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# TOPOLOGY PROCEEDINGS



Volume 13, 1988

Pages 83–91

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<http://topology.auburn.edu/tp/>

## COMPOSITIONS AND CONTINUOUS RESTRICTIONS OF CONNECTIVITY FUNCTIONS

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### Topology Proceedings

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**ISSN:** 0146-4124

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## COMPOSITIONS AND CONTINUOUS RESTRICTIONS OF CONNECTIVITY FUNCTIONS

Richard G. Gibson, Harvey Rosen, and Fred Roush

In [1] a connectivity function  $f: I \rightarrow I$  was constructed which had the property that the restriction of the function to any perfect set was not continuous. One purpose of this paper is to show that this does not hold true for connectivity functions defined on  $I^n$ ,  $n \geq 2$ , and having any metric space as its range. In [4] it is proved that for a continuum  $Y$ ,  $g: I \rightarrow Y$  is continuous on  $(0,1)$  if  $f: I^2 \rightarrow I$  is continuous and onto and the composition  $g \circ f$  is a connectivity function. It was asked if this result would hold if  $f$  is a connectivity function, (p. 1343). In the present paper we show that the answer is "yes" even if we only require  $f$  to be a Darboux function.

A function  $f: X \rightarrow Y$  is a *connectivity function* if for each connected subset  $C$  of  $X$  the graph of  $f$  restricted to  $C$  is connected in  $X \times Y$ . The function  $f$  is said to be *peripherally continuous* if for each  $x$  in  $X$  and for every pair of open sets  $U$  and  $V$  containing  $x$  and  $f(x)$ , respectively, there is an open set  $W$  contained in  $U$  and containing  $x$  such that  $f(\text{bd}(W)) \subset V$  where  $\text{bd}$  = boundary. The function  $f$  is a *Darboux function* if for each connected subset  $C$  of  $X$ ,  $f(C)$  is a connected subset of  $Y$ . For functions having domain  $I^n$ ,  $n \geq 2$ , and range a metric space, connectivity functions and peripherally continuous functions are the same [5].

By a neighborhood  $\bar{M}$  of  $x$ , we mean an open set  $M$  containing  $x$  with closure  $\bar{M}$ .

*Theorem 1.* If  $f: I^n \rightarrow Y$ ,  $n \geq 2$ , is a connectivity function, then for each  $x \in I^n$  there exists a perfect set  $K$  containing  $x$  such that  $f|K$  is continuous.

*Proof.* Select any  $x \in I^n$ . If  $x$  is not a corner of  $I^n$ , let  $L$  be any line segment such that  $x$  is not an endpoint. If  $x$  is a corner, let  $L$  be the union of two line segments such that  $x$  is an endpoint of each.

Let  $\bar{M}_1$  be a neighborhood with center  $x$  and  $\text{diameter}(\bar{M}_1) < 1$  such that  $\bar{M}_1$  contains neither endpoints of  $L$ . Let  $\bar{N}_1$  be a neighborhood of  $f(x)$  with  $\text{diameter}(\bar{N}_1) < 1$ . Since  $f$  is peripherally continuous, there exists a connected open set in  $M_1$  containing  $x$  with a connected boundary  $F \subset M_1$  such that  $f(F) \subset N_1$ . Choose  $x_0 \neq x_1$  in  $F \cap L$ . Let  $\bar{B}_0$  and  $\bar{B}_1$  be disjoint neighborhoods such that neither contains  $x$ ,  $x_0 \in B_0 \subset \bar{B}_0 \subset M_1$ ,  $\text{diameter}(\bar{B}_0) < 1$ ,  $x_1 \in B_1 \subset \bar{B}_1 \subset M_1$ , and  $\text{diameter}(\bar{B}_1) < 1$ .

If  $f(x_0) \neq f(x_1)$ , let  $\bar{D}_0$  and  $\bar{D}_1$  be disjoint neighborhoods such that  $f(x_0) \in D_0 \subset \bar{D}_0 \subset N_1$ ,  $\text{diameter}(\bar{D}_0) < 1$ ,  $f(x_1) \in D_1 \subset \bar{D}_1 \subset N_1$ , and  $\text{diameter}(\bar{D}_1) < 1$ . If  $f(x_0) = f(x_1)$ , let  $\bar{D}_0 = \bar{D}_1$ .

Then there exists a connected open set containing  $x_0$  and contained in  $B_0$  with a connected boundary  $C_0$  such that  $f(C_0) \subset D_0$ . Also there exists a connected open set containing  $x_1$  and contained in  $B_1$  with a connected boundary  $C_1$  such that  $f(C_1) \subset D_1$ . Now  $\bar{B}_0 \cup \bar{B}_1 \subset M_1$  and  $\bar{D}_0 \cup \bar{D}_1 \subset N_1$ .

Let  $x_{00}, x_{01} \in C_0 \cap L$  and  $x_{10}, x_{11} \in C_1 \cap L$ . Let  $\bar{B}_{00}$  and  $\bar{B}_{01}$  be disjoint neighborhoods such that  $x_{00} \in B_{00} \subset \bar{B}_{00} \subset B_0$ ,  $\text{diameter}(\bar{B}_{00}) < 1/2$ ,  $x_{01} \in B_{01} \subset \bar{B}_{01} \subset B_0$ , and  $\text{diameter}(\bar{B}_{01}) < 1/2$ .

If  $f(x_{00}) \neq f(x_{01})$ , let  $\bar{D}_{00}$  and  $\bar{D}_{01}$  be disjoint neighborhoods such that  $f(x_{00}) \in D_{00} \subset \bar{D}_{00} \subset D_0$ ,  $\text{diameter}(\bar{D}_{00}) < 1/2$ ,  $f(x_{01}) \in D_{01} \subset \bar{D}_{01} \subset D_0$ , and  $\text{diameter}(\bar{D}_{01}) < 1/2$ . If  $f(x_{00}) = f(x_{01})$ , let  $\bar{D}_{00} = \bar{D}_{01}$ .

Then there exists a connected open set containing  $x_{00}$  and contained in  $B_{00}$  with connected boundary  $C_{00}$  such that  $f(C_{00}) \subset D_{00}$ . Also there exists a connected set containing  $x_{01}$  and contained in  $B_{01}$  with connected boundary  $C_{01}$  such that  $f(C_{01}) \subset D_{01}$ .

By letting  $\bar{B}_{10}$  and  $\bar{B}_{11}$  have an analogous meaning and by continuing this procedure, we obtain a dyadic system

$$\begin{aligned} & \bar{B}_{c_1 \dots c_k} \quad \text{of non-empty closed sets satisfying} \\ & \bar{B}_{c_1 \dots c_k c_{k+1}} \subset \bar{B}_{c_1 \dots c_k}, \quad \text{diameter}(\bar{B}_{c_1 \dots c_k}) < 1/k, \\ & C_{c_1 \dots c_k} \subset \bar{B}_{c_1 \dots c_k}, \quad \bar{D}_{c_1 \dots c_k c_{k+1}} \subset \bar{D}_{c_1 \dots c_k}, \\ & f(C_{c_1 \dots c_k}) \subset D_{c_1 \dots c_k}, \quad \text{and diameter}(\bar{D}_{c_1 \dots c_k}) < 1/k. \end{aligned}$$

Now it follows that the set

$$A_1 = \bigcap_{k=1}^{\infty} \bar{B}_{c_1 \dots c_k}$$

where the union is taken over all systems of  $k$  digits  $c_1 \dots c_k$  is homeomorphic to the standard Cantor set and  $A_1 \subset L$ .

We now show that  $f|_{A_1}$  is continuous. Let  $a \in A_1$  and let  $a_n$  be a sequence in  $A_1$  such that  $a_n$  converges

to  $a$ . We need to show that  $f(a_n)$  converges to  $f(a)$ .

First we show that if  $x_{c_1 \dots c_m} \in C_{c_1 \dots c_m}$  and  $x_{c_1 \dots c_m}$  converges to  $a$ , then  $f(x_{c_1 \dots c_m})$  converges to  $f(a)$ . Now  $\bigcup_{n=0}^{\infty} C_{c_1 \dots c_m \dots c_{m+n}}$  is connected for each  $m$  and  $a \in \overline{\bigcup_{n=0}^{\infty} C_{c_1 \dots c_m \dots c_{m+n}}}$ . Since  $f$  is a Darboux function,

$$f(\overline{\bigcup_{n=0}^{\infty} C_{c_1 \dots c_m \dots c_{m+n}}}) \subset \overline{f(\bigcup_{n=0}^{\infty} C_{c_1 \dots c_m \dots c_{m+n}})} \subset \overline{D_{c_1 \dots c_m}}.$$

Now  $\text{diameter}(\overline{D_{c_1 \dots c_m}}) < 1/m$ . So

$$\text{dist}(f(a), f(x_{c_1 \dots c_m \dots c_{m+n}})) < 1/m \text{ for each } n = 0, 1, 2, \dots$$

So  $f(x_{c_1 \dots c_m})$  converges to  $f(a)$ .

Let  $\epsilon > 0$ . Since  $a_n$  converges to  $a$ , we can construct a sequence  $x_{d_1 \dots d_{m_n}}$  such that  $x_{d_1 \dots d_{m_n}}$  converges to  $a$

and

$$\text{dist}(f(x_{d_1 \dots d_{m_n}}), f(a_n)) < 1/2\epsilon.$$

From above  $f(x_{d_1 \dots d_{m_n}})$  converges to  $f(a)$ . Thus there

exists a positive integer  $P$  such that if  $n \geq P$ , then  $\text{dist}(f(x_{d_1 \dots d_{m_n}}), f(a)) < 1/2\epsilon$ . So if  $n \geq P$ , then

$$\text{dist}(f(a_n), f(a)) \leq \text{dist}(f(a_n), f(x_{d_1 \dots d_{m_n}})) +$$

$$\text{dist}(f(x_{d_1 \dots d_{m_n}}), f(a)) < 1/2\epsilon + 1/2\epsilon = \epsilon.$$

Therefore  $f(a_n)$  converges to  $f(a)$ .

By induction construct sequences  $\{\bar{M}_n\}$  with diameter( $\bar{M}_n$ )  $< 1/n$  and  $\{\bar{N}_n\}$  with diameter( $\bar{N}_n$ )  $< 1/n$  such that  $x$  is the center of  $\bar{M}_n$ ,  $f(x) \in N_n$ ,  $A_n$  is a Cantor set in  $L$ ,  $A_n$  is a subset of the complement of  $\bar{M}_{n+1}$ ,  $f|_{A_n}$  is continuous, and  $A_i \cap A_j = \emptyset$  whenever  $i \neq j$ .

Let  $K = \{x\} \cup (\bigcup_{n=1}^{\infty} A_n)$ . Then  $K$  is a Cantor set and  $f|_K$  is continuous.

We make the following remarks.

(1) If  $\bar{D}_{c_1 \dots c_m} \cap \bar{D}_{d_1 \dots d_m} = \emptyset$  whenever  $c_1 \dots c_m \neq d_1 \dots d_m$ , then  $f|_{A_n}$  is one-to-one and  $f|_{A_n}$  is a homeomorphism.

(2) In the proof of Theorem 1 it appears that we need something less than the requirement that the function be a connectivity function. The requirements for the function was that it be peripherally continuous and Darboux. With only these conditions, how general can the domain and range spaces be made so that the conclusion of the theorem is still true?

*Theorem 2.* If  $f: I^n \rightarrow I$ ,  $n \geq 2$ , is a Darboux and onto function and  $g: I \rightarrow Y$  is any function such that  $g \circ f: I^n \rightarrow Y$  is a connectivity function where  $Y$  is a metric space, then  $g$  is continuous except perhaps at 0 or 1.

*Proof.* Suppose  $p \in I^n$  and  $f(p) \in (0,1)$ . Let  $q, r \in I^n$  with  $f(q) = 0$  and  $f(r) = 1$ . Let  $A$  denote the line segment from  $p$  to  $q$  and  $B$  the line segment from  $p$  to  $r$ . Since  $g \circ f$  is a connectivity function, it is

peripherally continuous [6],[8]. Therefore given  $\varepsilon > 0$  and given an arbitrary  $a \in I^n$  and an  $\frac{\varepsilon}{4}$ -neighborhood  $G_a$  of  $g(f(a))$  in  $Y$ , there exists a connected open neighborhood  $W_a$  of  $a$  in  $I^n$  such that  $\text{bd}(W_a)$  is connected and  $g \circ f(\text{bd}(W_a)) \subset G_a$ . We may suppose that  $p \notin W_q$ ,  $q \notin W_p$ , and  $p, q \notin W_a$  for all  $a \in A - \{p, q\}$  and that  $p \notin W_r$ ,  $r \notin W_p$ , and  $p, r \notin W_a$  for all  $a \in B - \{p, r\}$ . For each  $a \in I^n$ ,  $\text{diameter}[g \circ f(\text{bd}(W_a))] < \frac{\varepsilon}{2}$  and  $f(\text{bd}(W_a))$  is an interval or singleton because  $f$  is a Darboux function.

We construct a chain from  $p$  to  $q$  as follows. Let  $\mathcal{U}_0$  be a finite subcover of the open cover  $\{W_a : a \in A\}$  for  $A$ . Let  $U_1 = W_p$ . Let  $x_1$  be the point of  $A \cap \text{bd}(U_1)$  that is closest to  $q$ . Some  $W_{a_1}$  in  $\mathcal{U}_0$  contains  $x_1$ . Now,  $\text{bd}(U_1) \cap \text{bd}(W_{a_1}) \neq \emptyset$ ; otherwise since  $x_1 \in W_{a_1} \cap \text{bd}(U_1)$ , then  $U_1 \subset W_{a_1}$ , in contradiction to  $p \notin W_{a_1}$ . Let  $U_2 = W_{a_1}$ . Let  $x_2$  be the point of  $A \cap \text{bd}(U_2)$  that is closest to  $q$ . Some  $W_{a_2}$  in  $\mathcal{U}_0$  contains  $x_2$ . Let  $U_3 = W_{a_2}$ . If  $\text{bd}(U_3) \cap \text{bd}(U_2) \neq \emptyset$ , then we would so far have a chain  $\{U_1, U_2, U_3\}$  from  $p$  to  $x_2$ . If  $\text{bd}(U_3) \cap \text{bd}(U_2) = \emptyset$ , then  $\text{bd}(U_3) \cap \text{bd}(U_1) \neq \emptyset$  because, otherwise, it would follow that  $U_1 \cup U_2 \subset U_3$ , in contradiction to  $p \notin W_{a_2}$ . Then we would have a chain  $\{U_1, U_3\}$  from  $p$  to  $x_2$ . Using induction and the finiteness of  $\mathcal{U}_0$ , we can finish constructing a chain  $\mathcal{U}_1 = \{U_{i_1}, U_{i_2}, \dots, U_{i_m}\} \subset \mathcal{U}_0$  from  $p$  to  $q$  where  $U_{i_1} = W_p$ ,  $U_{i_m} = W_q$ , and  $\text{bd}(U_{i_k}) \cap \text{bd}(U_{i_{k+1}}) \neq \emptyset$  for  $k = 1, 2, \dots, m - 1$ .

We show that the chain  $U_1$  from  $p$  to  $q$  can be chosen in such a way that  $f(\cup_{U \in U_1} \text{bd}(U)) \cap [0, f(p)) \neq \emptyset$ . Construct similarly, a sequence  $C_1, C_2, C_3, \dots$  of chains each from  $p$  to  $q$  so that each chain  $C_i$  has the same properties as  $U_1$ , but so that  $\text{mesh}(C_i) < \frac{1}{i}$  and the set  $C = \{p, q\} \cup (\cup_{i=1}^\infty \{\text{bd}(U) : U \in C_i\})$  is connected. Since  $f$  is Darboux,  $f(C) \supset [0, f(p)]$ . Therefore for some  $k$  and some  $U \in C_k$ ,  $f(\text{bd}(U)) \cap [0, f(p)) \neq \emptyset$ . Now  $U_1$  can be chosen to be  $C_k$ .

Let  $V_0$  be a finite subcover of the open cover  $\{W_a : a \in B\}$  for  $B$ . Similarly, we can construct a chain  $U_2 = \{V_{j_1}, V_{j_2}, \dots, V_{j_s}\} \subset V_0$  from  $p$  to  $r$  where  $V_{j_1} = W_p$ ,  $V_{j_s} = W_r$ ,  $\text{bd}(V_{j_k}) \cap \text{bd}(V_{j_{k+1}}) \neq \emptyset$  for  $k = 1, 2, \dots, s - 1$ , and  $f(\text{bd}(V)) \cap (f(p), 1] \neq \emptyset$  for some  $V \in U_2$ . Let  $U = U_1 \cup U_2$ .

Because  $\cup\{\text{bd}(U) : U \in U\}$  is a connected set  $J$ ,  $f(J)$  is by construction an interval whose interior contains  $f(p)$ . We claim there exists  $W_1$  in  $U$  such that  $f(\text{bd}(W_1))$  is nondegenerate and contains  $f(p)$ . There exists  $W$  in  $U$  such that  $f(p) \in f(\text{bd}(W))$ . If  $f(\text{bd}(W)) = \{f(p)\}$ , then there exist  $W_0, W_1 \in U$  such that  $f(\text{bd}(W_0)) = \{f(p)\}$ ,  $\text{bd}(W_0) \cap \text{bd}(W_1) \neq \emptyset$ , and  $f(\text{bd}(W_1))$  is nondegenerate. Therefore  $f(p) \in f(\text{bd}(W_1))$ . In case  $f(p) \in \text{int } f(\text{bd}(W_1))$ , we let  $K = f(\text{bd}(W_1))$ . But in case  $f(p)$  is instead an endpoint of  $f(\text{bd}(W_1))$ , it follows that there is  $W_2 \in U$  such that  $(f(p) \in f(\text{bd}(W_2)))$  and  $f(p) \in \text{int}[f(\text{bd}(W_1)) \cup$



$f(\text{bd}(W_2))]$ . For this case we let  $K = f(\text{bd}(W_1)) \cup f(\text{bd}(W_2))$ . In either case,  $f(p) \in \text{int}(K)$  and  $\text{diameter}(g(K)) < \epsilon$ . This implies  $g$  is continuous at  $f(p)$ .

We make the following observation.

(3) If  $f^{-1}(0)$  has a nondegenerate component  $C$ , then  $g$  is continuous at 0. To see this, first choose  $x \in I^n - C$  and let  $L$  be a line segment from  $x$  to a point of  $C$ .  $C$  is closed because  $f$  is a Darboux function [6]. Let  $y$  be the point of  $L \cap C$  closest to  $x$ . For each  $\epsilon > 0$ , there exists a connected open neighborhood  $W$  of  $y$  such that  $\text{bd}(W)$  is connected,  $C \cap \text{bd}(W) \neq \emptyset$ ,  $x \notin \bar{W}$ , and  $\text{diameter}(g \circ f(\text{bd}(W))) < \epsilon$ . It follows that  $\text{bd}(W)$  must meet  $L - C$  in at least one point  $z$  between  $x$  and  $y$ . Since  $\text{bd}(W) \not\subset C$  and  $\text{bd}(W)$  is connected,  $f(\text{bd}(W))$  is nondegenerate and therefore is an interval containing 0. Since  $\text{diameter}(g \circ f(\text{bd}(W))) < \epsilon$ ,  $g$  is continuous at 0.

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