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I. Introduction

In the paper wherein the almost continuous functions were first defined [9], Stallings showed that a connectivity function defined on an n-cell, $n > 1$, must be almost continuous but left open the question of whether the same implication holds for $n = 1$. It is now known that the implication reverses, for functions defined on the unit interval. For functions defined on the unit interval,

(*) *almost continuity implies connectivity,*

while the converse fails [3,4,5,8]. Herein, the concern is such dependency in a more general setting. When is it true that, if X is a continuum, then each almost continuous function defined on X is also a connectivity function, i.e., for which continua is condition (*) true?

We are concerned with only three types of continua: (1) dendrites, (2) continua in which each nondegenerate connected set is arcwise connected, and (3) hereditarily locally connected continua. It is known that (1) implies (2) and (2) implies (3) (see [6, p. 249, p. 301]). It was shown in [3] that (*) is not true for locally connected continua that are not hereditarily locally connected. The same techniques can be used to show that (*) implies (3). We show below, in Theorem 2, that (2) implies (*). An example of a continuum of type (3) but not of type (2) is given, leaving the question of whether it satisfies (*). In Theorem 3, it is shown that a connectivity function on a subarc of a dendrite can be extended to a connectivity function on the dendrite; that fact is used to show that, like the unit interval, the converse of (*) fails for dendrites. Whether or not the converse of (*) fails for a continuum of type (2) is left as an open question.

II. Main Results

Notation and terminology are generally standard. Where it is not or is not clear from context, it will be specifically defined. Thus, $I = [0,1]$ is the closed unit interval of the real line, R, and $K|M$ is the restriction of the subset K of a product space $X \times Y$ to the subset (or point) M of X. It is important to note that the function $f: X \to Y$ is a subspace of the product space $X \times Y$. All spaces considered are separable and metric. For definitions and properties of arcs, dendrites, etc., the reader is directed to a standard reference such as [6] or [10].

Definition 1. *The function* $f: X \to Y$ *is said to be a connectivity function provided that* $f|C$ *is connected whenever* C *is a connected subset of* X.

Definition 2. *The function* $f: X \to Y$ *is said to be almost continuous provided that, whenever* U *is an open set of* $X \times Y$ *containing* f, *then there is a continuous* $g: X \to Y$ *in* U.

Proposition 1. *If* $S \subset R$ *is connected, then the function* $f: S \to R$ *is a connectivity function if and only if* f *is connected.*

Proposition 2. *There is a connected function* $k: I \to R$ *that is dense in* $I \times R$ [5].

Theorem 1. *Suppose each nondegenerate connected subset of the space* X *is arcwise connected and* $f: X \to Y$ is *a function such that, if* $A \subset X$ *is an arc, then* $f|A$ *is connected. Then* f *is a connectivity function.*

Proof. Assume f is not a connectivity function. There is a nondegenerate subset $C \subset X$, the function $f|C$ is not connected, and there are points p and q of C such that $(p, f(p))$ and $(q, f(q))$ do not belong to the same component of $f|C$. Because C is arcwise connected, there is an arc $A \subset C$, with end points p and q, and $f|A$ intersects two components of $f|C$. This is contrary to hypothesis.

Theorem 2. *If each nondegenerate connected subset of the space* X *is arcwise connected and* $f: X \to R$ *is almost continuous, then* f *is a connectivity function.*

Proof. By the Corollary to Propositions 2 and 3 of [9], if A is an arc in X, then $f|A$ is connected. By Theorem 1, above, f is a connectivity function.

Theorem 3. *If* X *is a dendrite,* $A \subset X$ *is an arc, and* $f: X \to R$ *is a connectivity function, then there is a connectivity function* $F: X \to R$ *such that* $F|A = f$.

Proof. Suppose $A = [p,q]$ and e_0 and e_1 are two end points of X such that, if a is a point of (p,q), then e_0 is separated from q and e_1 is separated from p in $X - a$. Only one of the following cases is true: (a) $e_0 = p$ and $q = e_1$, (b) $e_0 = p$ and $q \neq e_1$, (c) $e_0 \neq p$ and $q = e_1$, (d) $e_0 \neq p$ and $q \neq e_1$.

In case (a), $F_0 = f$.

In case (b), there is a homeomorphism $h: I \to [q,e_1]$ such that $h(0) = q$ and $h(1) = e_1$. Then $F_0:[e_0,e_1] \to R$ is defined so that $F_0(x) = f(x)$, for x in A and $F_0(x) = k(h^{-1}(x))$, for x in $(q,e_1]$ and k from Proposition 2.

In case (c), there is a homeomorphism $h: I \to [p,e_0]$ such that, $h(0) = p$ and $h(1) = e_0$. Then $F_0[e_0,e_1] \to R$ is defined so that $F_0(x) = f(x)$, for x in A, and $F_0(x) = k(h^{-1}(x))$, for x in $[e_0,p]$ and k from Proposition 2.

In case (d), define $F_0|[p,e_1]$ as in case (b) and $F_0|[e_0,q]$ as in case (c).

In any of these cases, since its domain is an arc, F_0 is connected and must be a connectivity function. Denote by $e_2,e_3,e_4, \cdots$ the points of a countable dense subset of the end points of X, no one of which is e_0 or e_1. For each positive integer i, define $F_i[e_0,e_{i+1}]\to R$, by induction, as follows. The arc $[e_0,e_{i+1}]$ has a last point b_i of $\cup_{n=1}^{n=i}[e_0,e_n]$, along $[e_0,e_{i+1}]$, in the order from e_0 to e_{i+1}. There is a homeomorphism h_i: $I \to [b_i,e_{i+1}]$, where $h_i(0) = b_i$ and $h_i(1) = e_{i+1}$. Then $F_i(x) = F_{i-1}(x)$, for x in $\cup_{n=1}^{n=i}[e_0,e_n]$ and $F_i(x) = k(h_i^{-1}(x))$, for x in $(b_i,e_{i+1}]$ and k from Proposition 2.

With $G = F_1 \cup F_2 \cup F_3 \cup \cdots$, it is easy to see that G is connected. Suppose the domain, D, of G does not contain every non-end point of X. There is a point a of X, belonging to an arc $[e_0,e]$ in X, where e is an end point of X that is not in D. Since D + a is a connected subset of the dendrite X, it is arcwise connected and contains an arc from e_0 to e. Then the dendrite X contains two arcs from e_0 to e, which is impossible. Therefore, if x is a point of X and is not in D, then x is an end point of X and is in cl(D).

Knowing that G is connected is not enough to conclude that G is a connectivity function (see Note 2). Assume G is not a connectivity function. By Theorem 1, there is an arc $\alpha = [x,y]$ in the domain, D, of G such that $G|\alpha$ is not connected. There are two positive integers, i and j, with $[e_0,e_i]$ containing x and $[e_0,e_j]$ containing y. Then we have $\alpha = \beta \cup \gamma$, where β is a subarc of $[e_0,e_i]$ and has exactly one point in common with the subarc γ of $[e_0,e_j]$. For each positive integer n, the function $G|[e_0,e_n]$ is a connectivity function, so both $G|\beta$ and $G|\gamma$ are connected. However, $G|\alpha$ and $G|\gamma$ have a common point, so $G|\alpha$ is connected, contradicting our assumption.

Now take a function $F: X \to R$ such that $F|D = G$ and assume it is not a connectivity function. By Theorem 1, there is an arc α of X for which $F|\alpha$ is not connected. We may choose α so that one of its end points, x, is in D and the other, e, is an end point of X that is not in D. Since G is a connectivity function, $G|[x,e)$ is connected. Because e is an end point of the dendrite X, for every subarc $[z,e]$ of α there is a positive integer n and a subarc of $[b_n,e_n]$ in $[z,e]$. Thus, $F|e \subset (e \times R) \subset cl\{[x,e]\}$, so that $G|\alpha$ is connected. With this contradiction, the proof is complete.

Theorem 4. *If* X *is a dendrite, there is a connectivity function* $f: X \to R$ *that is not almost continuous.*

Proof. Suppose A is an arc in X. From either [4] or [8], it follows that there is a connectivity function $f: A \to R$ that is not almost continuous. Using Theorem 3, extend f to a connectivity function $F: X \to R$. There is an open set U of $A \times R$, containing f, but no continuous function from A into R. Since $(A \times R) - U$ is closed in $A \times R$, then $V = (X \times R) - [(A \times R) - U]$ is open in $X \times R$. If F were almost continuous, then there would be a continuous function $g: X \to R$ in V and the function $g|A: A \to R$ would be in V.

Note 1. *There is a hereditarily locally connected continuum with a connected nondegenerate subset that is not arcwise connected.*

To construct such a continuum, begin by denoting the unit interval of the x-axis, of R^3, by X_0. For each positive integer i, the continuous function $f_i: X_0 \to R$ is defined so that $f_i(x,0,0) = (x,0,0)$ if and only if either $x = 0$, $x = 1$, or there is a positive integer j such that $x = j/p_i$, where p_i is the i^{th} positive prime number; also, for x in X_0, $0 \leq f_i(x) \leq 1/2^i$. Rotate f_i about the x-axis through an angle of $\pi/(i+1)$ and call the resulting set X_i. Denote by p the point $(1/4,0,0)$. Add X_0 to the set $M = X_i \cup X_2 \cup X_3 \cup \cdots$ to get the hereditarily locally connected continuum X. Clearly, $C = M + p$ is a connected subset of X and it contains no arc from p to a point $q \neq p$.

Note 2. *There is a dendrite* X *and a function* f: X $\to$ R *that is connected but is not a connectivity function.*

First set $X_0 = I \times 0$. For each positive integer i, the continuum $X_i = (1/2^{(i-1)}) \times [0,(1/2^{(i-1)}]$ and the dendrite X is $\cup_{i=0}^{i=\infty} X_i$. Define the function $f: X \to R$ so that $f(0,0) = 1$; if $x > 0$, then $f(x,0) = 0$; and, if $y > 0$, then $f(x,y) = \sin(1/y)$. Clearly, f is connected but $f|X_0$ is not.

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