TOPOLOGY PROCEEDINGS

Volume 13, 1988

Pages 83–91

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

COMPOSITIONS AND CONTINUOUS RESTRICTIONS OF CONNECTIVITY FUNCTIONS

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In [1] a connectivity function $f: I \to I$ was constructed which had the property that the restriction of the function to any perfect set was not continuous. One purpose of this paper is to show that this does not hold true for connectivity functions defined on I^n, $n \geq 2$, and having any metric space as its range. In [4] it is proved that for a continuum Y, $g: I \to Y$ is continuous on $(0,1)$ if $f: I^2 \to I$ is continuous and onto and the composition $g \circ f$ is a connectivity function. It was asked if this result would hold if f is a connectivity function, (p. 1343). In the present paper we show that the answer is "yes" even if we only require f to be a Darboux function.

A function $f: X \to Y$ is a *connectivity function* if for each connected subset C of X the graph of f restricted to C is connected in $X \times Y$. The function f is said to be *peripherally continuous* if for each x in X and for every pair of open sets U and V containing x and $f(x)$, respectively, there is an open set W contained in U and containing x such that $f(bd(W)) \subset V$ where bd = boundary. The function f is a *Darboux function* if for each connected subset C of X, $f(C)$ is a connected subset of Y. For functions having domain I^n, $n \geq 2$, and range a metric space, connectivity functions and peripherally continuous functions are the same [5].

By a neighborhood $\overline{M}$ of x, we mean an open set M containing x with closure $\overline{M}$.

Theorem 1. *If* $f: I^n \to Y$, $n \geq 2$, *is a connectivity function, then for each* $x \in I^n$ *there exists a perfect set* K *containing* x *such that* $f|K$ *is continuous.*

Proof. Select any $x \in I^n$. If x is not a corner of I^n, let L be any line segment such that x is not an endpoint. If x is a corner, let L be the union of two line segments such that x is an endpoint of each.

Let $\overline{M}_1$ be a neighborhood with center x and diameter$(\overline{M}_1) < 1$ such that $\overline{M}_1$ contains neither endpoints of L. Let $\overline{N}_1$ be a neighborhood of f(x) with diameter$(\overline{N}_1) < 1$. Since f is peripherally continuous, there exists a connected open set in M_1 containing x with a connected boundary $F \subset M_1$ such that $f(F) \subset N_1$. Choose $x_0 \neq x_1$ in $F \cap L$. Let $\overline{B}_0$ and $\overline{B}_1$ be disjoint neighborhoods such that neither contains x, $x_0 \in B_0 \subset \overline{B}_0 \subset M_1$, diameter$(\overline{B}_0) < 1$, $x_1 \in B_1 \subset \overline{B}_1 \subset M_1$, and diameter$(\overline{B}_1) < 1$.

If $f(x_0) \neq f(x_1)$, let $\overline{D}_0$ and $\overline{D}_1$ be disjoint neighborhoods such that $f(x_0) \in D_0 \subset \overline{D}_0 \subset N_1$, diameter$(\overline{D}_0) < 1$, $f(x_1) \in D_1 \subset \overline{D}_1 \subset N_1$, and diameter$(\overline{D}_1) < 1$. If $f(x_0) = f(x_1)$, let $\overline{D}_0 = \overline{D}_1$.

Then there exists a connected open set containing x_0 and contained in B_0 with a connected boundary C_0 such that $f(C_0) \subset D_0$. Also there exists a connected open set containing x_1 and contained in B_1 with a connected boundary C_1 such that $f(C_1) \subset D_1$. Now $\overline{B}_0 \cup \overline{B}_1 \subset M_1$ and $\overline{D}_0 \cup \overline{D}_1 \subset N_1$.

Let $x_{00}, x_{01} \in C_0 \cap L$ and $x_{10}, x_{11} \in C_1 \cap L$. Let $\overline{B}_{00}$ and $\overline{B}_{01}$ be disjoint neighborhoods such that
$x_{00} \in B_{00} \subset \overline{B}_{00} \subset B_0$, diameter$(\overline{B}_{00}) < 1/2$,
$x_{01} \in B_{01} \subset \overline{B}_{01} \subset B_0$, and diameter$(\overline{B}_{01}) < 1/2$.

If $f(x_{00}) \neq f(x_{01})$, let $\overline{D}_{00}$ and $\overline{D}_{01}$ be disjoint neighborhoods such that $f(x_{00}) \in D_{00} \subset \overline{D}_{00} \subset D_0$, diameter$(\overline{D}_{00}) < 1/2$, $f(x_{01}) \in D_{01} \subset \overline{D}_{01} \subset D_0$, and diameter$(\overline{D}_{01}) < 1/2$. If $f(x_{00}) = f(x_{01})$, let $\overline{D}_{00} = \overline{D}_{01}$.

Then there exists a connected open set containing x_{00} and contained in B_{00} with connected boundary C_{00} such that $f(C_{00}) \subset D_{00}$. Also there exists a connected set containing x_{01} and contained in B_{01} with connected boundary C_{01} such that $f(C_{01}) \subset D_{01}$.

By letting $\overline{B}_{10}$ and $\overline{B}_{11}$ have an analogous meaning and by continuing this procedure, we obtain a dyadic system

$\overline{B}_{c_1 \ldots c_k}$ of non-empty closed sets satisfying

$\overline{B}_{c_1 \ldots c_k c_{k+1}} \subset \overline{B}_{c_1 \ldots c_k}$, diameter$(\overline{B}_{c_1 \ldots c_k}) < 1/k$,

$C_{c_1 \ldots c_k} \subset \overline{B}_{c_1 \ldots c_k}$, $\overline{D}_{c_1 \ldots c_k c_{k+1}} \subset \overline{D}_{c_1 \ldots c_k}$,

$f(C_{c_1 \ldots c_k}) \subset D_{c_1 \ldots c_k}$, and diameter $(\overline{D}_{c_1 \ldots c_k}) < 1/k$.

Now it follows that the set

$$A_1 = \cap_{k=1}^{\infty} \cup \overline{B}_{c_1 \ldots c_k}$$

where the union is taken over all systems of k digits $c_1 \ldots c_k$ is homeomorphic to the standard Cantor set and $A_1 \subset L$.

We now show that $f|A_1$ is continuous. Let $a \in A_1$ and let a_n be a sequence in A_1 such that a_n converges to a. We need to show that $f(a_n)$ converges to $f(a)$.

First we show that if $x_{c_1 \ldots c_m} \in C_{c_1 \ldots c_m}$ and $x_{c_1 \ldots c_m}$ converges to a, then $f(x_{c_1 \ldots c_m})$ converges to $f(a)$. Now $\cup_{n=0}^{\infty} C_{c_1 \ldots c_m \ldots c_{m+n}}$ is connected for each m and $a \in \overline{\cup_{n=0}^{\infty} C_{c_1 \ldots c_m \ldots c_{m+n}}}$. Since f is a Darboux function,

$$f(\overline{\cup_{n=0}^{\infty} C_{c_1 \ldots c_m \ldots c_{m+n}}}) \subset \overline{f(\cup_{n=0}^{\infty} C_{c_1 \ldots c_m \ldots c_{m+n}})} \subset \overline{D}_{c_1 \ldots c_m}.$$

Now diameter$(\overline{D}_{c_1 \ldots c_m}) < 1/m$. So

$\text{dist}(f(a), f(x_{c_1 \ldots c_m \ldots c_{m+n}})) < 1/m$ for each $n = 0, 1, 2, \ldots$.

So $f(x_{c_1 \ldots c_m})$ converges to $f(a)$.

Let $\varepsilon > 0$. Since a_n converges to a, we can construct a sequence $x_{d_1 \ldots d_{m_n}}$ such that $x_{d_1 \ldots d_{m_n}}$ converges to a and

$$\text{dist}(f(x_{d_1 \ldots d_{m_n}}), f(a_n)) < 1/2\varepsilon.$$

From above $f(x_{d_1 \ldots d_{m_n}})$ converges to $f(a)$. Thus there exists a positive integer P such that if $n \geq P$, then $\text{dist}(f(x_{d_1 \ldots d_{m_n}}), f(a)) < 1/2\varepsilon$. So if $n \geq P$, then

$$\text{dist}(f(a_n), f(a)) \leq \text{dist}(f(a_n), f(x_{d_1 \ldots d_{m_n}})) + \text{dist}(f(x_{d_1 \ldots d_{m_n}}), f(a)) < 1/2\varepsilon + 1/2\varepsilon = \varepsilon.$$

Therefore $f(a_n)$ converges to $f(a)$.

By induction construct sequences $\{\overline{M}_n\}$ with diameter$(\overline{M}_n) < 1/n$ and $\{\overline{N}_n\}$ with diameter$(\overline{N}_n) < 1/n$ such that x is the center of $\overline{M}_n$, $f(x) \in N_n$, A_n is a Cantor set in L, A_n is a subset of the complement of $\overline{M}_{n+1}$, $f|A_n$ is continuous, and $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Let $K = \{x\} \cup (\cup_{n=1}^{\infty} A_n)$. Then K is a Cantor set and $f|K$ is continuous.

We make the following remarks.

(1) If $\overline{D}_{c_1\ldots c_m} \cap \overline{D}_{d_1\ldots d_m} = \emptyset$ whenever $c_1\ldots c_m \neq d_1\ldots d_m$, then $f|A_n$ is one-to-one and $f|A_n$ is a homeomorphism.

(2) In the proof of Theorem 1 it appears that we need something less than the requirement that the function be a connectivity function. The requirements for the function was that it be peripherally continuous and Darboux. With only these conditions, how general can the domain and range spaces be made so that the conclusion of the theorem is still true?

Theorem 2. *If* $f: I^n \to I$, $n \geq 2$, *is a Darboux and onto function and* $g: I \to Y$ *is any function such that* $g \circ f: I^n \to Y$ *is a connectivity function where* Y *is a metric space, then* g *is continuous except perhaps at* 0 *or* 1.

Proof. Suppose $p \in I^n$ and $f(p) \in (0,1)$. Let $q, r \in I^n$ with $f(q) = 0$ and $f(r) = 1$. Let A denote the line segment from p to q and B the line segment from p to r. Since $g \circ f$ is a connectivity function, it is peripherally continuous [6],[8]. Therefore given $\varepsilon > 0$ and given an arbitrary $a \in I^n$ and an $\frac{\varepsilon}{4}$-neighborhood G_a of $g(f(a))$ in Y, there exists a connected open neighborhood W_a of a in I^n such that $bd(W_a)$ is connected and $g \circ f(bd(W_a)) \subset G_a$. We may suppose that $p \notin W_q$, $q \notin W_p$, and $p,q \notin W_a$ for all $a \in A - \{p,q\}$ and that $p \notin W_r$, $r \notin W_p$, and $p,r \notin W_a$ for all $a \in B - \{p,r\}$. For each $a \in I^n$, diameter$[g \circ f(bd(W_a))] < \frac{\varepsilon}{2}$ and $f(bd(W_a))$ is an interval or singleton because f is a Darboux function.

We construct a chain from p to q as follows. Let $\mathcal{U}_0$ be a finite subcover of the open cover $\{W_a : a \in A\}$ for A. Let $U_1 = W_p$. Let x_1 be the point of $A \cap bd(U_1)$ that is closest to q. Some W_{a_1} in $\mathcal{U}_0$ contains x_1. Now, $bd(U_1) \cap bd(W_{a_1}) \neq \emptyset$; otherwise since $x_1 \in W_{a_1} \cap bd(U_1)$, then $U_1 \subset W_{a_1}$, in contradiction to $p \notin W_{a_1}$. Let $U_2 = W_{a_1}$. Let x_2 be the point of $A \cap bd(U_2)$ that is closest to q. Some W_{a_2} in $\mathcal{U}_0$ contains x_2. Let $U_3 = W_{a_2}$. If $bd(U_3) \cap bd(U_2) \neq \emptyset$, then we would so far have a chain $\{U_1,U_2,U_3\}$ from p to x_2. If $bd(U_3) \cap bd(U_2) = \emptyset$, then $bd(U_3) \cap bd(U_1) \neq \emptyset$ because, otherwise, it would follow that $U_1 \cup U_2 \subset U_3$, in contradiction to $p \notin W_{a_2}$. Then we would have a chain $\{U_1,U_3\}$ from p to x_2. Using induction and the finiteness of $\mathcal{U}_0$, we can finish constructing a chain $\mathcal{U}_1 = \{U_{i_1},U_{i_2},\ldots,U_{i_m}\} \subset \mathcal{U}_0$ from p to q where $U_{i_1} = W_p$, $U_{i_m} = W_q$, and $bd(U_{i_k}) \cap bd(U_{i_{k+1}}) \neq \emptyset$ for $k = 1,2,\ldots,m-1$.

We show that the chain $\mathcal{U}_1$ from p to q can be chosen in such a way that $f(\bigcup_{U \in \mathcal{U}_1} bd(U)) \cap [0,f(p)) \neq \emptyset$. Construct similarly, a sequence $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \cdots$ of chains each from p to q so that each chain $\mathcal{C}_i$ has the same properties as $\mathcal{U}_1$, but so that $\text{mesh}(\mathcal{C}_i) < \frac{1}{i}$ and the set $C = \{p,q\} \cup (\cup(\cup_{i=1}^{\infty}\{bd(U): U \in \mathcal{C}_i\}))$ is connected. Since f is Darboux, $f(C) \supset [0,f(p)]$. Therefore for some k and some $U \in \mathcal{C}_k$, $f(bd(U)) \cap [0,f(p)) \neq \emptyset$. Now $\mathcal{U}_1$ can be chosen to be $\mathcal{C}_k$.

Let $\mathcal{V}_0$ be a finite subcover of the open cover $\{W_a: a \in B\}$ for B. Similarly, we can construct a chain $\mathcal{U}_2 = \{V_{j_1}, V_{j_2}, \ldots, V_{j_s}\} \subset \mathcal{V}_0$ from p to r where $V_{j_1} = W_p$, $V_{j_s} = W_r$, $bd(V_{j_k}) \cap bd(V_{j_{k+1}}) \neq \emptyset$ for $k = 1,2,\ldots,s-1$, and $f(bd(V)) \cap (f(p),1] \neq \emptyset$ for some $V \in \mathcal{U}_2$. Let $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$.

Because $\cup\{bd(U): U \in \mathcal{U}\}$ is a connected set J, f(J) is by construction an interval whose interior contains f(p). We claim there exists W_1 in $\mathcal{U}$ such that $f(bd(W_1))$ is nondegenerate and contains f(p). There exists W in $\mathcal{U}$ such that $f(p) \in f(bd(W))$. If $f(bd(W)) = \{f(p)\}$, then there exist $W_0, W_1 \in \mathcal{U}$ such that $f(bd(W_0)) = \{f(p)\}$, $bd(W_0) \cap bd(W_1) \neq \emptyset$, and $f(bd(W_1))$ is nondegenerate. Therefore $f(p) \in f(bd(W_1))$. In case $f(p) \in \text{int } f(bd(W_1))$, we let $K = f(bd(W_1))$. But in case f(p) is instead an endpoint of $f(bd(W_1))$, it follows that there is $W_2 \in \mathcal{U}$ such that $(f(p) \in f(bd(W_2))$ and $f(p) \in \text{int}[f(bd(W_1)) \cup$

$f(bd(W_2))]$. For this case we let $K = f(bd(W_1)) \cup f(bd(W_2))$. In either case, $f(p) \in int(K)$ and diameter$(g(K)) < \varepsilon$. This implies g is continuous at f(p).

We make the following observation.

(3) If $f^{-1}(0)$ has a nondegenerate component C, then g is continuous at 0. To see this, first choose $x \in I^n - C$ and let L be a line segment from x to a point of C. C is closed because f is a Darboux function [6]. Let y be the point of $L \cap C$ closest to x. For each $\varepsilon > 0$, there exists a connected open neighborhood W of y such that bd(W) is connected, $C \cap bd(W) \neq \emptyset$, $x \notin \overline{W}$, and diameter$(g \circ f(bd(W))) < \varepsilon$. It follows that bd(W) must meet L - C in at least one point z between x and y. Since $bd(W) \not\subset C$ and bd(W) is connected, f(bd(W)) is nondegenerate and therefore is an interval containing 0. Since diameter$(g \circ f(bd(W))) < \varepsilon$, g is continuous at 0.

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