TOPOLOGY PROCEEDINGS

Volume 13, 1988

Pages 125–136

http://topology.auburn.edu/tp/

REMARKS ON CLOSURE-PRESERVING SUM THEOREMS

by

J. C. Smith and R. Telgársky

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

REMARKS ON CLOSURE-PRESERVING SUM THEOREMS

J. C. Smith and R. Telgársky

1. Introduction

In 1975 Junnila and Potoczny [6] and Katuta [2] independently showed that, if a space X has a closure-preserving cover by compact sets, then X is metacompact. Thus began a study of spaces which have closure-preserving covers by "nice" (finite, countably compact, Lindelof, etc.) sets. The reader is referred to [1,2,4,5,6,8,11,12, 13,16,17,18,19,24] for these developments as well as their relationship with winning strategies in some topological games.

In 2 we show that if a space X has a closure-preserving cover by nowhere dense sets, then X need not be 1^{st} category. However, if these sets are also countably compact, then X must be 1^{st} category. The closure-preserving property of the cover is strengthened in a natural way in 3, and a somewhat more general result is obtained for the ideal of closed subsets of a space. The desired result for nowhere dense sets follows as a special application.[1] Finally in 4 closure-preserving sum theorems are obtained for the dimension functions dim and Ind for normal and totally normal spaces.

[1]The authors would like to thank the referee for his/her comments on this paper.

Definition 1.1. (i) A family $\mathcal{F}$ of subsets of a space X is closure-preserving (c-p) if $\cup\{\overline{F}: F \in \mathcal{F}'\} = \overline{\cup \mathcal{F}'}$ for every $\mathcal{F}' \subseteq \mathcal{F}$.

(ii) A c-p family $\mathcal{F}$ is special if there exists a point finite open collection $\mathcal{U}$ such that for $F \in \mathcal{F}$,

$$X - \overline{F} = \cup\{U \in \mathcal{U}: U \cap F = \phi\}.$$

In this case we say that $\mathcal{U}$ generates $\mathcal{F}$.

Definition 1.2. A family $\mathcal{J}$ of closed subsets of a space X is an ideal of closed sets if,

(i) for every finite $\mathcal{J}' \subseteq \mathcal{J}$, $\cup \mathcal{J}' \in \mathcal{J}$ and

(ii) if H is a closed set and $H \subseteq J \in \mathcal{J}$, then $H \in \mathcal{J}$.

We will denote the family of countable unions of members of $\mathcal{J}$ by $\sigma\mathcal{J}$.

Remark. As noted in [1], if $\mathcal{U}$ is a point finite open cover which generates a c-p $\mathcal{J}$-cover, then the family

$$\mathcal{C}_n = \{X_n \cap [\cap \mathcal{U}_x]: x \in X \text{ and } |\mathcal{U}_x| = n\}$$

consists of $\mathcal{J}$-small sets, and $\mathcal{C}_n$ is a discrete collection of relative closed sets in $X_n - X_{n-1}$, where

$$X_n = \{x \in X: |\mathcal{U}_x| \leq n\}.$$

Definition 1.3. A space X is called weakly $\overline{\theta}$-refinable if every open cover of X has an open refinement $\cup_{i=1}^{\infty} \mathcal{G}_i$ satisfying

(i) $\{G_i\}_{i=1}^{\infty}$ is point finite, where $G_i = \cup\mathcal{G}_i$ and

(ii) for each $x \in X$, there exists some n(x) such that $0 < \text{ord}(x, \mathcal{G}_{n(x)}) < \infty$.

A cover $\cup_{i=1}^{\infty} \mathcal{G}_i$ satisfying (i) and (ii) above is called a weak $\overline{\theta}$-cover.

It is well known that the class of weakly $\overline{\theta}$-refinable spaces lies strictly between the classes of θ-refinable and weakly θ-refinable spaces (see [8]).

All spaces will be assumed to be T_2.

2. Closure-preserving covers by nowhere dense sets

In this paper we consider spaces which have a closure-preserving cover by nowhere dense sets. Note that if $Q = \{r_1, r_2, \cdots\}$ is the set of rationals and $F_n = \{r_i\}_{i=1}^{n}$, then $\{F_i: i < \omega\}$ is a closure-preserving cover of Q by finite sets. Clearly, Q is a 1^{st} category space.

The following example shows that if a space X has a closure-preserving cover by nowhere dense sets, then X need not be of the 1^{st} category.

Example. Let $L = \{0,1\}^{\omega_1}$ where $\{0,1\}$ is the two point discrete space. The topology on Y is the countable box topology; that is, a basic open set in Y is a countable intersection of basic open sets in the usual product topology. Define $X = \{y \in Y: |\alpha < \omega_1: y(\alpha) \neq 0| < \omega_1\}$. Now if $F_\alpha = \{x \in X: x(\beta) = 0 \text{ for each } \beta \geq \alpha\}$, it is easy to check that $\{F_\alpha: \alpha < \omega_1\}$ is a closure-preserving cover of X by discrete closed sets. Furthermore each F_α is nowhere dense. We assert that X is not 1^{st} category. Indeed, let $\{G_n: n < \omega\}$ be a sequence of open dense subsets of X. Let B_0 be a basic open set such that $B_0 \subseteq G_0$, and define

$$C_0 = \{\alpha < \omega_1: x(\alpha) = y(\alpha) \text{ for all } x,y \in B_0\}.$$

Now by induction choose a basic open set $B_n \subseteq G_n \cap B_{n-1}$ and let

$$C_n = \{\alpha < \omega_1: x(\alpha) = y(\alpha) \text{ for all } x,y \in B_n\}.$$

Since $C = \bigcup_{n<\omega} C_n$ is countable, there exists a point $z \in X$ such that $z \in \bigcap_{n<\omega} B_n \subseteq \bigcap_{n<\omega} G_n$ and $z(\alpha) = 0$ for $\alpha > \sup C$. Therefore X is not 1^{st} category.

The following result is an easy consequence of Zorn's Lemma so the proof is omitted.

Lemma 2.1. *Let* $\mathcal{F} = \{F_\alpha: \alpha \in \Lambda\}$ *be any non-empty family of non-empty sets. Then there exists a maximal pairwise disjoint subfamily of* $\mathcal{F}$.

Definition 2.2. Let $\phi \in E \subseteq X$ and $\mathcal{F}$ a family of subsets of X.

(1) $\mathcal{M}(E)$ = a maximal pairwise disjoint subfamily of $\{F \cap E: F \in \mathcal{F}\}$.

(2) $M^*(E) = \cup \mathcal{M}(E)$.

The following uses a technique due to Telgársky and Yajima [16: Theorem 3.3].

Lemma 2.3. *Let* $\mathcal{C} = \{C_\alpha: \alpha \in \Lambda\}$ *be a closure-preserving cover of* X *by countably compact sets. If* $\{H_n\}_{n=1}^\infty$ *is any decreasing sequence of closed subsets of* X *satisfying* $H_{n+1} \subseteq X - M^*(H_n)$ *for each* n, *then* $\bigcap_{n=1}^\infty H_n = \phi$.

Proof. Suppose there exists a sequence $\{H_n\}_{n=1}^\infty$ of closed sets satisfying the above condition, but

$\cap_{n=1}^{\infty} H_n \neq \phi$. Then there exists some $C_{\alpha_0} \in \mathcal{C}$ such that $C_{\alpha_0} \cap H_n \neq \phi$ for each n. Furthermore $(C_{\alpha_0} \cap H_n) \not\subseteq \mathcal{M}(H_n)$; for otherwise, $C_{\alpha_0} \cap H_{n+1} = \phi$. Therefore for each n, there exists some $C_{\alpha_n} \in \mathcal{C}$ such that $(C_{\alpha_n} \cap H_n) \in \mathcal{M}(H_n)$ and $C_{\alpha_n} \cap (C_{\alpha_0} \cap H_n) \neq \phi$. Now choose $x_n \in C_{\alpha_n} \cap (C_{\alpha_0} \cap H_n)$ so that $x_n \in M^*(H_n)$ and hence $x_n \notin H_{n+1}$. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence of distinct points in C_{α_0} and must cluster at $y \in C_{\alpha_0}$. Now $y \in \overline{\{x_n : n \geq 1\}} \subseteq \cup_{n=1}^{\infty} C_{\alpha_n}$ and $y \in \cap_{n=1}^{\infty} H_n$ as well. But this is a contradiction, since $y \in (C_{\alpha_n} \cap H_n) \in \mathcal{M}(H_n)$ implies that $y \notin H_{n+1}$. Therefore it must be the case that $\cap_{n=1}^{\infty} H_n = \phi$.

Remark. It should be noted that the above proof only used the fact that every countable subfamily of $\mathcal{C}$ was closure-preserving.

Theorem 2.4. *Let* X *be a regular space with a closure-preserving cover* $\mathcal{C}$ *by countably compact nowhere dense sets. Then* X *is* 1^{st} *category.*

Proof. Since $M^*(X)$ is closed and nowhere dense in X, by Lemma 2.1 above, there exists a maximal pairwise disjoint family $\mathcal{H}(X)$ of regular closed subsets of X such that $\cup \mathcal{H}(X) \subseteq X - M^*(X)$. Furthermore, it is easy to see that $\cup \mathcal{H}(X)$ is dense in X. Define $\mathcal{G}_0 = \{\text{int}(H) : H \in \mathcal{H}(X)\}$ and $G_0 = \cup \mathcal{G}_0$. Note that G_0 is also dense in X. Now by induction we construct the sequence $(\mathcal{G}_0, G_0, \mathcal{G}_1, G_1, \cdots \mathcal{G}_n, G_n \cdots)$ where

$$\mathcal{G}_{n+1} = \{\text{int}(H) : H \in \mathcal{H}(\overline{G}) \text{ where } G \in \mathcal{G}_n\},$$

$$G_{n+1} = \cup \mathcal{G}_{n+1} \quad \text{and}$$

$\mathcal{H}(\overline{G})$ = a maximal pairwise disjoint family of regular closed subsets of X such that $\cup\mathcal{H}(\overline{G})$ is dense in $\overline{G} - M^*(\overline{G})$.

As before it is easy to see that G_n is open and dense in X.

Finally, if $H_0 = X$ and $H_{n+1} \in \mathcal{H}(H_n)$, by Lemma 2.3, $\cap_{n=0}^{\infty} H_n = \phi$. Therefore $\cap_{n=0}^{\infty} G_n = \phi$ and hence X is 1^{st} category.

Corollary 2.5. *Let* X *be a regular countably compact space. If* $\mathcal{F}$ *is a closure-preserving cover of* X, *then some member of* $\mathcal{F}$ *has non-empty interior.*

3. Special c-p covers

In [1] the authors studied spaces having c-p $\mathcal{F}$-covers induced by a point finite open cover. Here we get analogous results to Theorem 2.4 above by omitting the regularity and countably compact conditions.

The notion of B(D,λ)-refinability has been shown to play an important role in the study of weakly $\overline{\theta}$-refinable spaces introduced in [7].

Definition 3.1. A space X is B(D,λ)-refinable provided every open cover $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ of X has a refinement $\mathcal{E} = \cup\{\mathcal{E}_\beta = \{E(\beta,\gamma) : \gamma \in \Gamma\} : \beta < \lambda\}$ where satisfies

(i) $E(\beta,\gamma) \subseteq U_\gamma$ for each $\gamma \in \Gamma$, $\beta < \lambda$,

(ii) $\{\cup\mathcal{E}_\beta : \beta < \lambda\}$ partitions X,

(iii) for each $\beta < \lambda$, $\mathcal{E}_\beta$ is a closed discrete collection in $X - \cup\{\cup\mathcal{E}_\mu : \mu < \beta\}$, and

(iv) for each $\beta < \lambda$, $\cup\{\cup\mathcal{E}_\mu : \mu < \beta\}$ is closed in X.

In [7] the author has shown that every space which is $B(D,\omega)$-refinable is weakly $\overline{\theta}$-refinable and that every weakly $\overline{\theta}$-refinable space is $B(D,(\omega)^2)$-refinable. Throughout this paper λ will denote a countable ordinal.

Definition 3.2. Let $\mathcal{J}$ be an ideal of closed subsets of a space X. We say that $\mathcal{J}$ is <u>discretely additive</u> if

(i) $\mathcal{J}$ is closed under discrete unions, and

(ii) when $F \in \sigma\mathcal{J}$ and $\mathcal{D} \subseteq \mathcal{J}$ such that $\mathcal{D}$ is discrete in $X - F$, then $H = F \cup \{\cup\mathcal{D}\} \in \sigma\mathcal{J}$.

Theorem 3.3. *If a space* X *has a* $B(D,\lambda) - \mathcal{J}$ *cover for some countable ordinal* λ *and* $\mathcal{J}$ *is discretely additive, then* $X \in \sigma\mathcal{J}$.

Proof. Let $\mathcal{E} = \cup\{\mathcal{E}_\beta\colon \beta < \lambda\}$, where $\mathcal{E}_\beta = \{E(\beta,\gamma)\colon \gamma \in \Gamma\}$, be a $B(D,\lambda) - \mathcal{J}$ cover of X such that $\mathcal{J}$ is discretely additive. The proof is by induction on λ. It is easy for $\lambda = 1$; since $\mathcal{J}$ is closed under discrete unions, so $X \in \mathcal{J}$. Let $\beta < \lambda$ and assume true for all $\gamma < \beta$. If $\beta = \alpha_0 + 1$, define $F = \cup\{\cup\mathcal{E}_\mu\colon \mu \le \alpha_0\}$ so that $F \in \sigma\mathcal{J}$. Now $\mathcal{E}_\beta$ is discrete in $X - F$ and $\mathcal{E}_\beta \subseteq \mathcal{J}$, and hence $H_\beta = F \cup \{\cup\mathcal{E}_\beta\} \in \sigma\mathcal{J}$. If β is a limit ordinal, let $\alpha_i < \beta$ such that $\alpha_i < \alpha_{i+1}$ and $\sup\{\alpha_i\} = \beta$. Now for each i, $H_i = \cup\{\mathcal{E}_\mu\colon \mu \le \alpha_i\} \in \sigma\mathcal{J}$, so $H_\beta = \cup_{i=1}^\infty H_i \in \sigma\mathcal{J}$. Therefore $X \in \sigma\mathcal{J}$.

We now consider the ideal of closed nowhere dense subsets of a space X.

Lemma 3.4. *Let* $\mathcal{J}$ *be the ideal of closed nowhere dense subsets of a space* X. *Then* $\mathcal{J}$ *is discretely additive.*

Proof. Clearly, $\mathcal{J}$ is closed under discrete unions. Let $F \in \sigma\mathcal{J}$ and $\mathcal{D} = \{D_\alpha: \alpha \in \Lambda\} \subseteq \mathcal{J}$ such that $\mathcal{D}$ is discrete in X - F. Let W be any non-empty open subset of X such that $W \subseteq H = F \cup \{\cup\mathcal{D}\}$. If $U = \mathrm{int}(F)$, then $W \subseteq U$; otherwise, $W \cap D_\alpha \neq \phi$ for some $\alpha \in \Lambda$. Thus D_α fails to be nowhere dense in X. Therefore, $\cup\{(D_\alpha - U): \alpha \in \Lambda\} \in \mathcal{J}$; and hence $H \in \sigma\mathcal{J}$.

Corollary 3.5. *Let* X *be a space with* $B(D,\lambda)$*-cover by nowhere dense sets. Then* X *is* 1^{st} *category.*

Proof. Let $\mathcal{J}$ be the ideal of closed nowhere dense subsets of X. Then $\mathcal{J}$ is discretely additive by Lemma 3.4, and hence $X \in \sigma\mathcal{J}$ by Theorem 3.3.

Corollary 3.6. *If a space* X *has a special* c-p *cover by nowhere dense sets, then* X *is* 1^{st} *category.*

4. Applications to Dimension Theory

In [24] the author obtained the following result which also can be found using a technique similar to that of Theorem 2.4 above.

Theorem 4.1. [24]. *Let* X *be a normal space. If* X *has a* c-p *cover* $\mathcal{C}$ *by countably compact sets such that* $\dim(C) \leq n$ *for* $C \in \mathcal{C}$, *then* $\dim(X) \leq n$.

To obtain the special closure-preserving sum theorem for covering dimension as an application of Theorem 3.6 above, we need the following result found in [24].

Lemma 4.2. *Let* X *be a normal space and* F *a closed subset of* X *with* $\dim(F) \leq n$. *For each finite open cover* $\mathcal{U} = \{U_i: i \leq m\}$ *of* X, *there exists an open refinement* $\mathcal{V} = \{V_i: i \leq m\}$ *of* $\mathcal{U}$, *and an open set* $G \supset F$ *such that*

(i) $V_i \subseteq U_i$ *for each* $i \leq m$, *and*

(ii) $\mathrm{ord}(x, \mathcal{V}) \leq n + 1$ *for each* $x \in \overline{G}$.

Lemma 4.3. *Let* X *be a normal space and* F *a closed subset of* X *with* $\dim(F) \leq n$. *Let* $\mathcal{D} = \{D_\alpha: \alpha \in \Lambda\}$ *be a family of closed subsets of* X *such that*

(i) $\dim(D_\alpha) \leq n$ *for each* $\alpha \in \Lambda$, *and*

(ii) $\mathcal{D}$ *is discrete in* $X - F$.

Then $\dim(F \cup \{\cup\mathcal{D}\}) \leq n$.

Proof. Let $\mathcal{U} = \{U_i: i \leq m\}$ be a finite open (in X) cover of $H = F \cup \{\cup\mathcal{D}\}$. By Lemma 4.2 there exists an open refinement $\mathcal{V} = \{V_i: i \leq m\}$ of $\mathcal{U}$ and an open set $G \supseteq F$ such that $\mathrm{ord}(x, \mathcal{V}) \leq n + 1$ for all $x \in \overline{G}$. Now $\{D_\alpha - G: \alpha \in \Lambda\}$ is a discrete closed collection in X with $\dim(D_\alpha - G) \leq n$ for each $\alpha \in \Lambda$. Therefore $\dim(\cup \mathcal{D} - G) \leq n$ and hence $\mathcal{V}$ has a partial open refinement $\mathcal{W} = \{W_i: i \leq m\}$ covering $\{\cup \mathcal{D} - G)$, such that $W_i \subseteq V_i$ and $\mathrm{ord}(x, \mathcal{W}) \leq n + 1$ for $x \in (\cup \mathcal{D} - G)$. Define $W_i^* = W_i \cup (V_i - \overline{G})$ for $i \leq m$. It is easy to show that $\mathcal{W}^* = \{W_i^*: i \leq m\}$ is an open refinement of $\mathcal{U}$ covering H and $\mathrm{ord}(x, \mathcal{W}^*) \leq n + 1$ for $x \in H$. Therefore $\dim(H) \leq n$.

We now obtain the desired c-p sum theorem for covering dimension.

Theorem 4.4. *Let* X *be a normal space which has a* $B(D,\lambda)$ *cover* $\mathcal{E}$ *such that* $\dim(E) \leq n$ *for each* $E \in \mathcal{E}$. *Then* $\dim(X) \leq n$.

Proof. Let $\mathcal{J}$ be the ideal of closed subsets of X such that $\dim(J) \leq n$ for $J \in \mathcal{J}$. Since covering dimension satisfies the countable sum theorem for normal spaces, $\mathcal{J} = \sigma\mathcal{J}$. Therefore $\mathcal{J}$ is discretely additive by Theorem 4.3 above. Now from Theorem 3.3 it follows that $X \in \sigma\mathcal{J} = \mathcal{J}$. Hence $\dim(x) \leq n$.

Corollary 4.5. *Let* X *be a normal space. If* X *has a special* c-p *cover* $\mathcal{C}$, *such that* $\dim(C) \leq n$ *for each* $C \in \mathcal{C}$, *then* $\dim(X) \leq n$.

For totally normal spaces similar results hold for large inductive dimension using the lemma below. The proofs are straightforward and left for the reader. Yajima [18] has obtained a σ-closure-preserving sum theorem for Ind.

Lemma 4.6. *Let* X *be totally normal and* F *a closed subset of* X *such that* $\mathrm{Ind}(F) \leq n$. *If* $\mathrm{Ind}(K) \leq n$ *for each closed subset* K *of* X *such that* $K \cap F = \phi$, *then* $\mathrm{Ind}(K) \leq n$.

Theorem 4.7. *Let* X *be a totally normal space. If* X *has a special* c-p *cover* $\mathcal{C}$ *such that* $\mathrm{Ind}(C) \leq n$ *for* $C \in \mathcal{C}$, *then* $\mathrm{Ind}(X) \leq n$.

References

[1] H. Junnila J. C. Smith, and R. Telgársky, *Closure-preserving covers by small sets*, Top. Appl., 23 (1986) 237-262.

[2] Y. Katuta, *On spaces which admit closure-preserving covers by compact sets*, Proc. Japan Acad., 50 (1974) 826-828.

[3] A. R. Pears, *Dimension theory of general spaces*, Cambridge Univ. Press (1975).

[4] H. B. Potoczny, *A nonparacompact space which admits a closure-preserving cover of compact sets*, Proc. Amer. Math. Soc., 32 (1972) 309-311.

[5] H. B. Potoczny, *Closure-preserving families of compact sets*, Gen. Topology Appl. 3 (1973) 243-248.

[6] H. Potoczny and H. Junnila, *Closure-preserving families and metacompactness*, Proc. Amer. Math. Soc., 53 (1975) 523-529.

[7] J. C. Smith, *Irreducible spaces and property b_1*, Topology Proceedings, Vol. 5 (1980) 187-200.

[8] J. C. Smith, *Properties of weak $\overline{\theta}$-refinable spaces*, Proc. Amer. Math. Soc., 53 (1975) 511-517.

[9] J. C. Smith and L. L. Krajewski, *Expandability and collectionwise normality*, Trans. Amer. Math. Soc., 160 (1971) 437-451.

[10] J. C. Smith and R. Telgársky, *Closure-preserving covers of σ-products*, Proc. of Japan Acad., Vol. 63, No. 4 (1987) 118-120.

[11] H. Tamano, *A characterization of paracompactness*, Fund. Math., 72 (1971) 180-201.

[12] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math., 73 (1971) 59-74.

[13] R. Telgársky, *Closure-preserving covers*, Fund. Math., 85 (1974) 165-175.

[14] R. Telgársky, *Spaces defined by topological games*, Fund. Math., 88 (1975) 193-223.

[15] R. Telgársky, *Spaces defined by topological games, II*, Fund. Math., 116 (1983) 189-207.

[16] R. Telgársky and Y. Yajima, *On order locally finite and closure-preserving covers*, Fund. Math., 109 (1980) 211-216.

[17] Y. Yajima, *On spaces which have a closure-preserving cover by finite sets*, Pacific J. Math., 69 (1977) 571-578.

[18] Y. Yajima, *On order star-finite and closure-preserving covers*, Proc. Japan Acad., 55 (1979) 19-21.

[19] Y. Yajima, *A note on order locally finite and closure-preserving covers*, Bull. Acad. Polon. Sci. Ser. Math., 27 (1979) 401-405.

[20] Y. Yajima, *Topological games and products I*, Fund. Math., 113 (1981) 141-153.

[21] Y. Yajima, *Topological games and products II*, Fund. Math., 117 (1983a) 47-60.

[22] Y. Yajima, *Topological games and products III*, Fund. Math., 117 (1983b) 223-238.

[23] Y. Yajima, *Notes on topological games*, Fund. Math., 121 (1984) 31-40.

[24] Y. Yajima, *Topological games and applications*, (to appear).

Virginia Polytechnic Institute and State University

Blacksburg, Virginia 24061

Florida Atlantic University

Boca Raton, Florida 33431

University of Texas at El Paso

El Paso, Texas 79968