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by

Beverly Diamond, James Hatzenbuhler, and Don Mattson

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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ON WHEN A 0-SPACE IS RIMCOMPACT

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1. Introduction

In this paper all spaces are completely regular and Hausdorff. A space X is *rimcompact* if it has a base of open sets with compact boundaries, or *π-open* sets, and a *0-space* if it has a compactification KX in which the remainder KX \ X is 0-dimensional. It is well known that a rimcompact space is a 0-space, and that the converse is not true (see [Is], for example).

Substantial effort has been expended on the characterization and development of the properties of 0-spaces ([Di_2], [Di_3], [Is], and [Mc]). Let R(X) denote the set of points of X which do not have compact neighbourhoods. Since both locally compact and 0-dimensional spaces are rimcompact, it is natural to seek conditions on R(X), possibly involving local compactness and 0-dimensionality, which insure that if X is a 0-space then X is rimcompact. The main result (2.4) of this paper is the following: if R(X) is locally compact and 0-dimensional or is scattered, then the following are equivalent: X is a 0-space; X is rimcompact; X has a compactification with totally disconnected remainder. (In fact, if R(X) is rimcompact, then it is enough to require that $bd_X R(X)$ is locally compact and 0-dimensional (2.8).) The assumption of the disconnectedness of R(X) cannot be omitted; according to 3.1, if X is a 0-space and R(X) contains a metric continuum, then there is a non-rimcompact 0-space Y for which R(Y) = R(X). We also show that a scattered space is 0-dimensional if and only if it is a 0-space (2.6); an example of a scattered non-0-dimensional space appears in [So].

In the remainder of this section we define the various forms of connectedness used and mention the relationships between them. We present the main results in 2, and examples illuminating the results in 3.

A space is 0-*dimensional* if it has a base of sets that are both open and closed, or *open-closed; fully disconnected* if distinct points are contained in disjoint open-closed sets; *totally disconnected* if any nonempty connected set has cardinality one; and *scattered* if every nonempty subset has isolated points. Any totally disconnected locally compact space is 0-dimensional. It is easy to see that any scattered space is totally disconnected, while a number of the spaces constructed in 3 will be scattered but not fully disconnected. The space Q of rational numbers is an easy example of a 0-dimensional space that is not scattered.

2. The Main Results

The first two results provide the foundation for this section.

2.1. *Lemma. If* X *has a compactification* αX *with* $cl_{\alpha X}(\alpha X \setminus X)$ *0-dimensional, then* X *is rimcompact.*

Proof. It suffices to show that if $x \in R(X)$, then x has a base in X of π-open sets. Suppose that T is a closed subset of X, and that $x \in R(X) \setminus T$. Then $x \notin cl_{\alpha X}T$; let $S = cl_{\alpha X}T \cap cl_{\alpha X}(\alpha X \setminus X)$. The set S is closed in $cl_{\alpha X}(\alpha X \setminus X)$, and $x \in [cl_{\alpha X}(\alpha X \setminus X)] \setminus S$. Since $cl_{\alpha X}(\alpha X \setminus X)$ is compact and 0-dimensional, there is a compact open-closed set W of $cl_{\alpha X}(\alpha X \setminus X)$ with $x \in W$ and $W \cap S = \emptyset$. Then the sets W and $[cl_{\alpha X}(\alpha X \setminus X) \setminus W] \cup cl_{\alpha X}T$ are disjoint compact subsets of αX. There is a continuous function $f: \alpha X \to I$ with $f[W] = 0$ and $f[[cl_{\alpha X}(\alpha X \setminus X) \setminus W] \cup cl_{\alpha X}T] = 1$. Since $cl_{\alpha X}(\alpha X \setminus X) \subseteq f^{\leftarrow}(0) \cup f^{\leftarrow}(1)$, the set $f^{\leftarrow}[0,1/2) \cap X$ is an open neighbourhood of x with compact boundary in X.

2.2. *Lemma. Suppose that* T *is locally compact, and that* T *can be written as* $X \cup Y$, *where* $X \cap Y = \emptyset$, *and* X, Y *are totally disconnected. Then either* T *is 0-dimensional or there is a closed connected subset* C *of* T *such that* $cl_C(X \cap C) = cl_C(Y \cap C) = C$.

Proof. If T is locally compact and not 0-dimensional, there is a closed connected subset C of T with $|C| > 1$. Since X and Y are totally disconnected, $C \cap X$ and $C \cap Y$ are nonempty. The set $C \setminus cl_C(C \cap Y)$, as an open and thus locally compact subset of C contained in X, is an open 0-dimensional subset of the connected set C, and hence is empty. Similarly, $cl_C(C \cap X) = C$.

As defined above, $R(X)$ is the set of points of X which do not have a base of compact neighbourhoods in X. We refer to $X \setminus R(X)$, or the locally compact part of X, by $L(X)$. Note that $cl_X L(X) = X \setminus int_X R(X)$. For a non-limit ordinal $\alpha \geq 1$, $R^{\alpha+1}(X)$ will denote $R(R^{\alpha}(X))$. If β is a limit ordinal, then $R^{\beta}(X)$ is defined to be $\cap_{\alpha<\beta} R^{\alpha}(X)$. Note that either $R^{\gamma}(X) = \emptyset$ for some ordinal γ such that card $\gamma \leq$ card $R(X)$, or $R^{\alpha+1}(X) = R^{\alpha}(X) = R^{\alpha}0(X) \neq \emptyset$ for some ordinal α_0 and all $\alpha \geq \alpha_0$. In the latter case, the space $R^{\alpha}0(X)$ is nowhere locally compact.

2.3. *Theorem. Let* X *be a space for which* $R(X)$ *is totally disconnected and* $R^{\gamma}(X) = \emptyset$ *for some ordinal* γ. *The following are equivalent:*

a) X *has a compactification with totally disconnected remainder,*

b) X *has a compactification with* 0*-dimensional remainder,*

c) X *is rimcompact.*

Proof. Clearly c) implies b), and b) implies a). To shown that a) implies c), we show that with the hypotheses on X, if KX is any compactification of X with $KX \setminus X$ totally disconnected, then $cl_{KX}(KX \setminus X)$ is 0-dimensional. The result then follows from 2.1. Since $R(X)$ and $KX \setminus X$ are totally disconnected and $cl_{KX}(KX \setminus X)$ is compact, if $cl_{KX}(KX \setminus X)$ is not 0-dimensional, then according to 2.2 there is a connected compact subset C of $cl_{KX}(KX \setminus X)$ with $cl_C(C \cap R(X)) = cl_C(C \cap (KX \setminus X)) = C$.

Since $R(X) \setminus R^2(X)$ is locally compact, 0-dimensional, and open in $R(X)$, $C \cap R(X) \setminus R^2(X)$ is locally compact, 0-dimensional, and open in $C \cap R(X)$. The fact that $cl_C(C \cap R(X))$ is connected them implies that $C \cap R(X) \subseteq R^2(X)$. A similar argument implies that if $C \cap R(X) \subseteq R^\alpha(X)$, then $C \cap R(X) \subseteq R^{\alpha+1}(X)$. Suppose that β is a limit ordinal; if $C \cap R(X) \subseteq R^\alpha(X)$ for each $\alpha < \beta$, then $C \cap R(X) \subseteq R^\beta(X) = \cap_{\alpha<\beta} R^\alpha(X)$. It follows that $C \cap R(X) \subseteq R^\alpha(X)$ for every ordinal α. But $R^\gamma(X) = \emptyset$ for some γ, contradicting the fact that $C \cap R(X) \neq \emptyset$.

The next result follows immediately.

2.4. *Corollary. If* X *is a space for which* R(X) *is scattered or is locally compact and* 0-*dimensional, then the following are equivalent:*

a) X *has a compactification* KX *with* $KX \setminus X$ *totally disconnected,*

b) X *has a compactification* KX *with* $KX \setminus X$ 0-*dimensional,*

c) X *is rimcompact.*

Proof. It suffices to show that if R(X) is scattered and $R^\alpha(X) \neq \emptyset$, then $R^\alpha(X) \neq R^{\alpha+1}(X)$ for any ordinal α. This follows from the fact that any nonempty subset of R(X) has isolated points.

A space X is *pointwise rimcompact* if for distinct $x, y \in X$, there is an open set U of X with compact boundary such that $x \in U$, while $y \notin cl_X U$. A pointwise rimcompact space need not be rimcompact. For example, if X is the quotient space $\mathbf{R}/\{\mathbf{N}\}$, then X is pointwise rimcompact, since $|R(X)| = 1$, but is not rimcompact; the element $\{\mathbf{N}\}$ does not have a base of open sets with compact boundaries. It is easy to see that any fully disconnected space is pointwise rimcompact.

The next theorem shows that $\mathbf{R}/\{\mathbf{N}\}$ is not a 0-space, and thus that a space X satisfying the hypotheses of 2.3 or 2.4 does not necessarily satisfy any of the equivalent conditions listed in the statements of 2.3 and 2.4.

2.5. *Theorem. If* X *has a compactification with totally disconnected remainder, then the following are equivalent:*

a) X *is rimcompact,*

b) X *is pointwise rimcompact.*

Proof. b) $\Rightarrow$ a) Suppose that $x \in X$; let G_x denote $\cap\{cl_{\beta X}U$: U is π-open in X, $x \in U\}$. According to 2.2 of [Di_1], G_x is a compact connected subset of βX. If X is pointwise rimcompact, then $G_x \cap X = \{x\}$. Let KX be a compactification of X with totally disconnected remainder, and k: $\beta X \to KX$ the natural map. Then $k[G_x]$ is a connected subset of KX. Since $k[G_x] \cap X = \{x\}$, and $k[G_x] \cap (KX \setminus X)$ is totally disconnected, $k[G_x] \cap (KX \setminus X)$ is an open 0-dimensional subset of $k[G_x]$. It follows that $k[G_x] = \{x\}$, thus $G_x = \{x\}$. If $x \notin F$, where F is closed in X, then $x \notin cl_{\beta X}F$. A compactness argument and the fact that finite intersections of π-open sets are π-open show that there is a π-open set U of X with $x \in U$ and $F \cap cl_X U = \emptyset$.

R. C. Solomon ([So]) showed that scattered spaces need not be 0-dimensional. The next result contains a characterization of scattered spaces that are 0-dimensional. As mentioned in the introduction, any scattered space is totally disconnected.

2.6. *Corollary. If* X *is either scattered or fully disconnected, the following are equivalent:*

a) X *is* 0-*dimensional;*

b) X *is rimcompact;*

c) X *is a* 0-*space;*

d) X *has a compactification with totally disconnected remainder.*

Proof. If X is fully disconnected, then X is pointwise rimcompact, thus b), c), and d) are equivalent. It is easy to verify that a fully disconnected rimcompact space is 0-dimensional.

If X is scattered, then R(X) is scattered. According to 2.4, b), c) and d) are equivalent. In fact, according to the proof of 2.3, there is a compactification αX of X with $cl_{\alpha X}(\alpha X \setminus X)$ 0-dimensional. Since a scattered space is totally disconnected, L(X) is a locally compact totally disconnected space, and thus is 0-dimensional. An easy application of 2.2 to $\alpha X = cl_{\alpha X}(\alpha X \setminus X) \cup L(X)$ shows that αX is 0-dimensional, so that X is 0-dimensional.

Then any fully disconnected non-0-dimensional space is a pointwise rimcompact, nonrimcompact space. A totally disconnected rimcompact space need not be 0-dimensional (see the remarks following 3.2).

If a space X satisfies the conditions of 2.3, then L(X) is dense in X. The next lemma allows a weakening of this condition on X.

2.7. *Lemma. Suppose that* $bd_X R(X)$ *is locally compact. Then* X *is rimcompact if and only if* R(X) *and* $X \setminus int_X R(X)$ *are rimcompact.*

Proof. If X is rimcompact, then the closed subsets R(X) and $X \setminus int_X R(X)$ are rimcompact. Suppose then that the sets R(X) and $X \setminus int_X R(X)$ are rimcompact. It suffices to show that each point of $bd_X R(X)$ has a base in X of π-open sets. Choose $p \in bd_X R(X)$, and W open in X with $p \in W$. Since $bd_X R(X)$ is locally compact, $cl_{\beta X} bd_X R(X) \setminus bd_X R(X)$ is a compact subset of βX. Without loss of generality, $cl_{\beta X} W \cap [cl_{\beta X} bd_X R(X) \setminus bd_X R(X)] = \emptyset$, so that $cl_{\beta X} W \cap cl_{\beta X} bd_X R(X) = X \cap cl_{\beta X} W \cap cl_{\beta X} bd_X R(X) = cl_X W \cap bd_X R(X)$, hence this last set is compact.

Choose U_1 and U_2 to be open neighbourhoods of p in R(X) and $X \setminus int_X R(X) = cl_X L(X)$ respectively, with $bd_{R(X)} U_1$ and $bd_{clL(X)} U_2$ compact, and $U_1 \cup U_2 \subseteq W$. Consider the set $W_2 = V_1 \cup V_2$, where, for i = 1,2, $V_i = U_i \setminus [bd_X R(X) \setminus (U_1 \cap U_2)]$. Since $[U_1 \cup L(X)] \cap [U_2 \cup int_X R(X)] = [U_1 \cup U_2] \setminus [bd_X R(X) \setminus (U_1 \cap U_2)] = W_2$, the set W_2 is open in X. Now $bd_X(V_1 \cup V_2) \subseteq bd_X V_1 \cup bd_X V_2$. It is easy to see that

$bd_X V_1 \cap int_X R(X) \subseteq bd_X U_1$ and $bd_X V_2 \cap L(X) \subseteq bd_X U_2$, so that $bd_X V_1 \cup bd_X V_2 \subseteq bd_X U_1 \cup bd_X U_2 \cup (cl_X W \cap bd_X R(X))$. Since the latter set is compact, the proof is complete.

2.8. *Theorem. Suppose that* X *is a space in which* R(X) *is rimcompact and has locally compact,* 0-*dimensional boundary. The following are equivalent:*

a) X *has a compactification with totally disconnected remainder,*

b) X *has a compactification with* 0-*dimensional remainder,*

c) X *is rimcompact.*

Proof. Suppose that a compactification KX of X has totally disconnected remainder. Then $cl_{KX} L(X) \setminus cl_X L(X)$ is totally disconnected, and $R(cl_X L(X)) \subseteq bd_X R(X)$, so that $R(cl_X L(X))$ is locally compact and 0-dimensional. According to 2.4, $cl_X L(X)$ is rimcompact. The result now follows from 2.7.

3. Examples

The examples in this section will be based on the following construction. A collection of infinite subsets of **N** is called *almost disjoint* if the intersection of any two distinct members is finite. Let $\mathcal{R}$ denote a maximal family of almost disjoint subsets of **N**. The space $\mathbf{N} \cup \mathcal{R}$ will have the following topology: each point of **N** is isolated, and $\lambda \in \mathcal{R}$ has as an open base $\{\{\lambda\} \cup (\lambda \setminus F)$: F is a finite subset of $\mathbf{N}\}$. It is noted in 51 of [GJ] that such spaces are first countable, locally compact, 0-dimensional and pseudocompact. According to 2.1 of [Te], any compact metric space without isolated points is a homeomorphic to the remainder $\beta(\mathbf{N} \cup R) \setminus \mathbf{N} \cup R$ for a suitably chosen maximal almost disjoint family R.

The following examples make use of spaces X of the form $\beta(\mathbf{N} \cup R) \times \omega_1 \subseteq X \subseteq \beta(\mathbf{N} \cup R) \times (\omega_1 + 1)$ for some R (where ω_1 denotes the first uncountable ordinal); it follows from theorems 1 and 4 of [Gl] and 6.7 of [GJ] that for each such X, $\beta X = \beta(\mathbf{N} \cup R) \times (\omega_1 + 1)$.

The first example indicates that some disconnectedness in R(X) is necessary in 2.3.

3.1. *Example*. Suppose that X is a 0-space, and that R(X) contains a nontrivial metric continuum. There is a non-rimcompact 0-space Y for which R(Y) = R(X).

Choose C to be a nontrivial metric continuum in R(X), and R so that $\beta(\mathbf{N} \cup R) \setminus \mathbf{N} \cup R$ is homeomorphic to C. According to 3.2 of [Di_1], the space $Z = [\beta(\mathbf{N} \cup R) \times (\omega_1 + 1)] \setminus [(\mathbf{N} \cup R) \times \{\omega_1\}]$ is a non-rimcompact 0-space with R(Z) homeomorphic to C and $\beta Z \setminus Z$ 0-dimensional. Let Y_1 be the disjoint union of X and Z, and Y the quotient space of Y_1 obtained by identifying each point of C in X with its corresponding point in Z. The space Y is not rimcompact, since the closed subspace Z is not rimcompact. On the other hand, if KX is any compactification of X with 0-dimensional remainder, then the quotient space KY of the disjoint union of KX and βZ

obtained by a similar identifying of points of C is a compactification of Y having 0-dimensional remainder. Finally, $R(Y) = R(X)$.

The next example indicates that some form of the local compactness condition on $R(X)$, that $R^{\gamma}(X) = \emptyset$ for some γ, is also necessary in 2.3.

3.2. *Example.* There is a space X for which $R(X)$ and $\beta X \setminus X$ are totally disconnected, but X is not a 0-space.

Choose $\mathcal{R}$ so that $\beta(\mathbf{N} \cup \mathcal{R}) \setminus \mathbf{N} \cup \mathcal{R}$ is homeomorphic to the unit interval **I**, and define $X = [\beta(\mathbf{N} \cup \mathcal{R}) \times (\omega_1 + 1)] \setminus [(\mathbf{N} \cup \mathcal{R} \cup \mathbf{Q})] \times \{\omega_1\}]$, where **Q** denotes the rationals in $[0,1]$. Then $R(X)$ is 0-dimensional and nowhere locally compact. The space $\mathbf{N} \cup \mathcal{R} \cup \mathbf{Q}$ is totally disconnected but not 0-dimensional, for otherwise the closures in $\beta(\mathbf{N} \cup \mathcal{R} \cup \mathbf{Q}) = \beta(\mathbf{N} \cup \mathcal{R})$ of open-closed sets of $\mathbf{N} \cup \mathcal{R} \cup \mathbf{Q}$ separating 1/3 and 2/3 disconnect the unit interval **I**. Since $\beta X \setminus X$ is totally disconnected but not 0-dimensional, X is not a 0-space (see the introduction to [Di_1] for a justification of this last conclusion).

Note that if $\mathcal{R}$ and **Q** are as above, then the space $\mathbf{N} \cup \mathcal{R} \cup \mathbf{Q}$ is totally disconnected and rimcompact but not 0-dimensional, while the space $\mathbf{N} \cup \mathcal{R} \cup \{1/3, 2/3\}$ is scattered but not fully disconnected.

The last example indicates that the local compactness condition in 2.3 cannot be replaced with a stronger form of 0-dimensionality.

3.3. *Example*. There is a non-rimcompact 0-space X with R(X) strongly 0-dimensional (in the sense of dim).

Let R and $\mathbf{Q}$ be as in 3.2, and X = $[\beta(\mathbf{N} \cup R) \times (\omega_1 + 1)] \setminus [(R \cup \mathbf{Q})] \times \{\omega_1\}]$. Since R(X) is homeomorphic to the set of irrationals in [0,1], R(X) is strongly 0-dimensional.

To show that X is not rimcompact, we first show that if $p \in \mathbf{I} \setminus \mathbf{Q}$, then there is no neighbourhood U of the point p in $\beta(\mathbf{N} \cup R)$ with $1/3 \notin cl_{\beta(\mathbf{N}\cup R)}U$ and $(bd_{\beta(\mathbf{N}\cup R)}U) \cap R = \emptyset$. For if such a U exists, then $(bd_{\beta(\mathbf{N}\cup R)}U) \cap (\mathbf{N} \cup R) = \emptyset$, since points of $\mathbf{N}$ are isolated in $\beta(\mathbf{N} \cup R)$. That is, $U \cap (\mathbf{N} \cup R)$ is open-closed in $\mathbf{N} \cup R$, and $cl_{\beta(\mathbf{N}\cup R)}[U \cap (\mathbf{N} \cup R)]$ disconnects $\mathbf{I}$.

Suppose that X is rimcompact, and let V be a π-open subset of X with $p \in V$ for some $p \in R(X)$ while $(1/3, \omega_1) \notin cl_{\beta X}V$. Lemma 1 of [Sk] implies that $cl_{\beta X}V \cap (\beta X \setminus X)$ is open-closed in $\beta X \setminus X$. Then $cl_{\beta X}V \cap [\beta(\mathbf{N} \cup R) \times \omega_1]$ has the properties of U above, a contradiction.

We show that $R \cup \mathbf{Q}$ is 0-dimensional. Let $f: R \cup \mathbf{Q} \to \mathbf{I}$ be a continuous function. Since $\mathbf{Q}$ is countable, there is $r \in \mathbf{I}$ with $f^{\leftarrow}(r) \cap \mathbf{Q} = \emptyset$. Now points of R are isolated in $R \cup \mathbf{Q}$, so that $bd_{R\cup\mathbf{Q}}f^{\leftarrow}[0,r) \subseteq bd_{R\cup\mathbf{Q}}f^{\leftarrow}(r) = \emptyset$. Sets of the form $f^{\leftarrow}[0,r)$ form a base for any completely regular Hausdorff space, thus the proof is complete.

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College of Charleston

Charleston, South Carolina 29424

and

Moorhead State University
Moorhead, Minnesota 56560
and
Moorhead State University
Moorhead, Minnesota 56560