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1. Introduction

After its discovery by Ungar [8] in 1975, the so-called *Effros theorem* has become a central tool in the study of homogenous continua. (See e.g. [2], [7].) This important result (actually a corollary to the original result in Effros [4]) states that if a Polish transformation group G acts transitively on a Polish space X, then the evaluation map T_x: $G \to X$, given by $T_x(g) = g(x)$, is open for every $x \in X$. Stated in other terms, close points can be mapped to each other by homeomorphisms close to the identity map.

Effros' original proof used a Borel selection argument. An elementary straightforward proof was given by F. D. Ancel in [1]. The author presented a similar (independent) proof in Prague in January 1984; afterwards Jiří Vilímovský noted that it had similarity with the argument used by I. M. Dektjarev [3] for his result on almost open maps. In this note we present a proof of Effros' theorem based on Dektjarev's result. The most important case of this theorem stating that for a compact homogeneous metrizable space X, the evaluation map T_x: $H(X) \to X$ is open for every $x \in X$, would follow almost immediately, while in the general case one needs a simple additional argument.

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2. Preliminaries

In this note, X denotes a Polish space, and G denotes a Polish group acting transitively on X. We use on G the natural uniformity induced by the neighbourhoods of the identity. If μX, νY are uniform spaces, then a map $f\colon \mu X \to \nu Y$ is called *uniformly almost open* if for every $\mathcal{U} \in \mu$ there is a $\mathcal{V} \in \nu$ such that

$$\overline{f[\mathrm{St}(x,\mathcal{U})]} \supset \mathrm{St}(f(x),\mathcal{V})$$

for all $x \in X$. Likewise, such an f is called *uniformly open* if the above condition without closure bar is satisfied. A uniform space μX is called *supercomplete* (see [5]), if the uniform hyperspace $H(\mu X)$ of all closed subsets of X, with the Hausdorff uniformity, is complete. A complete metric space is supercomplete.

Let us now state Dektjarev's theorem.

Theorem 2.1 ([3]). *A uniformly almost open (multi-valued) mapping of a supercomplete uniform space into a uniform space, with closed graph, is uniformly open.*

3. The Result

In this section we give a short proof of Effros' theorem. With the help of Dektjarev's theorem, the arguments needed reduce to mere observations. In this way, Effros' theorem becomes a variant of the open mapping theorem for topological groups (see [6], p. 213), where one of the groups is replaced by a homogeneous space.

Theorem 3.1. *Let* X *be a Polish space, and let* G *be a Polish transformation group acting transitively on* X.

Then the evaluation map $T_x: G \to X$ *is open for every* $x \in X$.

Proof. First we shall prove that given any nonempty open set $B \subset G$, then $\text{int}\overline{T_x[B]} \neq \emptyset$. As G is separable, there exist $g_1, g_2, \cdots$ such that $G = \cup\{g_nB: n \in N\}$. Hence,

$$X = T_x[G] = \cup\{\overline{T_x[g_nB]}: n \in N\}.$$

Since X is Polish, it follows from Baire's Category Theorem that for some $k \in N$ the set $\overline{T_x[g_kB]}$ has a nonempty interior. Since g_k is a homeomorphism, we get $\text{int}_X\overline{T_x[B]} \neq \emptyset$.

Next we shall prove that in fact $x \in \text{int}_X\overline{T_x[B]}$ for any nonempty open $B \subset G$. Let V be a symmetric neighbourhood of the identity e in G such that $V^2 \subset B$. (We can assume that B is a neighbourhood of e.) Let $y \in \text{int}_X\overline{T_x[V]}$. Then there exists a sequence (h_n) of elements of V with $h_n(x) \to y$. But then $h_k(x) \in \text{int}_X\overline{T_x[V]}$ already for some k and consequently

$$x \in h_k^{-1}[\text{int}_X\overline{T_x[V]}] = \text{int}_X\overline{T_x[h_k^{-1}V]} \subset \text{int}_X\overline{T_x[B]},$$

as required. It follows that for any $g \in G$, we have $g(x) \in \text{int}_X\overline{T_x[gB]}$. Choose a complete compatible metric ρ for G, and define a new map $T'_x: G \to G \times X$ by setting $T'_x(g) = (g, g(x))$. Let $\mathcal{U}$ be a uniform cover of (G,ρ). Then there is $\varepsilon > 0$ such that the balls $B_\rho(g,\varepsilon)$ refine $\mathcal{U}$. For each $g \in G$, let $W_g = B_\rho(g,\varepsilon/2)$, and note that by the above there is an open set $V_g \subset X$ such that $g(x) \in V_g \subset \overline{T_x[W_g]}$. Consider the cover $\mathcal{V}$ of $T'_x[G]$ formed by the sets $(W_g \times V_g) \cap T'_x[G]$. As $T'_x[G]$ is metrizable and thus paracompact, $\mathcal{V}$ is a uniform cover in the fine uniformity $\mathcal{F}(T'_x[G])$. To show that T'_x is uniformly almost open, it is enough to show that $St(T'_x(g),\mathcal{V}) \subset \overline{T'_x[B_\rho(g,\varepsilon)]}$ for all $g \in G$. To see this, suppose that $(W_g \times V_g) \cap (W_h \times V_h) \neq \emptyset$. Then clearly $W_h \subset B_\rho(g,\varepsilon)$, and thus $V_h \subset \overline{T_x[W_h]} \subset \overline{T_x[B_\rho(g,\varepsilon)]}$, from which the claim follows. On the other hand, (G,ρ) is supercomplete, and it is obvious that (by continuity of the action of G on X) T'_x has closed graph. Therefore, T'_x is (uniformly) open by Dektjarev's theorem. But then T_x is open, too.

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