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by

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# REMARKS ON CLOSURE-PRESERVING SUM THEOREMS

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## 1. Introduction

In 1975 Junnila and Potoczny [6] and Katuta [2] independently showed that, if a space X has a closure-preserving cover by compact sets, then X is metacompact. Thus began a study of spaces which have closure-preserving covers by "nice" (finite, countably compact, Lindelof, etc.) sets. The reader is referred to [1,2,4,5,6,8,11,12, 13,16,17,18,19,24] for these developments as well as their relationship with winning strategies in some topological games.

In 2 we show that if a space X has a closure-preserving cover by nowhere dense sets, then X need not be $1^{st}$ category. However, if these sets are also countably compact, then X must be $1^{st}$ category. The closure-preserving property of the cover is strengthened in a natural way in 3, and a somewhat more general result is obtained for the ideal of closed subsets of a space. The desired result for nowhere dense sets follows as a special application.[1] Finally in 4 closure-preserving sum theorems are obtained for the dimension functions dim and Ind for normal and totally normal spaces.

[1]The authors would like to thank the referee for his/her comments on this paper.

*Definition* 1.1. (i) A family $\mathcal{F}$ of subsets of a space X is closure-preserving (c-p) if $\cup\{\overline{F}: F \in \mathcal{F}'\} = \overline{\cup \mathcal{F}'}$ for every $\mathcal{F}' \subseteq \mathcal{F}$.

(ii) A c-p family $\mathcal{F}$ is special if there exists a point finite open collection $\mathcal{U}$ such that for $F \in \mathcal{F}$,

$$X - \overline{F} = \cup\{U \in \mathcal{U}: U \cap F = \phi\}.$$

In this case we say that $\mathcal{U}$ generates $\mathcal{F}$.

*Definition* 1.2. A family $\mathcal{J}$ of closed subsets of a space X is an ideal of closed sets if,

(i) for every finite $\mathcal{J}' \subseteq \mathcal{J}$, $\cup \mathcal{J}' \in \mathcal{J}$ and

(ii) if H is a closed set and $H \subseteq J \in \mathcal{J}$, then $H \in \mathcal{J}$.

We will denote the family of countable unions of members of $\mathcal{J}$ by $\sigma\mathcal{J}$.

*Remark.* As noted in [1], if $\mathcal{U}$ is a point finite open cover which generates a c-p $\mathcal{J}$-cover, then the family

$$\mathcal{C}_n = \{X_n \cap [\cap \mathcal{U}_x]: x \in X \text{ and } |\mathcal{U}_x| = n\}$$

consists of $\mathcal{J}$-small sets, and $\mathcal{C}_n$ is a discrete collection of relative closed sets in $X_n - X_{n-1}$, where

$$X_n = \{x \in X: |\mathcal{U}_x| \leq n\}.$$

*Definition* 1.3. A space X is called weakly $\overline{\theta}$-refinable if every open cover of X has an open refinement $\cup_{i=1}^{\infty} \mathcal{G}_i$ satisfying

(i) $\{G_i\}_{i=1}^{\infty}$ is point finite, where $G_i = \cup\mathcal{G}_i$ and

(ii) for each $x \in X$, there exists some $n(x)$ such that $0 < \operatorname{ord}(x, \mathcal{G}_{n(x)}) < \infty$.

A cover $\cup_{i=1}^{\infty} \mathcal{G}_i$ satisfying (i) and (ii) above is called a <u>weak $\bar{\theta}$-cover</u>.

It is well known that the class of weakly $\bar{\theta}$-refinable spaces lies strictly between the classes of $\theta$-refinable and weakly $\theta$-refinable spaces (see [8]).

All spaces will be assumed to be $T_2$.

## 2. Closure-preserving covers by nowhere dense sets

In this paper we consider spaces which have a closure-preserving cover by nowhere dense sets. Note that if $Q = \{r_1, r_2, \cdots\}$ is the set of rationals and $F_n = \{r_i\}_{i=1}^{n}$, then $\{F_i: i < \omega\}$ is a closure-preserving cover of Q by finite sets. Clearly, Q is a $1^{st}$ category space.

The following example shows that if a space X has a closure-preserving cover by nowhere dense sets, then X need not be of the $1^{st}$ category.

*Example.* Let $L = \{0,1\}^{\omega_1}$ where $\{0,1\}$ is the two point discrete space. The topology on Y is the countable box topology; that is, a basic open set in Y is a countable intersection of basic open sets in the usual product topology. Define $X = \{y \in Y: |\alpha < \omega_1: y(\alpha) \neq 0| < \omega_1\}$. Now if $F_\alpha = \{x \in X: x(\beta) = 0 \text{ for each } \beta \geq \alpha\}$, it is easy to check that $\{F_\alpha: \alpha < \omega_1\}$ is a closure-preserving cover of X by discrete closed sets. Furthermore each $F_\alpha$ is nowhere dense. We assert that X is not $1^{st}$ category. Indeed, let $\{G_n: n < \omega\}$ be a sequence of open dense subsets of X. Let $B_0$ be a basic open set such that $B_0 \subseteq G_0$, and define

$$C_0 = \{\alpha < \omega_1 \colon x(\alpha) = y(\alpha) \text{ for all } x,y \in B_0\}.$$

Now by induction choose a basic open set $B_n \subseteq G_n \cap B_{n-1}$ and let

$$C_n = \{\alpha < \omega_1 \colon x(\alpha) = y(\alpha) \text{ for all } x,y \in B_n\}.$$

Since $C = \bigcup_{n<\omega} C_n$ is countable, there exists a point $z \in X$ such that $z \in \bigcap_{n<\omega} B_n \subseteq \bigcap_{n<\omega} G_n$ and $z(\alpha) = 0$ for $\alpha > \sup C$. Therefore X is not $1^{st}$ category.

The following result is an easy consequence of Zorn's Lemma so the proof is omitted.

*Lemma* 2.1. *Let $\mathcal{F} = \{F_\alpha \colon \alpha \in \Lambda\}$ be any non-empty family of non-empty sets. Then there exists a maximal pairwise disjoint subfamily of $\mathcal{F}$.*

*Definition* 2.2. Let $\phi \in E \subseteq X$ and $\mathcal{F}$ a family of subsets of X.

(1) $\mathcal{M}(E)$ = a maximal pairwise disjoint subfamily of $\{F \cap E \colon F \in \mathcal{F}\}$.

(2) $M^*(E) = \cup \mathcal{M}(E)$.

The following uses a technique due to Telgársky and Yajima [16: Theorem 3.3].

*Lemma* 2.3. *Let $\mathcal{C} = \{C_\alpha \colon \alpha \in \Lambda\}$ be a closure-preserving cover of* X *by countably compact sets. If $\{H_n\}_{n=1}^{\infty}$ is any decreasing sequence of closed subsets of* X *satisfying $H_{n+1} \subseteq X - M^*(H_n)$ for each* n, *then $\bigcap_{n=1}^{\infty} H_n = \phi$.*

*Proof*. Suppose there exists a sequence $\{H_n\}_{n=1}^{\infty}$ of closed sets satisfying the above condition, but

$\cap_{n=1}^{\infty} H_n \neq \phi$. Then there exists some $C_{\alpha_0} \in \mathcal{C}$ such that $C_{\alpha_0} \cap H_n \neq \phi$ for each n. Furthermore $(C_{\alpha_0} \cap H_n) \notin \mathcal{M}(H_n)$; for otherwise, $C_{\alpha_0} \cap H_{n+1} = \phi$. Therefore for each n, there exists some $C_{\alpha_n} \in \mathcal{C}$ such that $(C_{\alpha_n} \cap H_n) \in \mathcal{M}(H_n)$ and $C_{\alpha_n} \cap (C_{\alpha_0} \cap H_n) \neq \phi$. Now choose $x_n \in C_{\alpha_n} \cap (C_{\alpha_0} \cap H_n)$ so that $x_n \in M^*(H_n)$ and hence $x_n \notin H_{n+1}$. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence of distinct points in $C_{\alpha_0}$ and must cluster at $y \in C_{\alpha_0}$. Now $y \in \overline{\{x_n: n \geq 1\}} \subseteq \cup_{n=1}^{\infty} C_{\alpha_n}$ and $y \in \cap_{n=1}^{\infty} H_n$ as well. But this is a contradiction, since $y \in (C_{\alpha_n} \cap H_n) \in \mathcal{M}(H_n)$ implies that $y \notin H_{n+1}$. Therefore it must be the case that $\cap_{n=1}^{\infty} H_n = \phi$.

*Remark.* It should be noted that the above proof only used the fact that every countable subfamily of $\mathcal{C}$ was closure-preserving.

*Theorem* 2.4. *Let* X *be a regular space with a closure-preserving cover* $\mathcal{C}$ *by countably compact nowhere dense sets. Then* X *is* $1^{st}$ *category.*

*Proof.* Since $M^*(X)$ is closed and nowhere dense in X, by Lemma 2.1 above, there exists a maximal pairwise disjoint family $\mathcal{H}(X)$ of regular closed subsets of X such that $\cup \mathcal{H}(X) \subseteq X - M^*(X)$. Furthermore, it is easy to see that $\cup \mathcal{H}(X)$ is dense in X. Define $\mathcal{G}_0 = \{\text{int}(H): H \in \mathcal{H}(X)\}$ and $G_0 = \cup \mathcal{G}_0$. Note that $G_0$ is also dense in X. Now by induction we construct the sequence $(\mathcal{G}_0, G_0, \mathcal{G}_1, G_1, \cdots \mathcal{G}_n, G_n \cdots)$ where

$$\mathcal{G}_{n+1} = \{\text{int}(H): H \in \mathcal{H}(\overline{G}) \text{ where } G \in \mathcal{G}_n\},$$
$$G_{n+1} = \cup \mathcal{G}_{n+1} \quad \text{and}$$

$\mathcal{H}(\overline{G})$ = a maximal pairwise disjoint family of regular closed subsets of X such that $\cup\mathcal{H}(\overline{G})$ is dense in $\overline{G} - M^*(\overline{G})$.

As before it is easy to see that $G_n$ is open and dense in X.

Finally, if $H_0 = X$ and $H_{n+1} \in \mathcal{H}(H_n)$, by Lemma 2.3, $\cap_{n=0}^{\infty} H_n = \phi$. Therefore $\cap_{n=0}^{\infty} G_n = \phi$ and hence X is $1^{st}$ category.

*Corollary* 2.5. *Let* X *be a regular countably compact space. If* $\mathcal{J}$ *is a closure-preserving cover of* X*, then some member of* $\mathcal{J}$ *has non-empty interior.*

## 3. Special c-p covers

In [1] the authors studied spaces having c-p $\mathcal{J}$-covers induced by a point finite open cover. Here we get analogous results to Theorem 2.4 above by omitting the regularity and countably compact conditions.

The notion of B(D,λ)-refinability has been shown to play an important role in the study of weakly $\overline{\theta}$-refinable spaces introduced in [7].

*Definition* 3.1. A space X is B(D,λ)-refinable provided every open cover $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ of X has a refinement $\mathcal{E} = \cup\{\mathcal{E}_\beta = \{E(\beta,\gamma) : \gamma \in \Gamma\} : \beta < \lambda\}$ where satisfies

(i) $E(\beta,\gamma) \subseteq U_\gamma$ for each $\gamma \in \Gamma$, $\beta < \lambda$,

(ii) $\{\cup\mathcal{E}_\beta : \beta < \lambda\}$ partitions X,

(iii) for each $\beta < \lambda$, $\mathcal{E}_\beta$ is a closed discrete collection in $X - \cup\{\cup\mathcal{E}_\mu : \mu < \beta\}$, and

(iv) for each $\beta < \lambda$, $\cup\{\cup\mathcal{E}_\mu : \mu < \beta\}$ is closed in X.

In [7] the author has shown that every space which is $B(D,\omega)$-refinable is weakly $\overline{\theta}$-refinable and that every weakly $\overline{\theta}$-refinable space is $B(D,(\omega)^2)$-refinable. Throughout this paper $\lambda$ will denote a countable ordinal.

*Definition* 3.2. Let $\mathcal{J}$ be an ideal of closed subsets of a space X. We say that $\mathcal{J}$ is <u>discretely additive</u> if

(i) $\mathcal{J}$ is closed under discrete unions, and

(ii) when $F \in \sigma\mathcal{J}$ and $\mathcal{D} \subseteq \mathcal{J}$ such that $\mathcal{D}$ is discrete in $X - F$, then $H = F \cup \{\cup\mathcal{D}\} \in \sigma\mathcal{J}$.

*Theorem* 3.3. *If a space* X *has a* $B(D,\lambda) - \mathcal{J}$ *cover for some countable ordinal* $\lambda$ *and* $\mathcal{J}$ *is discretely additive, then* $X \in \sigma\mathcal{J}$.

*Proof*. Let $\mathcal{E} = \cup\{\mathcal{E}_\beta\colon \beta < \lambda\}$, where $\mathcal{E}_\beta = \{E(\beta,\gamma)\colon \gamma \in \Gamma\}$, be a $B(D,\lambda) - \mathcal{J}$ cover of X such that $\mathcal{J}$ is discretely additive. The proof is by induction on $\lambda$. It is easy for $\lambda = 1$; since $\mathcal{J}$ is closed under discrete unions, so $X \in \mathcal{J}$. Let $\beta < \lambda$ and assume true for all $\gamma < \beta$. If $\beta = \alpha_0 + 1$, define $F = \cup\{\cup\mathcal{E}_\mu\colon \mu \leq \alpha_0\}$ so that $F \in \sigma\mathcal{J}$. Now $\mathcal{E}_\beta$ is discrete in $X - F$ and $\mathcal{E}_\beta \subseteq \mathcal{J}$, and hence $H_\beta = F \cup \{\cup\mathcal{E}_\beta\} \in \sigma\mathcal{J}$. If $\beta$ is a limit ordinal, let $\alpha_i < \beta$ such that $\alpha_i < \alpha_{i+1}$ and $\sup\{\alpha_i\} = \beta$. Now for each i, $H_i = \cup\{\mathcal{E}_\mu\colon \mu \leq \alpha_i\} \in \sigma\mathcal{J}$, so $H_\beta = \cup_{i=1}^{\infty} H_i \in \sigma\mathcal{J}$. Therefore $X \in \sigma\mathcal{J}$.

We now consider the ideal of closed nowhere dense subsets of a space X.

*Lemma* 3.4. *Let $\mathcal{J}$ be the ideal of closed nowhere dense subsets of a space* X. *Then $\mathcal{J}$ is discretely additive.*

*Proof.* Clearly, $\mathcal{J}$ is closed under discrete unions. Let $F \in \sigma\mathcal{J}$ and $\mathcal{D} = \{D_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{J}$ such that $\mathcal{D}$ is discrete in X - F. Let W be any non-empty open subset of X such that $W \subseteq H = F \cup \{\cup\mathcal{D}\}$. If $U = \mathrm{int}(F)$, then $W \subseteq U$; otherwise, $W \cap D_\alpha \neq \phi$ for some $\alpha \in \Lambda$. Thus $D_\alpha$ fails to be nowhere dense in X. Therefore, $\cup\{(D_\alpha - U) : \alpha \in \Lambda\} \in \mathcal{J}$; and hence $H \in \sigma\mathcal{J}$.

*Corollary* 3.5. *Let* X *be a space with* $B(D,\lambda)$*-cover by nowhere dense sets. Then* X *is* $1^{st}$ *category.*

*Proof.* Let $\mathcal{J}$ be the ideal of closed nowhere dense subsets of X. Then $\mathcal{J}$ is discretely additive by Lemma 3.4, and hence $X \in \sigma\mathcal{J}$ by Theorem 3.3.

*Corollary* 3.6. *If a space* X *has a special* c-p *cover by nowhere dense sets, then* X *is* $1^{st}$ *category.*

## 4. Applications to Dimension Theory

In [24] the author obtained the following result which also can be found using a technique similar to that of Theorem 2.4 above.

*Theorem* 4.1. [24]. *Let* X *be a normal space. If* X *has a* c-p *cover* $\mathcal{C}$ *by countably compact sets such that* $\dim(C) \leq n$ *for* $C \in \mathcal{C}$, *then* $\dim(X) \leq n$.

To obtain the special closure-preserving sum theorem for covering dimension as an application of Theorem 3.6 above, we need the following result found in [24].

*Lemma* 4.2. *Let* X *be a normal space and* F *a closed subset of* X *with* $\dim(F) \leq n$. *For each finite open cover* $\mathcal{U} = \{U_i: i \leq m\}$ *of* X, *there exists an open refinement* $\mathcal{V} = \{V_i: i \leq m\}$ *of* $\mathcal{U}$, *and an open set* $G \supset F$ *such that*

*(i)* $V_i \subseteq U_i$ *for each* $i \leq m$, *and*

*(ii)* $\mathrm{ord}(x, \mathcal{V}) \leq n + 1$ *for each* $x \in \overline{G}$.

*Lemma* 4.3. *Let* X *be a normal space and* F *a closed subset of* X *with* $\dim(F) \leq n$. *Let* $\mathcal{D} = \{D_\alpha: \alpha \in \Lambda\}$ *be a family of closed subsets of* X *such that*

*(i)* $\dim(D_\alpha) \leq n$ *for each* $\alpha \in \Lambda$, *and*

*(ii)* $\mathcal{D}$ *is discrete in* $X - F$.

*Then* $\dim(F \cup \{\cup\mathcal{D}\}) \leq n$.

*Proof*. Let $\mathcal{U} = \{U_i: i \leq m\}$ be a finite open (in X) cover of $H = F \cup \{\cup\mathcal{D}\}$. By Lemma 4.2 there exists an open refinement $\mathcal{V} = \{V_i: i \leq m\}$ of $\mathcal{U}$ and an open set $G \supseteq F$ such that $\mathrm{ord}(x, \mathcal{V}) \leq n + 1$ for all $x \in \overline{G}$. Now $\{D_\alpha - G: \alpha \in \Lambda\}$ is a discrete closed collection in X with $\dim(D_\alpha - G) \leq n$ for each $\alpha \in \Lambda$. Therefore $\dim(\cup \mathcal{D} - G) \leq n$ and hence $\mathcal{V}$ has a partial open refinement $\mathcal{W} = \{W_i: i \leq m\}$ covering $\{\cup \mathcal{D} - G)$, such that $W_i \subseteq V_i$ and $\mathrm{ord}(x, \mathcal{W}) \leq n + 1$ for $x \in (\cup \mathcal{D} - G)$. Define $W_i^* = W_i \cup (V_i - \overline{G})$ for $i \leq m$. It is easy to show that $\mathcal{W}^* = \{W_i^*: i \leq m\}$ is an open refinement of $\mathcal{U}$ covering H and $\mathrm{ord}(x, \mathcal{W}^*) \leq n + 1$ for $x \in H$. Therefore $\dim(H) \leq n$.

We now obtain the desired c-p sum theorem for covering dimension.

*Theorem* 4.4. *Let* X *be a normal space which has a* B(D,$\lambda$) *cover* $\mathcal{E}$ *such that* dim(E) $\leq$ n *for each* E $\in \mathcal{E}$. *Then* dim(X) $\leq$ n.

*Proof*. Let $\mathcal{J}$ be the ideal of closed subsets of X such that dim(J) $\leq$ n for J $\in \mathcal{J}$. Since covering dimension satisfies the countable sum theorem for normal spaces, $\mathcal{J} = \sigma\mathcal{J}$. Therefore $\mathcal{J}$ is discretely additive by Theorem 4.3 above. Now from Theorem 3.3 it follows that X $\in \sigma\mathcal{J} = \mathcal{J}$. Hence dim(x) $\leq$ n.

*Corollary* 4.5. *Let* X *be a normal space. If* X *has a special* c-p *cover* $\mathcal{C}$, *such that* dim(C) $\leq$ n *for each* C $\in \mathcal{C}$, *then* dim(X) $\leq$ n.

For totally normal spaces similar results hold for large inductive dimension using the lemma below. The proofs are straightforward and left for the reader. Yajima [18] has obtained a $\sigma$-closure-preserving sum theorem for Ind.

*Lemma* 4.6. *Let* X *be totally normal and* F *a closed subset of* X *such that* Ind(F) $\leq$ n. *If* Ind(K) $\leq$ n *for each closed subset* K *of* X *such that* K $\cap$ F = $\phi$, *then* Ind(K) $\leq$ n.

*Theorem* 4.7. *Let* X *be a totally normal space. If* X *has a special* c-p *cover* $\mathcal{C}$ *such that* Ind(C) $\leq$ n *for* C $\in \mathcal{C}$, *then* Ind(X) $\leq$ n.

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