological group called a solenoid. Since each $n_i\colon R_{i+1} \to R_i$ is a topological group isomorphism $L_{\underline{n}}$ is a topological group isomorphic to R. Let $\pi_\infty = \varprojlim \pi_i\colon L_{\underline{n}} \to S_{\underline{n}}$ be the induced map. Then C_0, the composant of the identity element 0 in $S_{\underline{n}}$ is the one to one continuous image of $L_{\underline{n}}$ under π_∞. For each i let

$$Z_i' = \pi_\infty \circ \phi_i^{-1}(Z_i).$$

Note that Z_i' is a topological group in $C_0 \subset S_{\underline{n}}$ that is group isomorphic to Z.

To describe the topology of the topological group $S_{\underline{n}}$ it suffices to describe a neighbourhood basis at the identity element 0 of $S_{\underline{n}}$.

Note that $\Psi_i^{-1}(0)$ is a Cantor set (i.e. a zero-dimensional, compact set without isolated points) and $\Psi_i^{-1}(0) \cap C_0 = Z_i'$. Also, Z_i' is a countable dense set in

$\Psi_i^{-1}(0)$. Hence, $Z_i^!$ is homeomorphic to the set of rational numbers.

Since $S_{\underline{n}}$ is an inverse limit of the simple closed curves R/Z_i a basic neighbourhood of 0 in $S_{\underline{n}}$ is of the form $\Psi_i^{-1}(U_i)$ where U_i is an open interval which is a basic neighbourhood of $\Psi_i(0)$ in the simple closed curve R/Z_i. Let V_i be the open interval in R_i about 0 which projects by π_i one to one onto U_i. Then $\Psi_i^{-1}(U_i)$ is the algebraic sum of the interval $\pi_\infty \circ \phi_i^{-1}(V_i)$ (without endpoints and containing 0) in C_0 and the Cantor set $\Psi_i^{-1}(0)$. Hence, $\Psi_i^{-1}(U_i) \cap C_0$ is a basic neighbourhood of 0 in C_0 and is the algebraic sum of the interval $\pi_\infty \circ \phi_i^{-1}(V_i)$ with $Z_i^! = \Psi_i^{-1}(0) \cap C_0$.

To describe the topology of the topological group $Z_i^!$ it suffices to describe a neighbourhood basis at the identity element 0. A basic neighbourhood of 0 in $Z_i^!$ is

$$\Psi_{i+j}^{-1}(U_{i+j}) \cap Z_i^! = (\pi_\infty \circ \phi_{i+j}^{-1}(V_{i+j}) + Z_{i+j}^!) \cap Z_i^!$$
$$= Z_{i+j}^! = n_i \cdot n_{i+1} \cdot \cdots n_{i+j-1} Z_i^!$$

for all sufficiently small basic open neighbourhoods U_{i+j} of $\Psi_{i+j}(0)$ in R/Z_{i+j} and V_{i+j} the component of 0 in $\pi_{i+j}^{-1}(U_{i+j})$. Now, $Z_i^!$ is a topological group which is group isomorphic to the group Z by $\xi_i\colon Z \to Z_i^!$. If we give Z the topology with basic neighbourhoods of the identity having the form $n_i \cdot n_{i+1} \cdot \cdots n_{i+j} Z$ then ξ_i becomes a topological group isomorphism.

3. Homeomorphisms of Composants and Maps of Integers

Let $\underline{n} = \{n_i\}_{i=1}^{\infty}$ and $\underline{m} = \{m_i\}_{i=1}^{\infty}$ be two sequences of integers greater than 1. Let C_0 be the composant of the identity 0 in $S_{\underline{n}}$ and let $\tilde{C}_0$ be the composant of the identity 0 in $S_{\underline{m}}$. Let $Z_i' = C_0 \cap \Psi_i^{-1}(0)$ and $\tilde{Z}_i = \tilde{C}_0 \cap \Psi_i^{-1}(0)$.

We will establish a correspondence between the set of homeomorphisms of C_0 onto $\tilde{C}_0$ and the set of all one to one, order preserving, open continuous mappings of some Z_j' to $\tilde{Z}_i'$ (i and j are not fixed).

Let $h: C_0 \to \tilde{C}_0$ be a homeomorphism. Since $\tilde{C}_0$ is homogeneous we may suppose h carries the identity 0 of C_0 onto the identity of $\tilde{C}_0$. Since the mapping $x \to -x$ is a homeomorphism of $\tilde{C}_0$ we may suppose h preserves order. We show that h can be used to define a one to one, order preserving, open, continuous function $f: Z_j' \to \tilde{Z}_i$ for some i and j.

Let $V_0 + Z_1'$ be a small basic open neighbourhood of the identity in C_0. Since h is open there is a basic open neighbourhood $U + \tilde{Z}_i'$ of the identity in $\tilde{C}_0$ such that $U + \tilde{Z}_i' \subset h(V_0 + Z_1')$. Since h is continuous there exists a neighbourhood $V_1 + Z_j' \subset V_0 + Z_1'$ of the identity in C_0 such that $h(V_1 + Z_j') \subset U + \tilde{Z}_i'$. Then

$$Z_j' \subset V_1 + Z_j' \overset{h}{\to} U + \tilde{Z}_i' \overset{p}{\to} \tilde{Z}_i'$$

where p is the second coordinate projection. Then p is open. Now, $h' = p \circ h|Z_j'$ maps Z_j' to $\tilde{Z}_i'$. Since $p \circ h$: $V_1 + Z_j' \to \tilde{Z}_i'$ is open and $p \circ h$ factors through h' it follows that h' is open. Since p and h are order preserving h' is order preserving. Finally, since

$$h \circ h^{-1}|U + \tilde{Z}_i': U + \tilde{Z}_i' \to U + \tilde{Z}_i'$$

is the identity and distinct components of $V_1 + Z_j'$ lie in distinct components of $V_0 + Z_j'$ it follows that h' is one to one. Hence, h defines a one to one, order preserving, continuous, open mapping $h': Z_j' \to \tilde{Z}_i'$.

We shall need the following proposition:

Proposition 3.1. *If* $f: Z_j' \to \tilde{Z}_i'$ *is a one to one, order preserving continuous, open function then there exists an integer* k *such that* $f(a+1) - f(a) \leq k$ *for all* $a \in Z_j'$.

Proof. Since f is open there exists an integer q such that $\tilde{Z}_{i+q}' \subset f(Z_j')$. Now,

$$\tilde{Z}_{i+q}' = m_i \cdot m_2 \cdot \cdots m_{i+q-1} \tilde{Z}_1'.$$

Let $k = m_1 \cdot m_2 \cdot \cdots m_{i+q-1}$. Then for $a \in Z_j'$ $f(a+1) - f(a) \leq k$ since f is order preserving.

Now, suppose $f: Z_j' \to \tilde{Z}_i'$ is a one to one, order preserving, continuous, open function. Since $\tilde{Z}_i'$ is homogeneous we may suppose $f(0) = 0$. We denote by 1 a generator of Z_j'. Let $g: C_0 \to \tilde{C}_0$ be the linear extension of f, i.e. $g(x) = f(k) + (f(k+1) - f(k))(x-k)$ if $k \leq x \leq k+1$ for $k \in Z_j'$. Clearly, g is one to one, preserves order and carries C_0 onto $\tilde{C}_0$. We must prove g is continuous and open.

Define $r: Z_j' \to \tilde{Z}_i'$ by $r(a) = f(a+1) - f(a)$. By Proposition 3.1 $r(Z_j')$ is a finite discrete set. Since r is continuous r is a locally constant function. Also, $g(x) = f(k) + r(k)(x-k)$ where $k \leq x \leq k+1$, $k \in Z_j'$.

Let $x \in C_0 \setminus Z_j'$. Then $k < x < k+1$ for some $k \in Z_j'$. Let U be a basic open neighbourhood of $g(x)$ in $\tilde{C}_0$. Then $U = (-\varepsilon, \varepsilon) + g(x) + \tilde{Z}_{i+q_1}'$ for some $\varepsilon > 0$ and some positive integer q_1. Since f is continuous and $f(0) = 0$ there exists a positive integer q_2 such that the neighbourhood $k + Z_{j+q_2}'$ of k in Z_j' maps into the neighbourhood

$f(k) + \tilde{Z}'_{i+q_1}$ of $f(k)$ in $\tilde{C}_0$. We may also suppose r is constant on $k + Z'_{j+q_2}$. Then $V = (-\frac{\varepsilon}{r(k)}, \frac{\varepsilon}{r(k)}) + x + Z'_{j+q_2}$ is a neighbourhood of x in C_0. We have

$$\begin{aligned} g(V) &= f(k + Z'_{j+q_2}) + [(-\tfrac{\varepsilon}{r(k)} + x - k, \tfrac{\varepsilon}{r(k)} + x - k)]r(k) \\ &\subset f(k) + Z'_{i+q_1} + (-\varepsilon,\varepsilon) + (x - k)r(k) \\ &= (-\varepsilon,\varepsilon) + g(x) + \tilde{Z}'_{i+q_1} \end{aligned}$$

and $g(V)$ is open.

If $x \in Z'_j$ let $U = (-\varepsilon,\varepsilon) + f(x) + \tilde{Z}'_{i+q_1}$, $U_- = (-\varepsilon,0] + f(x) + \tilde{Z}'_{i+q_1}$ and $U_+ = [0,\varepsilon) + f(x) + \tilde{Z}'_{i+q_1}$. One can then carry through the above argument for each of U_- and U_+ to get an open neighbourhood V of x such that $g(V)$ is open and contained in U.

This completes the proof that g is both open and continuous. So g is the required homeomorphism of Z'_j into $\tilde{Z}'_i$.

Theorem 3.2. *If* h, g: $C_0 \to \tilde{C}_0$ *are homeomorphisms and* $h - g$ *is bounded then* h *is homotopic to* g.

Proof. Suppose $-a \leq h(x) - g(x) \leq a$ for all $x \in C_0$. Let H: $C_0 \times I \to \tilde{C}_0$ be the linear homotopy

$$H(x,t) = H_t(x) = (1 - t)h(x) + tg(x) = h(x) + t(g - h)(x).$$

To prove H is an homotopy from h to g it clearly suffices to show that the function H_t is a continuous function for each $t \in I$.

Since $g - h$ is bounded and $t \in I$ $t(g - h)$ is bounded. Since the topology on each bounded interval of $\tilde{C}_0$ is the usual topology it follows that $t(g - h)$ is continuous since $g - h$ is continuous. It follows that H_t is continuous since it is the sum of two continuous functions.

If $x < y$ in C_0 then $h(x) < h(y)$ and $g(x) < g(y)$ in $\tilde{C}_0$ since h and g are one to one and order preserving. Hence,

$$H_t(x) = (1 - t)h(x) + tg(x) < (1 - t)h(y) + tg(y) = H_t(y).$$

We have proved that H_t is one to one and order preserving. The theorem is proved.

Proposition 3.3. *If* h, g: $C_0 \to \tilde{C}_0$ *are homeomorphisms and* H: $C_0 \times I \to \tilde{C}_0$ *is a homotopy from* h *to* g *then* $h - g$ *is bounded.*

Proof. The set $H(\{0\} \times I)$ is an arc of some length a. Let V be a neighbourhood of $H(\{0\} \times I)$ in $\tilde{C}_0$ whose components have length less than 2a. Since $\{0\}$ is compact there exists a neighbourhood U of 0 in C_0 such that $H(U \times I) \subset V$. Hence, for $x \in U$ $h(x)$ and $g(x)$ lie in an arc in V. So $(h - g)|U$ is bounded. Let i be an integer with $Z_i' \subset U$. Then $(h - g)|Z_i'$ is bounded.

Since h is a homeomorphism $\{h(i + 1) - h(i) | i \in Z_i'\}$ is bounded by Proposition 3.1. Let 1 be a generator of Z_i'. If $z \in C_0$ then $j \leq x < j + 1$ for some $j \in Z_i'$. So $h(j) \leq h(x) < h(j + 1)$ and $g(j) \leq g(x) < g(j + 1)$. Thus, $h - g$ is bounded.

Corollary 3.4. *If* h, g: $C_0 \to \tilde{C}_0$ *are homeomorphisms and* h *is homotopic to* g *then* h^{-1} *is homotopic to* g^{-1}.

Proof. We may suppose h is order preserving. By Proposition 3.1 there exists an integer k such that

$$h^{-1}(j + 1) - h^{-1}(j) \leq k \text{ for } j \in Z_1'.$$

Let $a \in Z_1'$ such that

$$-a \leq h(x) - g(x) \leq a$$

for $x \in C_0$ by Proposition 3.3.

For $y \in \tilde{C}_0$

$$h^{-1}(h(g^{-1}(a))) - h^{-1}(g(g^{-1}(a))) = g^{-1}(a) - h^{-1}(a).$$

So $-ak \leq g^{-1}(a) - h^{-1}(a) \leq ak$.

Theorem 3.5. *If* h, g: $C_0 \to \tilde{C}_0$ *are homeomorphisms such that* h - g *is bounded then* h *is isotopic to* g.

Proof. By 3.3 and 3.4 there exists a number a such that

$$-a \leq h(x) - g(x) \leq a \text{ for } x \in C_0$$

and

$$-a \leq h^{-1}(y) - g^{-1}(y) \leq a \text{ for } y \in \tilde{C}_0.$$

Let H: $C_0 \times I \to \tilde{C}_0$ be the linear homotopy from h to g defined in 3.2. It was proved in 3.2 that each $H_t: C_0 \to \tilde{C}_0$ is continuous, one to one and order preserving. It remains to prove that H_t is open.

Let $x_0 \in C_0$.

Let W be a basic open neighbourhood of 0 in $\tilde{C}_0$ such that components of $\tilde{C}_0 \setminus W$ have length greater than 10a. Let U be a basic open neighbourhood of 0 in C_0 such that components of $C_0 \setminus U$ have length greater than 10a and since h and g are continuous $h(U + x_0) \subset W + h(x_0)$ and $g(U + x_0) \subset W + g(x_0)$. Since h and g are open there exists $V \subset W$ a basic open neighbourhood of 0 in $\tilde{C}_0$ such that

$V + h(x_0) \subset h(U + x_0)$ and $V + g(x_0) \subset g(U + x_0)$. We prove $H_t(U + x_0) \supset V + H_t(x_0)$.

Define homeomorphisms

$$h': C_0 \to \tilde{C}_0 \text{ and } g': C_0 \to \tilde{C}_0$$

by $h'(x) = h(x + x_0) - h(x_0) \in W$ and $g'(x) = g(x + x_0) - g(x_0) \in W$ for $x \in C_0$. Then $-2a < h'(x) - g'(x) < 2a$ for $x \in C_0$. Hence, if $x \in U$ then $h'(x)$ and $g'(x)$ lie in the same component of W and $h(x + x_0)$ and $g(x + x_0)$ lie in the same component of $W + h(x_0)$. Let $y \in V$ then $y + h(x_0) = h(x + x_0)$ and $y + g(x_0) = g(x' + x_0)$ for some $x, x' \in U$. Hence, $x + x_0$ and $x' + x_0$ lie in the same component of $U + x_0$. Thus, x and x' lie in the same component of U. We may suppose $x < x'$ and $h(x + x_0) < g(x' + x_0)$. Then, $h(x_0) < g(x_0)$. It follows that $g(x' + x_0) < g(x + x_0)$ and $h(x' + x_0) < h(x + x_0)$ since g and h are order preserving. Now,

$$\begin{aligned} H_t(x + x_0) &= (1 - t)h(x + x_0) + t\,g(x + x_0) \\ &= (1 - t)(y + h(x_0)) + t(g(x' + x_0) - t(g(x' + x_0) \\ &\quad - g(x + x_0))) \\ &= (1 - t)(y + h(x_0)) + t(y + g(x_0)) - t(g(x' + x_0) \\ &\quad - g(x + x_0)) \\ &= y + H_t(x_0) - t(g(x' + x_0) - g(x + x_0)) < y \\ &\quad + H_t(x_0). \end{aligned}$$

Similarly, $H_t(x' + x_0) > y + H_t(x_0)$.

Since x and x' are contained in an arc in U it follows that $H_t(x'') = y + H_t(x_0)$ for some $x'' \in U$. Hence, $V + H_t(x_0) \subset H_t(U + x_0)$ and the theorem is proved.

Theorem 3.6. *Let* h: $C_0 \to \tilde{C}_0$ *be a homeomorphism,* f: $Z'_j \to \tilde{Z}'_i$ *is a continuous, one to one, order preserving, open mapping induced by* h *and* g: $C_0 \to \tilde{C}_0$ *is the homeomorphism of* C_0 *onto* $\tilde{C}_0$ *induced by* f *as in the paragraph following Proposition* 3.1. *Then* h *is isotopic to* g.

Proof. We proved that $h|Z'_j - f$ is bounded. Let 1 be a generator of Z'_j. If $x \in C_0$ then $k < x < k + 1$ for some $k \in Z'_j$. Now,

$$h(k) < h(x) < h(k + 1) \text{ and } f(k) < g(k) < f(k + 1).$$

Hence, h - g is bounded. The theorem now follows by Theorem 3.5.

A homeomorphism h: $C_0 \to \tilde{C}_0$ is said to be *regular* if there exists a linear homeomorphism g: $C_0 \to \tilde{C}_0$ such that h is homotopic to g.

Remark 3.7. If g, h: $C_0 \to \tilde{C}_0$ are homotopic linear homeomorphisms then f - g is constant.

Remark 3.8. If g: $C_0 \to \tilde{C}_0$ is a linear map then g is uniformly continuous and, hence, g extends to a linear map $\bar{g}$: $S_{\underline{n}} \to S_{\underline{m}}$. Similarly, g^{-1} extends to a linear map $\overline{g^{-1}}$: $S_{\underline{m}} \to S_{\underline{n}}$. Then $\overline{g^{-1}} \circ \bar{g}$ is the identity on C_0. By continuity it is the identity on $S_{\underline{n}}$. So $\bar{g}$ is one to one. Hence, since $S_{\underline{n}}$ is compact $\bar{g}$ is a homeomorphism.

4. Lifting Homeomorphisms of Composants of $K_{\underline{n}}$

Let $S_{\underline{n}}$ be a solenoid. Let $\mathcal{N}_{\underline{n}} = \mathcal{N}$: $S_{\underline{n}} \to K_{\underline{n}}$ be the quotient map onto the quotient space $K_{\underline{n}}$ where point inverses under $\mathcal{N}_{\underline{n}}$ are the pairs $\{x,-x\}$ for $x \in S_{\underline{n}}$. The decomposition of $S_{\underline{n}}$ into $\{\{x,-x\}: x \in S_{\underline{n}}\}$ is upper semi-continuous so $K_{\underline{n}}$

is a continuum. We call $K_{\underline{n}}$ a *simplest Knaster indecomposable continuum*. The map $\mathcal{N}_{\underline{n}}$ folds the composant C_0 of 0 in $S_{\underline{n}}$ so $\mathcal{N}_{\underline{n}}(C_0)$ is the one to one continuous image of $[0,\infty)$ and $\mathcal{N}_{\underline{n}}(0)$ is an end point of $K_{\underline{n}}$. If all but finitely many of the integers $\{n_i\}_{i=1}^{\infty}$ are odd then $\mathcal{N}_n$ also folds the composant of the point $(a_1,\cdots,a_r,\pi,\pi,\pi,\cdots) = a$ so that $\mathcal{N}_{\underline{n}}(a)$ is also an endpoint of $K_{\underline{n}}$. All other composants of $S_{\underline{n}}$ are mapped one to one onto composants of $K_{\underline{n}}$.

Let D be a composant of $K_{\underline{n}}$ which is the one to one continuous image of a line. Give D some orientation.

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in D which converges to a point $x \in D$. Let $V + Z_k'$ be a basic neighbourhood of 0 such that the closure of $V + Z_k' + x$ does not contain an endpoint of $K_{\underline{n}}$. We may suppose each $x_i \in V + Z_k' + x$. We decompose the sequence $\{x_i\}_{i=1}^{\infty}$ into three disjoint subsequences $\{x_{1,j}\}_{j=1}^{\infty}$, $\{x_{2,j}\}_{j=2}^{\infty}$ and $\{x_{3,j}\}_{j=1}^{\infty}$. The sequence $\{x_{1,j}\}_{j=1}^{\infty}$ is a sequence in a compact interval of D. We call such a sequence a type I sequence. The orientation of the component of $V + Z_k' + x$ containing $x_{2,j}$ is the same as that of the component of $V + x$ for large j. Such a sequence is called a type II sequence. The orientation of the component of $V + Z_k' + x$ containing $x_{3,j}$ is opposite to that of the component of $V + x$ for large j. Such a sequence will be called a type III sequence. Such a decomposition is called a decomposition of type (t). Any two such divisions of $\{x_i\}_{i=1}^{\infty}$ differ in at most finitely many elements. If ϕ is a homeomorphism of D onto a composant $\tilde{D}$ of $\tilde{K}_m$ then $\{\phi(x_{1,j})\}$, $\{\phi(x_{2,j})\}$ and $\{\phi(x_{3,j})\}$ is a decomposition of $\{\phi(x_j)\}$ of type (†). Hence, this division is topological.

Let C be a composant of $S_{\underline{n}}$ such that $\mathcal{N}_{\underline{n}}(C) = D$. Let -C denote the inverse component to C. Let $\{y_i\}_{i=1}^{\infty}$ be a sequence in C ∪ (-C) which converges to y ∈ C. Then the sequence $\{y_i\}_{i=1}^{\infty}$ may be decomposed into three disjoint sequences $\{y_{1,j}\}_{j=1}^{\infty}$, $\{y_{2,j}\}_{j=1}^{\infty}$ and $\{y_{3,j}\}_{j=1}^{\infty}$ such that the sequence $\{y_{1,j}\}$ is contained in a compact interval of C. We call $\{y_{1,j}\}$ a type I sequence. Each subsequence of the sequence $\{y_{2,j}\}$ is an unbounded sequence of C. We call $\{y_{2,j}\}$ a type II sequence. Each subsequence of $\{y_{3,j}\}$ is an unbounded sequence in -C. We call $\{y_{3,j}\}$ a type III sequence. Two such divisions of $\{y_n\}$ differ in at most a finite number of elements.

The composant C is the one to one image of a line so we can assign to it an orientation. This orientation is continuous on C since C projects by small open maps to a circle. Note that the orientation on C can be extended continuously to each composant of $S_{\underline{n}}$.

Next we show that $\{\mathcal{N}_{\underline{n}}(y_{1,j})\}$, $\{\mathcal{N}_{\underline{n}}(y_{2,j})\}$ and $\{\mathcal{N}_{\underline{n}}(y_{3,j})\}$ is a decomposition of the sequence $\{\mathcal{N}_{\underline{n}}(y_i)\}$ of the type (†). That $\{\mathcal{N}_{\underline{n}}(y_{1,j})\}$ lies in a compact interval in D is clear. Clearly, also, no subsequence of $\{\mathcal{N}_{\underline{n}}(y_{2,j})\}$ or $\{\mathcal{N}_{\underline{n}}(y_{3,j})\}$ is contained in a bounded interval of D. That each small interval about a point $\mathcal{N}_{\underline{n}}(y_{2,j})$ for large j has the same orientation in D as a small interval in D containing $\mathcal{N}_{\underline{n}}(y)$ follows from the fact that intervals close to each other in C have the same orientation in C and $\mathcal{N}_{\underline{n}}$ is continuous.

Note that the orientation of $D = \mathcal{N}_{\underline{n}}(-C) = \mathcal{N}_{\underline{n}}(C)$ introduced from C by $\mathcal{N}_{\underline{n}}$ is opposite to the orientation introduced from $-C$ by $\mathcal{N}_{\underline{n}}$. It follows that for large j an interval in D containing $\mathcal{N}_{\underline{n}}(y_{3,j})$ has opposite orientation to that of an interval in D containing $\mathcal{N}_{\underline{n}}(y)$. Hence, $\{\mathcal{N}_{\underline{n}}(y_{3,j})\}$ is a type III sequence.

Theorem 4.1. *Suppose* $K_{\underline{n}}$ *and* $K_{\underset{\sim}{\underline{m}}}$ *are simplest Knaster indecomposable continua and* $h\colon D \to \tilde{D}$ *is a homeomorphism of a composant* D *of* $K_{\underline{n}}$ *without an endpoint onto a composant* $\tilde{D}$ *of* $K_{\underline{m}}$. *Let* C *and* $-C$ *be the composants of* $S_{\underline{n}}$ *which project by* $\mathcal{N}_{\underline{n}}$ *onto* D *and let* $\tilde{C}$ *and* $-\tilde{C}$ *be the composants of* $S_{\underline{m}}$ *which project by* $\mathcal{N}_{\underline{m}}$ *onto* $\tilde{D}$. *Then* h *can be lifted uniquely to a homeomorphism* $\bar{h}\colon C \cup (-C) \to \tilde{C} \cup (-\tilde{C})$ *such that* $\bar{h}(C) = \tilde{C}$.

Proof. The existence of a unique one to one function $\bar{h}\colon C \cup (-C) \to \tilde{C} \cup (-\tilde{C})$ such that $\bar{h}(C) = \tilde{C}$ and $\mathcal{N}_{\underline{m}} \circ \bar{h} = h \circ \mathcal{N}_{\underline{n}}$ is clear. Note that $\bar{h}(-x) = -\bar{h}(x)$ for $x \in C \cup (-C)$.

We must prove that $\bar{h}$ is a homeomorphism. It suffices to prove $\bar{h}$ is continuous.

Let $y \in C \cup (-C)$ and let $\{y_i\}$ be a sequence in $C \cup (-C)$ which converges to x. Without loss of generality $y \in C$. Note that the sequence $\{\bar{h}(y_i)\}$ has at most two limit points in $S_{\underline{m}}$ namely $\bar{h}(y)$ and $\bar{h}(-y)$ since $\{\mathcal{N}_{\underline{m}} \circ \bar{h}(y_i)\}$ converges to $h \circ \mathcal{N}_{\underline{n}}(y)$ by commutativity. We shall prove $\lim \bar{h}(y_i) = \bar{h}(y)$.

Let $\{y_{1,j}\}$, $\{y_{2,j}\}$ and $\{y_{3,j}\}$ be a decomposition of the sequence $\{y_i\}$ into type I, type II and type III sequences respectively. Then $\{\bar{h}(y_{1,j})\}$ converges to $\bar{h}(y)$ since $\bar{h}$ carries a bounded sequence in C to a bounded sequence in $\tilde{C}$.

The type II sequence $\{y_{2,j}\}$ goes to the sequence $\{h \circ \mathcal{N}_{\underline{n}}(y_{2,j})\}$ which is a type II sequence in $\tilde{D}$ since both $\mathcal{N}_{\underline{n}}$ and h preserve type II sequences. If $\{\bar{h}(y_{2,j})\}$ were to converge to $\bar{h}(-y) \in -\tilde{C}$ then it would be a type III sequence converging to $\bar{h}(-y)$. But $\mathcal{N}_{\underline{m}}$ preserves type III sequences. Hence, $\{\bar{h}(y_{2,j})\}$ converges to $\bar{h}(y)$. Similarly, $\{\bar{h}(y_{3,j})\}$ converges to $\bar{h}(y)$. The theorem is proved.

5. Regular Homeomorphisms of Compasants of Knaster Continua

Let $K_{\underline{n}}$ and $K_{\underline{m}}$ be simplest Knaster indecomposable continua. Let $D \subset K_{\underline{n}}$ and $\tilde{D} \subset K_{\underline{m}}$ be composants without endpoints. Let $h: D \to \tilde{D}$ be a homeomorphism. In section 4 we proved that h lifts to a homeomorphism

$$\bar{h}: C \cup (-C) \to \tilde{C} \cup (-\tilde{C})$$

where C is a composant of $S_{\underline{n}}$ and $\tilde{C}$ is a composant of $S_{\underline{m}}$ and $\bar{h}(c) = \tilde{C}$.

Let C_0 be the composant of the identity 0 in $S_{\underline{n}}$ and let $\tilde{C}_0$ be the composant of the identity in $S_{\underline{m}}$. Let $a \in C$. For $x \in C_0$ define

$$g(x) = \bar{h}(x + a) - \bar{h}(a).$$

Then $g(0) = 0$. Clearly, $g: C_0 \to \tilde{C}_0$ is a homeomorphism. We say $h: D \to \tilde{D}$ is *regular* if $g: C_0 \to \tilde{C}_0$ is regular. Notice that this definition is independent of the choice of a and of the lifting $\bar{h}$.

Theorem 5.1. *If* $h: D \to \tilde{D}$ *is a regular homeomorphism then* $K_{\underline{n}}$ *and* $K_{\underline{m}}$ *are homeomorphic and* D *and* $\tilde{D}$ *are in the same position.*

Proof. Let C (resp. $\tilde{C}$) be a composant of $S_{\underline{n}}$ (resp. $S_{\underline{m}}$) such that $\mathcal{N}_{\underline{n}}(C) = D$ (resp. $\mathcal{N}_{\underline{m}}(\tilde{C}) = \tilde{D}$). Let $\bar{h}\colon C \cup (-C) \to \tilde{C} \cup (-\tilde{C})$ be a lifting of h so $\bar{h}(C) = C$. Let $a \in C$ and define $g\colon C_0 \to \tilde{C}_0$ by $g(x) = \bar{h}(x + a) - \bar{h}(a)$ for $x \in C_0$.

Since g is a regular homeomorphism there exists a linear homeomorphism $f\colon S_{\underline{n}} \to S_{\underline{m}}$ such that $f|C_0$ is homotopic to g. Note that $f(C_0) = \tilde{C}_0$. We may suppose $f(0) = 0$.

Define $f' = S_{\underline{n}} \to S_{\underline{m}}$ by

$$f'(y) = f(y - a) + \bar{h}(a).$$

Then f' is a linear homeomorphism (but $f'(0) = f(-a) + \bar{h}(a)$ is not necessarily zero).

Since $f'(a) = f(0) + \bar{h}(a) = 0 + \bar{h}(a) \in \tilde{C}$ we have $f'(C) = \tilde{C}$.

We prove next that $f'(-C) = -\tilde{C}$. For $x \in C$

$$\begin{aligned}\bar{h}(x) - f'(x) &= \bar{h}(x) - f(x - a) - \bar{h}(a)\\ &= g(x - a) - f(x - a) \in C_0.\end{aligned}$$

Since $x - a \in C_0$ and $f|C_0$ and g are homotopic we have $f|C_0 - g$ is bounded. Hence, $(f|C_0 - g)(C_0)$ is contained in a compact interval J of $\tilde{C}_0$. By continuity $(\bar{h} - f')(C \cup (-C))$ is contained in J since C is dense in $C \cup (-C)$. So $f'(-C) \subset -\tilde{C}$.

Since f' is linear $f'(x) = p(x) + f'(0)$ for each $x \in S_{\underline{n}}$ where $p\colon S_{\underline{n}} \to S_{\underline{m}}$ is linear and $p(0) = 0$.

For $x \in C$

$$\begin{aligned}f'(x) + f'(-x) &= p(x) + f'(0) + p(-x) + f'(0)\\ &= 2f'(0) \in \tilde{C}_0 \text{ since } f'(x) \in \tilde{C} \text{ and } f'(-x) \in -\tilde{C}.\end{aligned}$$

Hence, $2f'(0) = 2\alpha$ where $\alpha \in \tilde{C}_0$.

Define $\bar{f}: S_{\underline{n}} \to S_{\underline{m}}$ by

$$\bar{f}(x) = f'(x) - \alpha.$$

Then $\bar{f}(-x) = f'(-x) - \alpha$ but $f'(-x) = 2\alpha - f'(x)$ so $\bar{f}(-x) = \alpha - f'(x) = -(f'(x) - \alpha) = -\bar{f}(x)$. Hence, $\bar{f}$ is a linear homeomorphism of $S_{\underline{n}}$ onto $S_{\underline{m}}$, $\bar{f}(C) = \tilde{C}$, $\bar{f}(-C) = -\tilde{C}$ and $\bar{f}(-x) = -\bar{f}(x)$.

Define a homeomorphism

$$f^*: K_{\underline{n}} \to K_{\underline{m}}$$

by $f^*(x) = \mathcal{N}_{\underline{m}} \circ \bar{f}(\mathcal{N}_{\underline{n}}^{-1}(x))$. Then $f^*(D) = \tilde{D}$ since $\bar{f}(C) = \tilde{C}$.

Remark 5.2. The converse to Theorem 5.1 is true. If D and $\tilde{D}$ are in the same position in $K_{\underline{n}}$ then Debski [5] has shown that there exists a regular homeomorphism of $K_{\underline{n}}$ which takes D onto $\tilde{D}$.

Remark 5.3. If D is a composant of $K_{\underline{n}}$ without an endpoint then there exist by [5] at most countably many composants of $K_{\underline{n}}$ which are homeomorphic to D under a regular homeomorphism. Hence, there exists a family of cardinality c of composants of $K_{\underline{n}}$ such that no two members of the family are homeomorphic under a regular homeomorphism.

6. Questions

We list a few open questions about composants of Knaster continua and solenoids.

(1) *(Bellamy)* Do there exist in $K_{\underline{2}}$ two composants without endpoints which are *not* homeomorphic?

(2) Are two homeomorphic composants of $K_{\underline{n}}$ in the same position?

(3) If $C \subset S_{\underline{n}}$ and $\tilde{C} \subset S_{\underline{m}}$ are homeomorphic composants is $S_{\underline{n}}$ homeomorphic to $S_{\underline{m}}$?

(4) Is each homeomorphism $h: C \to \tilde{C}$ of composants of solenoids homotopic to a linear homeomorphism $\bar{h}: C \to \tilde{C}$ (i.e. $\bar{h}(x) = ax + b$ for each x)?

A position solution to Question 4 would imply a positive solutions to Questions 1, 2 and 3.

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