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MAXIMAL RIMCOMPACT IMAGES

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1. Introduction and Known Results

All spaces considered are completely regular and Hausdorff. Recall that a space X is *rimcompact* if X has a base of open sets with compact boundaries ([Is]). A space X is *almost rimcompact* if X possesses a compactification KX in which each point of $KX\setminus X$ has a base of open sets of KX whose boundaries lie in X. Each rimcompact space is almost rimcompact ([Mo_1]); the converse is not true ([Is]) (see [Di_1] and [Di_4] for the internal characterization and a discussion of almost rimcompactness). A space X is a 0-*space* if X possesses a compactification with zero-dimensional remainder; there are 0-spaces which are not almost rimcompact ([Di_1]).

A *map* is a continuous surjection. A function $f: X \to Y$ is *closed* if whenever F is closed in X, then f[F] is closed in Y. If a map f is closed, and $f^{\leftarrow}(y)$ ($bd_X f^{\leftarrow}(y)$ respectively) is compact for $y \in Y$, then f is *perfect* (*rimperfect* respectively). A map $f: X \to Y$ is *monotone* if $f^{\leftarrow}(y)$ is connected for each $y \in Y$.

In the following, L(X) will denote the locally compact part of X.

In an investigation of maps from almost rimcompact spaces onto rimcompact spaces, the following was proved (2.5 of [Di_2]).

1.1 *Theorem. Suppose that* X *is the perfect preimage of a rimcompact space. Then there is a rimcompact space* Z *and a perfect monotone map* $g: X \to Z$ *such that*

a) $\overleftarrow{g}[g[L(X)]] = L(X)$, *and* $g|_{L(X)}$ *is a homeomorphism.*

b) *if* Y *is any rimcompact space, and* $f: X \to Y$ *is perfect, then there is a perfect map* $h: Z \to Y$ *such that* $h \circ g = f$.

Rimcompactness is not generally preserved in perfect images and preimages without the addition of other conditions; in the presence of these other conditions, rimperfect maps usually suffice.

1.1 is proved with "rimcompact" replaced by "almost rimcompact" or "0-space." Slightly weaker conclusions hold when "perfect" is replaced by "rimperfect"; the map $g: X \to Z$ need not be rimperfect or monotone.

The following will be used without mention: if F is closed in X, then $bd_{\beta X}cl_{\beta X}F = cl_{\beta X}bd_X F$. This is true in any *perfect* compactification (see [Sk] or [Is] for the definition); the inclusion $cl_{KX}bd_X F \subseteq bd_{KX}cl_{KX}F$ holds in any compactification KX of X. If X is a 0-space, then F_0X denotes the maximal compactification of X having zero-dimensional remainder.

2. The Main Results

The main theorem is based on the following two results.

2.1 *Lemma. Suppose that* $f: X \to Z$ *is rimperfect, and that maps* $g: X \to Y$ *and* $h: Y \to Z$ *exist so that* $h \circ g = f$. *Then* h *is rimperfect. In fact, if* KY, KZ *are any* *compactifications of* Y, Z *respectively such that* h *extends to* H: KY → KZ, *then* $bd_{KY}H^{\leftarrow}(z) \subseteq Y$ *for* $z \in Z$.

Proof. The map h is clearly closed. Let F: βX → KZ and G: βX → KY denote the natural maps extending f and g respectively. Then F = H ∘ G. Since $H^{\leftarrow}(z) = G[F^{\leftarrow}(z)]$, it suffices to show that $bd_{KY}G[F^{\leftarrow}(z)] \subseteq Y$. As f is closed, $F^{\leftarrow}(z) = cl_{\beta X}f^{\leftarrow}(z)$ (1.1, 1.2 of [Iw]). Hence $bd_{\beta X}F^{\leftarrow}(z) = bd_{\beta X}cl_{\beta X}f^{\leftarrow}(z) = cl_{\beta X}bd_X f^{\leftarrow}(z) = bd_X f^{\leftarrow}(z) \subseteq X$, so that $G[bd_{\beta X}F^{\leftarrow}(z)] \subseteq Y$. The map G is closed, thus $bd_{KY}G[F^{\leftarrow}(z)] \subseteq G[F^{\leftarrow}(z)]$. Suppose that $p \in G[F^{\leftarrow}(z)] \setminus G[bd_{\beta X}F^{\leftarrow}(z)]$. Now $G^{\leftarrow}G[F^{\leftarrow}(z)] = F^{\leftarrow}(z)$, so that $G^{\leftarrow}(p) \subseteq F^{\leftarrow}(z) \setminus bd_{\beta X}F^{\leftarrow}(z)$. That is, $G^{\leftarrow}(p) \subseteq int_{\beta X}F^{\leftarrow}(z)$; since G is closed, $p \in int_{\beta X}G[F^{\leftarrow}(z)]$. This proves that $bd_{KY}G[F^{\leftarrow}(z)] \subseteq G[bd_{\beta X}F^{\leftarrow}(z)]$. The fact that h is rimperfect follows from the observation that h has an extension H: βY → βZ.

A corollary of the above is the following: if h extends to H: KY → KZ, then $cl_{KY}h^{\leftarrow}(z) \cap cl_{KY}(Y \setminus h^{\leftarrow}(z)) \subseteq Y$. The extension of h to H is necessary; the statement can be made in general only for a perfect compactification KY of Y, even if Y = X and g is the identity map.

We need the following straightforward generalization of Lemma 3 of [Mo_2], which states that if f: Y → Z is rimperfect, and an open set U of Z has compact boundary, then $bd_Y f^{\leftarrow}[U]$ is compact.

2.2 *Lemma. Suppose that* h: Y → Z *is rimperfect and extends to* H: KY → KZ. *If* U *is open in* KZ *with* $bd_{KZ}U \subseteq Z$, *then* $bd_{KY}H^{\leftarrow}[U] \subseteq Y$.

Proof. Since $bd_{KZ}U \subseteq Z$, $bd_{KY}H^{\leftarrow}[U] \subseteq H^{\leftarrow}[bd_{KZ}U] = H^{\leftarrow}[(bd_{KZ}U) \cap Z] = \cup\{H^{\leftarrow}(z): z \in (bd_{KZ}U) \cap Z\}$. According to 2.1, $bd_{KY}H^{\leftarrow}(z) \subseteq Y$. Then $bd_{KY}H^{\leftarrow}[U] \subseteq \cup\{H^{\leftarrow}(z) \setminus int_{KY}H^{\leftarrow}(z): z \in (bd_{KZ}U) \cap Z\} \subseteq Y$.

Thus if h extends to $H: KY \to KZ$, and $bd_Z U$ is compact, then $bd_{KY}cl_{KY}h^{\leftarrow}[U] \subseteq Y$. Once again the extension of h to H is necessary unless KY is a perfect compactification of Y.

2.3 *Theorem. Suppose that for* $\alpha \in A$, $f_\alpha: X \to X_\alpha$ *is rimperfect, where* X_α *is rimcompact (almost rimcompact, a 0-space respectively). Let* $g: X \to \Pi_{\alpha\in A}X_\alpha$ *be the diagonal map. Then* $g[X]$ *is rimcompact (almost rimcompact, has a compactification with totally disconnected remainder respectively).*

Proof. Let $F_\alpha: \beta X \to F_O X_\alpha$ denote the extension of f_α, for $\alpha \in A$, and $G: \beta X \to \Pi_{\alpha\in A}F_O X_\alpha$ the diagonal map. Then $cl_\Pi g[X] = G[\beta X]$ and $G|_X = g$.

For $\alpha \in A$, let h_α and H_α denote the restriction of Π_α to $g[X]$ and $G[\beta X]$ respectively, where $\Pi_\alpha: \Pi_{\alpha\in A}F_O X_\alpha \to F_O X_\alpha$ is the projection map. Clearly $h_\alpha \circ g = f_\alpha$ and $H_\alpha \circ G = F_\alpha$. Since f_α is rimperfect, by 2.1 h_α is rimperfect.

Suppose that X_α is rimcompact for each $\alpha \in A$; we wish to show that $g[X]$ is rimcompact. Choose $\langle x_\alpha \rangle \in U$, where U is open in $g[X]$. There is an open set U' of $\Pi_{\alpha\in A}X_\alpha$ such that $U' \cap g[X] = U$, and a finite subset F of A such that $\langle x_\alpha \rangle \in \Pi_{\alpha\in F}U_\alpha \times \Pi_{\alpha\notin F}X_\alpha \subseteq U'$, where U_α is open in X_α. For

$\alpha \in F$, choose an open set W_α of X_α with compact boundary such that $x_\alpha \in W_\alpha \subseteq cl_{X_\alpha} W_\alpha \subseteq U_\alpha$. As h_α is rimperfect, according to Lemma 3 of [Mo_2] (as mentioned preceding 2.2), $bd_{g[X]} h_\alpha^{\leftarrow}[W_\alpha]$ is compact. In addition, $h_\alpha^{\leftarrow}[W_\alpha] \subseteq [U_\alpha \times \Pi_{\beta \neq \alpha} X_\beta] \cap g[X]$, thus $\bigcap_{\alpha \in F} h_\alpha^{\leftarrow}[W_\alpha] \subseteq [\Pi_{\alpha \in F} U_\alpha \times \Pi_{\alpha \notin F} X_\alpha] \cap g[X]$. Since the set $\bigcap_{\alpha \in F} h_\alpha^{\leftarrow}[W_\alpha]$ has compact boundary, $\langle x_\alpha \rangle$ has a base of such sets in $g[X]$.

Suppose that X_α is almost rimcompact for $\alpha \in A$. We show that points of $G[\beta X] \setminus g[X]$ have a base of open sets of $G[\beta X]$ whose boundaries are contained in $g[X]$. Choose $\langle p_\alpha \rangle \in U \setminus g[X]$, where U is open in $G[\beta X]$. There is a finite subset F of A and a set of the form $\Pi_{\alpha \in F} U_\alpha \times \Pi_{\alpha \notin F} F_o X_\alpha$ (where U_α is open in $F_o X_\alpha$) such that $\langle p_\alpha \rangle \in [\Pi_{\alpha \in F} U_\alpha \times \Pi_{\alpha \in F} F_o X_\alpha] \cap G[\beta X] \subseteq U$. Suppose that for $\alpha \in F$, $p_\alpha \in X_\alpha$. According to 2.1, $bd_{G[\beta X]} H_\alpha^{\leftarrow}(p_\alpha) \subseteq g[X]$, so that $\langle p_\alpha \rangle \in int_{G[\beta X]} H_\alpha^{\leftarrow}(p_\alpha)$ (which equals $int_{G[\beta X]}[(\{p_\alpha\} \times \Pi_{\beta \neq \alpha} F_o X_\beta) \cap G[\beta X]]$). Let $W'_\alpha = int_{G[\beta X]} H_\alpha^{\leftarrow}(p_\alpha)$. Note that since $p_\alpha \in U_\alpha$, $W'_\alpha \subseteq [U_\alpha \times \Pi_{\beta \neq \alpha} F_o X_\beta] \cap G[\beta X]$. On the other hand, if for $\alpha \in F$, $p_\alpha \in F_o X_\alpha \setminus X_\alpha$, there is an open set W_α of $F_o X_\alpha$ with $bd_{F_o X} W_\alpha \subseteq X_\alpha$ and $p_\alpha \in W_\alpha \subseteq cl_{F_o X} W_\alpha \subseteq U_\alpha$. It follows from 2.2 that $bd_{G[\beta X]} H_\alpha^{\leftarrow}[W_\alpha] \subseteq g[X]$, while $\langle p_\alpha \rangle \in H_\alpha^{\leftarrow}[W_\alpha] \subseteq [U_\alpha \times \Pi_{\beta \neq \alpha} F_o X_\beta] \cap G[\beta X]$. In this case, let $W'_\alpha = H_\alpha^{\leftarrow}[W_\alpha]$. Finally, if $W' = \bigcap_{\alpha \in F} W'_\alpha$, then $bd_{G[\beta X]} W' \subseteq g[X]$, and $W' \subseteq [\Pi_{\alpha \in F} U_\alpha \times \Pi_{\alpha \notin F} F_o X_\alpha] \cap G[\beta X]$. Thus $g[X]$ is almost rimcompact.

Suppose that for $\alpha \in A$, X_α is a 0-space. We wish to show that the connected component C_p in $G[\beta X] \setminus g[X]$ of $p = \langle p_\alpha \rangle \in G[\beta X] \setminus g[X]$ equals $\{p\}$. It suffices to show that

$H_\alpha[C_p] = \{p_\alpha\}$ for each $\alpha \in A$. Suppose that $g_\alpha \in X_\alpha$. According to 2.1, $H_\alpha^{\leftarrow}(g_\alpha) \cap (G[\beta X] \setminus g[X])$ is clopen in $G[\beta X] \setminus g[X]$, hence $C_p \subseteq H_\alpha^{\leftarrow}(g_\alpha)$ or $C_p \cap H_\alpha^{\leftarrow}(g_\alpha) = \phi$. Then for any $g_\alpha \in X_\alpha$, $H_\alpha[C_p] = \{g_\alpha\}$ or $g_\alpha \notin H_\alpha[C_p]$. In particular, if $p_\alpha \in X_\alpha$, then $H_\alpha[C_p] = \{p_\alpha\}$, and if $p_\alpha \in F_o X_\alpha \setminus X_\alpha$, $H_\alpha[C_p] \cap X_\alpha = \phi$. In the latter case, $H_\alpha[C_p]$ is a connected subset of the zero-dimensional space $F_o X_\alpha \setminus X_\alpha$; once again $H_\alpha[C_p] = \{p_\alpha\}$. Thus $G[\beta X] \setminus g[X]$ is totally disconnected.

Since the product of rimcompact spaces is rarely a 0-space (see [Di_3]), some argument of the sort above is needed in the proof of 2.3. The map g in 2.3 need not be closed, even if g is 1-1 (see example 3.1).

The hypothesis in 2.3 that X_α is a 0-space is stronger than is necessary in order to conclude that g[X] has a compactification with totally disconnected remainder; the conclusion holds if X_α has a compactification with totally disconnected remainder. The space g[X] is constructed as in 2.3, with $\Pi_{\alpha\in A} X_\alpha \subseteq \Pi_{\alpha \in A} SX_\alpha$, where SX_α is the maximal compactification of X_α having totally disconnected remainder. Then $SX_\alpha \setminus X_\alpha$ is totally disconnected rather than zero-dimensional, sufficient for the proof. We do not know if 2.3 holds with "0-space" throughout.

2.4 *Corollary. For any completely regular space* X, *there exists a rimcompact space* Z *and a continuous map* g: X → Z *such that*

1) $g^{\leftarrow}[g[L(X)]] = L(X)$ *and* $g|_{L(X)}$ *is a homeomorphism,*

2) *If* Y *is any rimcompact space and* f: X → Y *is rimperfect, then there exists a rimperfect map* h: Z → Y *such that* f = h ∘ g.

Proof. The map collapsing X to a single point is rimperfect. Let $\mathcal{S}$ be the collection of all rimcompact spaces which are the image of X under a rimperfect map. Define two such images (X_1,f_1) and (X_2,f_2) to be equivalent if there is a homeomorphism $h: X_1 \to X_2$ such that $h \circ f_1 = f_2$. Since all maps are onto, the collection $\mathcal{S}$ is a set, up to equivalence. The existence of Z then follows from 2.3.

Suppose that $x \in U \subseteq cl_X U \subseteq L(X)$, and that $cl_X U$ is compact. There is a continuous function j: X → [0,1] such that j(x) = 0 and $j[X \setminus cl_X U] = 1$. Such a map is clearly rimperfect, thus the family of rimperfect maps on X with rimcompact range separates points of L(X) from closed sets of X. The theorem follows.

2.5 *Corollary*. 2.4 *holds if "rimcompact" is replaced everywhere by "almost rimcompact" or by "has a compactification with totally disconnected remainder."*

The following is essentially 2.6 of [Di_2].

2.6 *Lemma*. *Suppose that* f: X → Y *is perfect, where* Y *is rimcompact (almost rimcompact, a* 0*-space respectively). Then there are a rimcompact (almost rimcompact,* 0*-space respectively)* Z *and perfect maps* g: X → Z, h: Z → Y *such that* h ∘ g = f *and* g *is monotone.*

2.7 *Theorem*. *Suppose that* X *is the perfect preimage of a rimcompact space. Then there exists a rimcompact* *space* Z *and a perfect monotone map* $g: X \to Z$ *such that*

1) $g^{\leftarrow}[g[L(X)]] = L(X)$, *and* $g|_{L(X)}$ *is a homeomorphism,*

2) *if* Y *is any rimcompact space and* $f: X \to Y$ *is perfect, then there is a perfect map* $h: Z \to Y$ *such that* $h \circ g = f$.

The pair (Z,g) *is unique up to homeomorphism of* Z. *The result holds if "rimcompact" is replaced everywhere by "almost rimcompact" or "0-space."*

Proof. Suppose that $f_{\alpha'}: X \to X_{\alpha'}$ is perfect. Then the diagonal map $f: X \to g[X]$ (as in 2.3) and $h_{\alpha'}: g[X] \to X_{\alpha'}$ are perfect (see 3.7.10 of $[E_n]$). The result in the rimcompact or almost rimcompact case then follows from 2.3, 2.4 and 2.6.

In the case in which (for $\alpha \in A$) X_α is a 0-space, for $p \in F_o X_\alpha \setminus X_{\alpha'}$, $H^{\leftarrow}_{\alpha'}(p)$ is a compact subset of the totally disconnected set $G[\beta X] \setminus g[X]$, hence $H^{\leftarrow}_{\alpha'}(p)$ is zero-dimensional. Since $h_{\alpha'}$ is perfect, $H^{\leftarrow}_{\alpha'}[F_o X_{\alpha'} \setminus X_{\alpha'}] = G[\beta X] \setminus g[X]$. It is easy to show that if $f: X \to Y$ is perfect, $f^{\leftarrow}(y)$ is zero-dimensional for $y \in Y$, and Y is zero-dimensional, then X is zero-dimensional (see, for example, [Ny]). The above, combined with the fact that $H_{\alpha'}|_{G[\beta X] \setminus g[X]}$ is closed, suffice to show that $G[\beta X] \setminus g[X]$ is zero-dimensional. Thus g[X] is a 0-space.

It remains to show the uniqueness of (Z,g). Suppose that there exist Z' and $g': X \to Z'$ having the properties of Z and g. Then there exist perfect maps $h': Z' \to Z$ and $h: Z \to Z'$ so that $h' \circ g' = g$ and $h \circ g = g'$. It follows that $h' \circ h \circ g = h' \circ g' = g$, thus $h' \circ h: Z \to Z$ is the identity map on Z, and h is a homeomorphism. •

As mentioned in the introduction, the above result for the rimcompact case appears in [Di_2]. The proof in that paper made use of decompositions; a necessary inductive step was omitted in that proof.

In light of the uniqueness of Z in 2.7, it would be interesting to determine if the Z of 2.4 is unique.

Suppose that a space X maps perfectly onto at least one rimcompact space Y. A rimcompact perfect image Z of X can be constructed as in 2.3 by considering the family of rimperfect maps on X with rimcompact range; a second rimcompact perfect image Z' of X can be constructed, again as in 2.3, by considering the family of perfect maps on X with rimcompact range. According to the following, Z and Z' are equivalent. (Note that in the above discussion and the next result, "rimcompact" can be replaced by "almost rimcompact" or "0-space.")

2.8 *Theorem. Let* Z *and* Z' *be as above. Then* $Z \simeq Z'$.

Proof. This follows from the uniqueness of Z in 2.7, and the fact that all perfect maps with rimcompact image factor through Z' as constructed.

2.9 *Corollary. Suppose that the family of rimperfect maps on* X *with rimcompact (almost rimcompact) range separates points and closed sets of* X. *Then* X *is rimcompact (almost rimcompact respectively).*

Proof. This follows directly from 2.3.

According to the comments following the proof of 2.3, 2.9 holds if "rimcompact range" is replaced by "ranges having compactifications with totally disconnected remainder." There is a straightforward direct proof of 2.9 for the rimcompact case; a direct proof using the internal characterization of almost rimcompactness is possible but more difficult than 2.3. The approach of 2.3 appears to be the only reasonable one in the totally disconnected case.

Example 3.1 will indicate that separating points of X is not sufficient in 2.9.

3. Examples

The first two examples indicate that the weaker conclusions drawn about the properties of the map $g: X \to Z$ when constructing Z as in 2.4 with rimperfect maps rather than with perfect maps are necessarily weaker. The third example indicates that although the space Z constructed in the former way exists for every space X, it may be in some sense trivial.

3.1 *Example.* There is a nonrimcompact space X for which the family of rimperfect maps with rimcompact range separates points on X, so that $g: X \to Z$ is 1-1, where Z is as in 2.4. The map g is not closed.

Let $\mathcal{R}$ denote a maximal almost disjoint collection of infinite subsets of the natural numbers N. The space $N \cup \mathcal{R}$ has the topology described in 5I of [GJ]; each point of N is isolated and $\lambda \in \mathcal{R}$ has as an open base $\{\{\lambda\} \cup (\lambda \setminus F)$: F is a finite subset of N$\}$. The space $N \cup \mathcal{R}$ is locally compact and zero-dimensional. According to 2.1 of [Te], there is a family $\mathcal{R}$ so that $\beta(N \cup \mathcal{R}) \setminus N \cup \mathcal{R}$ is homeomorphic to the unit interval I. Let $X = N \cup \mathcal{R} \cup \{0\}$. The point 0 has no base of open sets with boundaries contained in X, so that X is not rimcompact. Points of X are separated by the rimperfect characteristic functions of clopen sets of X. The space Z of 2.4 is homeomorphic to the one-point compactification of $N \cup \mathcal{R}$; each of the above rimperfect functions collapses [0,1] to a point. Thus the map $g: X \to Z$ is 1-1 but not closed.

3.2 *Example*. Let $X' = N \cup \mathcal{R} \cup \{0,1\}$ where $\mathcal{R}$ is as in 3.1. The space Z of 2.4 is again the one-point compactification of $N \cup \mathcal{R}$; the map $g: X \to Z$ is not monotone.

3.3 *Example*. There exists a space X for which the Z described in 2.4 has cardinality 1. Choose X to be any completely regular space having no nontrivial open set with compact boundary; that is, any open set with compact boundary must have closure equal to X. The only rimperfect map on X is that collapsing X to a point. Since X has nonconstant continuous functions into [0,1], the space Z described in 2.4 is not a "largest" rimcompact continuous image of X; Z merely lies above every rimcompact rimperfect image of X.

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