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Dedicated to Professor Yukihiro Kodama on his 60th birthday

1. Introduction

It is known that locally connected, rim-compact, normal Moore spaces are metrizable (in fact it was proved that locally connected, rim-compact, normal submetacompact spaces are paracompact), see [Bl]. In this paper, we shall prove that under $2^{\omega} < 2^{\omega_1} < 2^{\omega_2}$ locally connected, rim-Lindelöf, normal, submetaLindelöf spaces of character $\leqq 2^{\omega}$ are paracompact and that under $2^{\omega} < 2^{\omega_1}$ locally connected, rim-Lindelöf, normal, submetaLindelöf spaces of character $\leqq 2^{\omega}$ and tightness $\leqq \omega$ are paracompact (thus locally connected, rim-Lindelöf, normal Moore spaces are metrizable if $2^{\omega} < 2^{\omega_1}$ is assumed).

First we review topological and set theoretical notations. All topological spaces are assumed to be regular T_1. A subset S of a topological space is said to be *normalized* if for every $S' \subset S$, S' and $S - S'$ can be separated by disjoint open sets. A subset S of a topological space is said to be *separated* if for every x of S there is a neighborhood U_x of x such that $\{U_x : x \in S\}$ is disjoint. For a point x of a space X, $\chi(x,X)$ denotes the least infinite cardinality κ such that x has a neighborhood base of cardinality $\leqq \kappa$.

For a cardinal κ, a space is *κ-Lindelöf* if every open cover has a subcover of cardinality $\leq\kappa$. Note that ω-Lindelöf is Lindelöf in the usual sense.

A space is *submetaLindelöf* if for every open cover, there is a countable family $\{\mathcal{U}_n: n \in \omega\}$ of open covers refining it such that for every x in X there is an n in ω such that $|(\mathcal{U}_n)_x| \leq \omega$, where $(\mathcal{U}_n)_x = \{U \in \mathcal{U}_n: x \in U\}$ and $|A|$ denotes the cardinality of A for every set A.

A space is *rim-κ-Lindelöf* if every point has a neighborhood base consisting of open sets with κ-Lindelöf boundaries.

A space is *κ-compact* if there is no closed discrete subspace of cardinality κ.

For an ordinal α and a set X, ${}^{\alpha}X$ denotes the set of all functions from α to X and X^{α} denotes the cardinality of ${}^{\alpha}X$. Furthermore ${}^{<\alpha}X$ denotes the set $\cup_{\beta<\alpha}{}^{\beta}X$ and $X^{<\alpha}$ denotes the cardinality of ${}^{<\alpha}X$. For a cardinal κ, $[X]^{\leq\kappa}$ ($[X]^{<\kappa}$) denotes the set $\{Y \subset X: |Y| \leq \kappa\}$ ($\{Y \subset X: |Y| < \kappa\}$, respectively). A subset of an ordinal is said to be *club* if it is closed in the ordinal with the order topology and unbounded in it. For a function f, $f|A$ denotes the restriction of f to A. For other set theoretical or topological notions or notations, see [E], [J] and [K].

2. Results

To prove our results, first we introduce Φ and N, and present basic facts without proof. Here Φ was introduced in [DS]. For further reference, see Ch. 14 of [Sh].

1. *Definitions.* Let κ be an uncountable regular cardinal, λ be a cardinal, and S be a subset of κ.

$\Phi(\kappa,\lambda,S)$ denotes the following assertion:

For every F: ${}^{<\kappa}\lambda \to 2$, there exists a g in ${}^{\kappa}2$ such that for every f in ${}^{\kappa}\lambda$, $\{\alpha \in S: F(f|\alpha) = g(\alpha)\}$ is stationary in κ. Incidentally such S must be stationary in κ, if $\Phi(\kappa,\lambda,S)$

Furthermore when λ is an infinite cardinal, we define $N(\kappa,\lambda,S)$ as follows:

For every topological space X and every normalized sequence $\{x_\alpha: \alpha \in S\}$ of distinct points, if for every α in S, $\chi(x_\alpha,X) \leqq \lambda$, then there is a stationary subset S' of S such that $\{x_\alpha: \alpha \in S'\}$ is separated.

The proofs 1) and 2) of the following lemma are easy by the definition of Φ. The proofs of 3) and 4) are similar to [DS], and the proofs of 5) and 6) are also similar to [Ta].

From now on we always assume that κ is an infinite cardinal.

2. *Lemma. The following results hold:*

1) *If* $S \subset S' \subset \kappa^+$ *and* $\Phi(\kappa^+,2,S)$ *hold, then so does* $\Phi(\kappa^+,2,S')$.

2) *If* S *is a stationary subset of* κ^+, *then* $\Phi(\kappa^+,2,S)$ *holds iff* $\Phi(\kappa^+,2,S \cap C)$ *holds for every club* C *of* κ^+ *iff* $\Phi(\kappa^+,2,S \cap C)$ *holds for some club* C *of* κ^+.

3) *If* $2^\kappa < 2^{\kappa^+}$ *holds, then so does* $\Phi(\kappa^+,2,\kappa^+)$.

4) *Let* $\{S_\alpha: \alpha < \kappa\}$ *be a family of subsets of* κ^+. *If* $\Phi(\kappa^+,2,\cup_{\alpha<\kappa}S_\alpha)$ *holds, then there is an* $\alpha < \kappa$ *such that* $\Phi(\kappa^+,2,S_\alpha)$ *holds.*

5) *For every subset* S *of* κ^+, $\Phi(\kappa^+,2,S)$ *holds iff so does* $\Phi(\kappa^+,2^\kappa,S)$.

6) *For every subset* S *of* κ^+, *if* $\Phi(\kappa^+,2^\kappa,S)$ *holds, then so does* $N(\kappa^+,2^\kappa,S)$.

Next, applying the techniques of [B1], [B2] and the previous lemma, we shall prove our theorems. The next lemma is proved in [A].

3. *Lemma* ([A]). *Let* X *be a submetaLindelöf*, κ^+*-compact space. Then* X *is* κ*-Lindelöf.*

4. *Lemma.* $[2^\kappa < 2^{\kappa^+}]$ *Let* X *be a locally connected normal space of character* $\leq 2^\kappa$, *and let* $\mathcal{U}$ *be a family of* $\leq\kappa$*-many open subsets with* κ*-Lindelöf boundaries. Then* $\partial(\cup\mathcal{U})$ *is* κ^+*-compact.*

Proof. Assume indirectly that there is a closed discrete subset $\{x_\alpha : \alpha \in \kappa^+\}$ of $\partial(\cup\mathcal{U})$. By $2^\kappa < 2^{\kappa^+}$ and 3), 5) and 6) of 2, there is a stationary subset S of κ^+ such that $\{x_\alpha : \alpha \in S\}$ is separated. Since X is normal and locally connected, there is a discrete family $\{B_\alpha : \alpha \in S\}$ of connected open sets such that $x_\alpha \in B_\alpha$ for each $\alpha \in S$. Since the cardinality of $\mathcal{U}$ does not exceed κ, there are a stationary subset S' of S and a U in $\mathcal{U}$ such that $B_\alpha \cap U \neq 0$ for every α in S'. Thus $B_\alpha \cap \partial U \neq 0$ for α in S', by the connectedness of B_α's. This contradicts to the κ-Lindelöfness of ∂U.

5. *Lemma* ([B2]). *Let* X *be a submetaLindelöf space and* E *be a subset of* X *such that each* x *in* X *has a neighborhood* U_x *such that the cardinality of* $U_x \cap E$ *is of* $\leq\kappa$.

Then E *is a union of at most* κ*-many closed discrete subsets of* X.

6. *Lemma.* $[2^\kappa < 2^{\kappa^+}]$ *Let* X *be a locally connected, submetaLindelöf, normal space of character* $\leq 2^\kappa$*, and* K *be a connected closed subspace of* X. *If* $\mathcal{U}$ *is an open cover of* K *of cardinality* κ^+ *such that the boundary of each member of* $\mathcal{U}$ *is* κ*-Lindelöf, then there is a subfamily of* $\mathcal{U}$ *which covers* K *and is of cardinality* $\leq\kappa$.

Proof. Assume indirectly that $\mathcal{U}$ has no subcover of K of cardinality $\leq\kappa$. Then by using induction on κ^+, we may assume that $\mathcal{U}$ is $\{U_\alpha: \alpha < \kappa^+\}$ such that $K \cap (U_\alpha - \cup_{\beta<\alpha}U_\beta) \neq 0$ for each $\alpha < \kappa^+$. Since K is connected, fix $x_\alpha \in \text{cl}(K \cap \cup_{\beta<\alpha}U_\beta) - \cup_{\beta<\alpha}U_\beta$ for each $\alpha \in \kappa^+$. Let $f(\alpha) = \min\{\beta < \kappa^+: x_\alpha \in U_\beta\}$ for each $\alpha < \kappa^+$, then $C = \{\alpha < \kappa: \forall\beta < \alpha(f(\beta) < \alpha))\}$ is club in κ^+. Then points of $E = \{x_\alpha: \alpha \in C\}$ are all distinct. Then $\mathcal{U}' = \mathcal{U} \cup \{X - K\}$ is an open cover of X and each member of $\mathcal{U}'$ meets E at most $\leq\kappa$-many points. Hence by 5, E is a union of at most κ-many closed discrete subsets, say $E = \cup_{\beta<\kappa}E_\beta$, where E_β's are closed discrete. Let $C_\beta = \{\alpha \in C: x_\alpha \in E_\beta\}$. Since $2^\kappa < 2^{\kappa^+}$ holds, so does $\Phi(\kappa^+,2,\kappa^+)$ by 3) of 2. Then by 2) of 2, $\Phi(\kappa^+,2,C)$ holds. Again by 4) of 2, $\Phi(\kappa^+,2,C_\beta)$ holds for some $\beta < \kappa$. Finally by 5) and 6) of 2, $N(\kappa^+,2^\kappa,C_\beta)$ holds. Hence there is a stationary subset S of C_β such that $\{x_\alpha: \alpha \in S\}$ is separated. Since X is normal and locally connected, take a discrete family $\{B_\alpha: \alpha \in S\}$ of connected open sets such that $x_\alpha \in B_\alpha$ for every $\alpha \in S$. Since for every $\alpha \in S$, $x_\alpha \in \text{cl}(\cup_{\beta<\alpha}U_\beta)$, we can define a regressive function g on S (i.e. $g(\alpha) < \alpha$ for each $\alpha \in S$) such that $U_{g(\alpha)} \cap B_\alpha \neq 0$. Hence by the pressing down lemma, there are a stationary subset S' and S and a $\gamma < \kappa^+$ such that $g(\alpha) = \gamma$ for every $\alpha \in S'$. By the connectedness of B_α's, $B_\alpha \cap \partial U_\gamma \neq 0$ for $\alpha \in S$ and $\alpha > \gamma$. But this contradicts to the κ-Lindelöfness of ∂U_γ.

7. *Theorem.* $[2^\kappa < 2^{\kappa^+} < 2^{\kappa^{++}}]$ *Let* X *be a connected, locally connected, rim-*κ*-Lindelöf, submetaLindelof, normal space of character* $\leqq 2^\kappa$. *Then* X *is* κ*-Lindelöf.*

Proof. To prove this theorem, we shall show that such a space is κ^+-compact. Then by 3, it is κ-Lindelöf. Assume that such X is not κ^+-compact. Then there is a closed discrete subspace $\{x_\alpha : \alpha < \kappa^+\}$. By $2^\kappa < 2^{\kappa^+}$ and the fact that X is normal and of character $\leqq 2^\kappa$, there is a stationary subset S of κ^+ such that $E = \{x_\alpha : \alpha \in S\}$ is separated. Applying normality, local connectedness and rim-κ-Lindelöfness, take a discrete family $\mathcal{U} = \{U_\alpha : \alpha \in S\}$ of connected open sets such that ∂U_α is κ-Lindelöf and $x_\alpha \in U_\alpha$ for each $\alpha \in S$. Since X is locally connected and rim-κ-Lindelöf, take a family $\mathcal{B}$ of connected open sets with κ-Lindelöf boundaries such that $X - E = \cup\mathcal{B}$. By the connectedness of X, for α and α' of S, fix $\mathcal{B}(\alpha,\alpha') \in [\mathcal{B}]^{<\omega}$, say $\{B_0,\cdots,B_n\}$, such that $B_0 \cap U_\alpha \neq 0$, $B_n \cap U_{\alpha'} \neq 0$ and $B_i \cap B_{i+1} \neq 0$ for $i \in n$. Let $\mathcal{U}_0$ be the family $\mathcal{U} \cup \cup\{\mathcal{B}(\alpha,\alpha') : \alpha,\alpha' \in S\}$ of $\leqq\kappa^+$-many connected open sets with κ-Lindelöf boundaries. Then $\cup\mathcal{U}_0$ is connected. Then applying 4 to $2^{\kappa^+} < 2^{\kappa^{++}}$, $\partial(\cup\mathcal{U}_0)$ is κ^{++}-compact. By submetaLindelöfness and 3, $\partial(\cup\mathcal{U}_0)$ is κ^+-Lindelöf. Hence there is a family $\mathcal{U}_1$ of

κ^+-many connected open (in X) sets with κ-Lindelöf boundaries such that $\cup\mathcal{U}_1 \supset \partial(\cup\mathcal{U}_0)$ and $\cup\mathcal{U}_1 \cap E = 0$. Define $K = cl(\cup\mathcal{U}_0)$, then K is connected closed. Then $\mathcal{U}_2 = \mathcal{U}_0 \cup \mathcal{U}_1$ covers K and $|\mathcal{U}_2| \leqq \kappa^+$, but $\cup(\mathcal{U}_2 - \mathcal{U}) \cap E = 0$. Thus by 6, there is a subfamily of $\mathcal{U}_2$ which covers K and is of cardinality $\leqq\kappa$. Hence there is a subfamily of $\mathcal{U}$ which covers E and is of cardinality $\leqq\kappa$. But this contradicts to $|E| = \kappa^+$. The theorem is proved.

8. *Corollary.* $[2^\omega < 2^{\omega_1} < 2^{\omega_2}]$ *Let* X *be a connected, locally connected, rim-Lindelöf, submetaLindelöf, normal space of character* $\leqq 2^\omega$. *Then* X *is Lindelöf.*

9. *Corollary.* $[2^\kappa < 2^{\kappa^+} < 2^{\kappa^{++}}]$ *Let* X *be a locally connected, rim-*κ*-Lindelöf, submetaLindelöf, normal space of character* $\leqq 2^\kappa$. *Then* X *is a free union of* κ*-Lindelöf subspaces.*

Proof. Apply 7 in each component.

10. *Corollary.* $[2^\omega < 2^{\omega_1} < 2^{\omega_2}]$ *Let* X *be a locally connected, rim-Lindelöf, submetaLindelöf, normal space of character* $\leqq 2^\omega$. *Then* X *is a free union of Lindelöf subspaces. Hence* X *is strongly paracompact.*

11. *Theorem.* $[2^\kappa < 2^{\kappa^+}]$ *Let* X *be a connected, locally connected, rim-*κ*-Lindelöf, submetaLindelöf, normal space of character* $\leqq 2^\kappa$ *and tightness* $\leqq\kappa$ *(especially, of character* $\leqq\kappa$*). Then* X *is* κ*-Lindelöf.*

Proof. Let $\mathcal{U}$ be a cover of X by connected open sets with κ-Lindelöf boundaries. By induction on $\alpha < \kappa^+$, we shall define $\mathcal{U}_\alpha \in [\mathcal{U}]^{\leqq\kappa}$ such that $\cup\mathcal{U}_\alpha$ is connected and $cl(\cup\mathcal{U}_\alpha) \subset \cup\mathcal{U}_{\alpha+1}$. Assume that for every $\beta < \alpha$, $\mathcal{U}_\beta$ has been defined. If α is limit, put $\mathcal{U}_\alpha = \cup\{\mathcal{U}_\beta: \beta < \alpha\}$. Then it is easy to show that $\cup\mathcal{U}_\alpha$ is connected using the connectedness of $\cup\mathcal{U}_\beta$ for every $\beta < \alpha$. Assume $\alpha = \beta + 1$. Since $\mathcal{U}_\beta$'s are of cardinality $\leqq\kappa$, $\partial(\cup\mathcal{U}_\beta)$ is κ-Lindelöf by 3 and 4. Thus there is a $\mathcal{U}'$ in $[\mathcal{U}]^{\leqq\kappa}$ such that $\mathcal{U}'$ covers $\partial(\cup\mathcal{U}_\beta)$ and for every U in $\mathcal{U}'$, $U \cap \partial(\cup\mathcal{U}_\beta) \neq 0$ holds. Put $\mathcal{U}_\alpha = \mathcal{U}_\beta \cup \mathcal{U}'$. Then it is easy to show that $\cup\mathcal{U}_\alpha$ is connected. Thus we have defined $\mathcal{U}_\alpha$ for every $\alpha < \kappa^+$.

Since X is of tightness $\leqq\kappa$, $cl(\cup(\cup\{\mathcal{U}_\alpha: \alpha < \kappa^+\})) = \cup\{\cup\mathcal{U}_\alpha: \alpha < \kappa^+\}$. Therefore it is clopen in X. Thus by the connectedness of X, $\cup\{\mathcal{U}_\alpha: \alpha < \kappa^+\}$ is a cover of X and of cardinality $\leqq\kappa^+$. Then by 6, it has a subcover of cardinality $\leqq\kappa$. Thus the theorem is proved.

Using 11, we can prove similar results of 8, 9, and 10 under the assumption $2^\kappa < 2^{\kappa^+}$ (or $2^\omega < 2^{\omega_1}$). In particular as a corollary, we can prove:

12. *Corollary.* $[2^\omega < 2^{\omega_1}]$ *Locally connected, rim-Lindelöf, normal Moore spaces are strongly paracompact (thus metrizable).*

Remark. Assume $\omega_1 < 2^\omega$ and the Martin's axiom. Then the bubble space derived from a Q-set of reals (see [T]) is locally connected, rim-Lindelöf, normal, non-metrizable Moore space. But $2^\omega = 2^{\omega_1}$ holds.

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