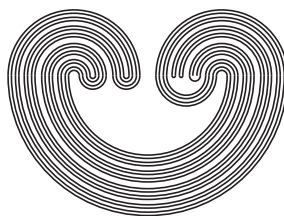


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## CLASSIFYING SHAPE FIBRATIONS AND PRO-FIBRATIONS II

by

HAROLD M. HASTINGS

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**Mail:** Topology Proceedings  
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Auburn University, Alabama 36849, USA

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## CLASSIFYING SHAPE FIBRATIONS AND PRO-FIBRATIONS II

Harold M. Hastings

### 1. Introduction

This paper is a sequel to *Classifying shape fibrations and pro-fibrations* by Hastings and Waner [HW]. We shall use the categories *Top of compactly generated spaces*,  $\text{Top}^{\mathbb{N}}$  of towers of compactly generated spaces and levelwise maps, and *pro-Top* of towers of compactly generated spaces and maps given by

$$\text{pro-Top}(\{X_m\}, \{Y_n\}) = \lim_n \text{colim}_n \{\text{Top}(X_m, Y_n)\}.$$

There is an evident functor  $\text{Top}^{\mathbb{N}} \rightarrow \text{pro-Top}$  extending the identity map on objects. In [HW], the authors first used May's techniques to classify fibrations in  $\text{Top}^{\mathbb{N}}$ , and then applied these results to classify fibrations in *pro-Top* under suitable finiteness conditions. Unfortunately, this classification involved the colimit of a complicated diagram of maps into classifying spaces in  $\text{Top}^{\mathbb{N}}$ . In the present paper, the authors define a *monoid of equivalences* directly in  $\text{Top}^{\mathbb{N}}$ . This yields a classification result in terms of May's two-sided bar construction. We thank Mahendra Jani, José M. R. Sanjurjo and Stefan Waner for helpful conversations. We thank the Universidad de Complutense of Madrid for their hospitality and partial support for some of this work.

## 2. Pro-spaces

We review some of the basic ideas about pro-spaces and their homotopy theory used in this paper.

2.1 *Homotopy theory.* The homotopy theories of  $\text{Top}^{\mathbb{N}}$  and  $\text{pro-Top}$  and the corresponding homotopy categories  $\text{Ho}(\text{Top}^{\mathbb{N}})$  and  $\text{Ho}(\text{pro-Top})$  were defined by D. A. Edwards and the first-named author [EH] using Quillen's [Q] closed model structures.

2.2 *Internal mapping functor.* We shall also use the following internal mapping functor on  $\text{pro-Top}$ . Let  $\text{Map}$  denote the internal mapping functor in  $\text{Top}$ ;  $\text{Map}(A, B)$  is defined by suitably topologizing the set of functions from  $A$  to  $B$  [Hs, St].

2.3 *Definition.* For any pro-spaces  $\{X_m\}$  and  $\{Y_n\}$ , define

$$\text{MAP}(\{X_m\}, \{Y_n\}) = \{\text{colim}_m \{\text{Map}(\{X_m\}, \{Y_n\})\}\},$$

a pro-space indexed by  $n$ .

Then  $\text{MAP}(\{X_m\}, \{Y_n\})$  is a pro-space, and  $\text{Map}$  extends to a functor

$$\text{MAP}: \text{Pro-Top}^{\text{op}} \times \text{pro-Top} \rightarrow \text{pro-Top}.$$

It is easy but tedious to prove that  $\text{MAP}$  is an internal mapping functor on  $\text{pro-Top}$ .

2.4 *Proposition.* For any three pro-spaces  $X$ ,  $Y$ , and  $Z$ , there is a natural equivalence

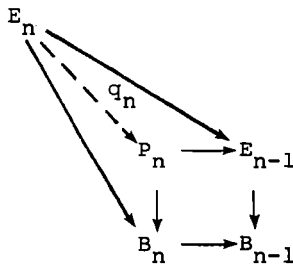
$$\text{MAP}(X, \text{MAP}(Y, Z)) \rightarrow \text{MAP}(X \times Y, Z)$$

in  $\text{pro-Top}$ .

2.5. *Geometric realization.* Although the "Milnor realization" [M1, see also Hs]  $R$  does not prolong to a functor from the category  $ss(\text{pro-Top})$  of simplicial objects in  $\text{pro-Top}$  to  $\text{pro-Top}$ , it is easy to see the following.

2.6. *Proposition.* *The Milnor realization prolongs to a functor, also denoted  $R$ , from the full sub-category of simplicial objects of finite simplicial degree in  $\text{pro-Top}$  to  $\text{pro-Top}$ .*

2.7. *Fibrations and fibrant objects.* We recall [EH] that fibrations in  $\text{pro-Top}$  are retracts of fibrations in  $\text{Top}^N$ , and that a map  $p: E \rightarrow B$  in  $\text{Top}^N$  is a fibration in  $\text{Top}^N$  if  $p_0: E_0 \rightarrow B_0$  is a fibration in  $\text{Top}$ , and for all  $n > 0$ , the induced maps  $q_n$  in the following diagrams (in which the spaces  $P_n$  are pullbacks) are fibrations.



These fibrations satisfy the *covering homotopy extension property* with respect to maps which are levelwise cofibrations and homotopy equivalences (and, more generally, retracts of such maps) [EH]. They therefore play the role of Hurewicz fibrations. See also [Q].

A pro-space  $X$  is called *fibrant* if the natural map  $X \rightarrow *$  is a fibration, which is equivalent to the bonding maps of  $X$  being fibrations.

We also note that towers of fibrations (levelwise fibrations) need not be fibrations in  $\text{Top}^{\mathbb{N}}$ . However, we have the following [EH].

2.8. *Proposition [EH].* Let  $B$  be a connected pro-space, let  $p: E \rightarrow B$  be a tower of fibrations, let  $b \in B$  be a one-point pro-space, and let  $F = p^{-1}(b)$ . Then  $p$  factors through a fibration  $p': E' \rightarrow B$  in  $\text{Top}^{\mathbb{N}}$ , with fibre homotopy equivalent to  $F$ , and  $E'$  homotopy equivalent to  $E$  (both in  $\text{Ho}(\text{Top}^{\mathbb{N}})$ ).

More generally, we may  $\Gamma$ -fy spaces and maps using the following.

2.9. *Proposition [EH].* Let  $f: X \rightarrow Y$  be a map in  $\text{pro-Top}$ . Then  $f$  factors through a fibration  $f': \Gamma X \rightarrow Y$  in  $\text{Top}^{\mathbb{N}}$ , with  $\Gamma X$  homotopy equivalent to  $X$  (both in  $\text{Ho}(\text{Top}^{\mathbb{N}})$ ).

Any pro-space  $X$  may be replaced by a fibrant pro-space (tower of fibrations) by applying the above  $\Gamma$ -construction to the map  $X \rightarrow *$ .

Fibrations of pro-spaces of the levelwise homotopy type of CW complexes of *bounded dimension* also satisfy a Dold theorem (Theorem 4.2 in [HW]). The dimensionality restriction is a consequence of the behavior of the Whitehead theorem [EH, Ch. 5] in  $\text{pro-top}$ : there is a

Whitehead theorem for pro-spaces of the levelwise homotopy type of CW complexes of *bounded* dimension, but there are counterexamples to Whitehead theorem without dimension bounds.

2.10. *Proposition.* Let  $B$  be a path connected pro-space, let  $p: E \rightarrow B$  and  $p': E' \rightarrow B$  be fibrations, and let  $f: E \rightarrow E'$  be a map over  $B$ . Suppose that all pro-spaces have the levelwise homotopy type of CW complexes of bounded dimension. If the restriction of  $f$  to a fibre of  $p$  is an equivalence in  $\text{Ho}(\text{pro-Top})$ , then  $f$  is a (fibrewise) equivalence in  $\text{Ho}(\text{pro-Top})$ .

2.11. *Definition.* A map  $p: E \rightarrow B$  in  $\text{Top}^{\mathbb{N}}$  is called a quasi-fibration if each level  $P_n \rightarrow E_n$  is a quasi-fibration in  $\text{Top}$ .

As in Proposition 2.8,  $\Gamma$ -fication preserves fibres of quasi-fibrations up to pro-homotopy equivalence.

2.12. *Proposition.* Let  $B$  be a connected pro-space, let  $p: E \rightarrow B$  be a quasi-fibration, let  $b \in B$  be a one-point pro-space, and let  $F = p^{-1}(b)$ . Then  $p$  factors through a fibration  $p': E' \rightarrow B$  in  $\text{Top}^{\mathbb{N}}$ , with fibre homotopy equivalent to  $F$ , and  $E'$  homotopy equivalent to  $E$  (both in  $\text{Ho}(\text{Top}^{\mathbb{N}})$ ).

Quasi-fibrations are also preserved under the usual pushout and finite colimit constructions.

We conclude this section by defining monoid and group objects in  $\text{pro-Top}$ .

2.13. *Definitions.* A monoid object  $M$  in  $\text{pro-Top}$  consists of a pro-space  $M$ , a "unit" one-point pro-space  $e \subset M$ , and an associative "multiplication map"  $\mu: M \times M \rightarrow M$  for which  $e$  is a unit (the composite maps  $M \xrightarrow{\cong} e \times M \rightarrow M$  and  $M \xrightarrow{\cong} M \times e \rightarrow M$  are the identity on  $M$ ).

A *group-like monoid object* is a monoid object as above with the additional property that for all one-point pro-spaces  $m$  contained in  $M$ , left and right multiplication by  $m$  are equivalences in  $\text{pro-Top}$ .

A *group object* in  $\text{pro-Top}$  is a monoid object with the additional property that there is an inverse map  $\nu: M \rightarrow M$  with respect to the multiplication  $\mu$ .

Clearly towers of monoids or groups are monoid objects or group objects respectively. For any pro-space  $X$ , the function pro-space  $\text{MAP}(X, X)$  is a monoid object. Monoids of self-equivalences will be defined below.

### 3. Principal quasi-fibrations, fibre bundles and principal bundles

We define principal quasi-fibrations, and also for possible later use, we propose definitions of fibre bundles and principal bundles in  $\text{pro-Top}$ . Fibre bundles and principal bundles are defined because many examples of quasi-fibrations and principal quasi-fibrations in  $\text{pro-Top}$  are in fact fibre bundles and principal bundles, respectively. Some of these examples are given below.

3.1. *Definitions.* Let  $M$  be a pro-monoid, and let  $X$  be a pro-space. A right action of  $M$  on  $X$  is a map  $\alpha: X \times M \rightarrow X$ , which is associative, and for which the identity in  $M$  is a unit. Left actions are defined analogously. A (right or left) action is called principal if for each map  $* \rightarrow X$ , the orbit of the image is isomorphic to  $M$ . A (right or left) action is called homotopy principal if the orbits (as defined above) are pro-homotopy equivalent to  $M$ .

3.2. *Definition.* Let  $M$  be a monoid object in pro-Top, let  $E$  be a pro-space with right action by  $M$ , let  $B$  be the quotient space of  $E$  under the action, that is,  $B = E \times_M *$ , and let  $p: E \rightarrow B$  be the quotient map. If  $p$  is a quasi-fibration, then it is called a right principal quasi-fibration. Left principal quasi-fibrations are defined analogously.

3.3. *Remarks.* In particular, towers of principal fibrations are principal quasi-fibrations. (We do not define principal fibrations because a tower of fibrations need not be a fibration in the model structure on pro-Top. However, by modifying Proposition 2.8, in the case of homotopically finite-dimensional, connected pro-spaces, one may replace principal quasifibrations by fibrations with homotopy equivalent fibres. We shall see more examples below.



3.4. *Fibre bundles.* We sketch an etale definition of fibre bundles.

First define etale coverings in pro-Top by mimicking the definition in [AM]. Let  $X$  be a pro-space. We now call a family of towers  $\{U_\alpha\}$  and levelwise maps  $U_\alpha \rightarrow X$  is an etale cover of  $X$  if

- (1) for each  $\alpha$ , at each level  $n$ , the restriction  $U_{\alpha,n} \rightarrow X_n$  is a homeomorphism onto an open set, and  
 (2) the induced map from the coproduct in  $\text{Top}^{\mathbb{N}}$

$$\coprod_{\alpha} U_{\alpha} \rightarrow X$$

is surjective. Note that levelwise maps were used in order to handle coproducts, which do not exist in general in pro-Top.

We shall use the notation  $\{U_\alpha \rightarrow X\}$  for the above etale cover. We can now define Steenrod fibre bundles in pro-Top.

3.5. *Definition.* A map of pro-spaces  $E \rightarrow B$  is called a Steenrod fibre bundle if there is an etale covering  $\{U_\alpha \rightarrow X\}$  such that the pullback over  $\coprod_{\alpha} U_{\alpha}$  is isomorphic to a coproduct  $\coprod_{\alpha} U_{\alpha} \times F$ , and the usual compatibility conditions [St] hold.

The definition of principal fibre bundle is now clear. Although fibre bundles are levelwise fibrations, they in general need not be fibrations in pro-Top.

3.6. *Examples.* (1) Let  $G$  be a group-object in pro-Top. For example,  $G$  might be the Lie series

associated with a compact topological group ( $G$  is the inverse limit of its Lie series, and the Lie series is defined up to equivalence of pro-Lie groups). Let  $\alpha: X \times G \rightarrow X$  be an associative, unitary, right action of  $G$  on a pro-space  $X$ . Let  $X/G$  be the "orbit pro-space";  $X/G = X \times_G *$ , where the point  $*$  is considered as a trivial pro-space with trivial action. For any map  $*$  to  $X/G$ , one can form the pullback of  $X$  over  $*$ . By definition of the action  $\alpha$ , this pullback is isomorphic to  $G$ . Then the quotient map  $X \rightarrow X/G$  is a principal fibre bundle, see [EH2].

(2) Open principal fibrations in the sense of J. Cohen [Ch] correspond to principal fibre bundles under a modification of the Lie series construction [EH2]. To do this, let  $G$  be a compact topological group, let  $\alpha: X \times G \rightarrow X$  be an associative, unitary action, such that the map to the orbit space  $X \rightarrow X/G$  is an open principal fibration. Let  $\{G_n\}$  be the Lie series associated with  $G$ . Then let  $X_n = X \times_G G_n$  for each  $n$ . This construction yields a tower of principal bundles  $\{G_n \rightarrow X_n \rightarrow X_n/G_n = X/G\}$ , and thus a principal quasi-fibration. This example and the one above are related by the inverse functors *Lie series* (on the category of compact topological groups) and *lim* (on the category of towers of Lie groups).

(3) Let  $\{F_n \rightarrow E_n \rightarrow B_n\}$  be a tower of Steenrod fibre bundles. Then it is clear that this tower is also a Steenrod fibre bundle in pro-Top.

Let  $G \rightarrow E \rightarrow B$  be a principal Steenrod fibre bundle and let  $F$  be a pro-space with principal right  $G$ -action. Then one may form the associated bundle  $F \rightarrow E \times_G F \rightarrow B$ , which is easily shown to be a Steenrod fibre bundle.

#### 4. The bar construction for pro-monoids

It would be desirable to directly prolong the classical bar construction and May's two-sided bar construction to pro-Top. Unfortunately, these bar constructions require infinite colimits (of sequences of cofibrations; these colimits serve as homotopy colimits), and infinite colimits do not in general exist in pro-Top.

However, these bar constructions admit natural filtrations (see especially [St], also [Hs2]), in which each level requires only finite colimits (which do exist in pro-Top). Recall that for a group  $G$  in Top, the usual classifying space construction,  $BG$ , admits a filtration.

$$(4.1). \quad F_0 BG \subset F_1 BG \subset \cdots \subset F_p BG \subset \cdots \subset \operatorname{colim}_p F_p BG =$$

$BG$ , and for any CW complex  $X$  of dimension  $\leq p$ ,

$$(4.2). \quad [X, F_p BG] \overset{\sim}{\cong} [X, F_p BG] \overset{\sim}{\cong} [X, BG].$$

Equivalently, principal  $G$ -bundles over any CW complex  $X$  of dimension  $\leq p$  are classified by  $[X, F_{p+1} BG]$ . This suggests that we restrict the classification theorem in pro-Top to *finite-dimensional* base pro-spaces, and construct the classifying object as a sequence of pro-spaces and cofibrations.

$$(4.3). \quad F_0BG \subset F_1BG \subset \cdots \subset F_pBG \subset \cdots .$$

Although the sequence (4.3) does not have a homotopy colimit (use the counter-example [EH] to an infinite-dimensional Whitehead theorem, a highly twisted fibration in pro-Top), it does have a weak homotopy colimit in the sense that for any finite-dimensional pro-space  $X$  of the homotopy type of a tower of CW complexes, the sequence

$$(4.4). \quad \{Ho(\text{pro-Top}) (X, F_pBG)\}$$

of sets of homotopy classes of maps is eventually constant (see Proposition 4.8, below).

4.5. *Two-sided bar construction.* We review May's [Ma] two-sided bar construction in the context of pro-Top.

Let  $M$  be a monoid-object in pro-Top, let  $E$  and  $F$  be pro-spaces which admit a right and respectively left  $M$ -actions. Then, following May, define the simplicial bar construction  $B_*(E, M, F)$ , or more simply just  $B_*$  when there is no chance of confusion, as a simplicial object

$$(4.6). \quad E \times F \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} E \times M \times F \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} E \times M \times M \times F \cdots$$

As usual, the  $p$ -skeleton of  $B_*$ ,  $F_pB_*$ , is the sub-simplicial object generated by non-degenerate simplices of simplicial degree  $\leq p$ .

We therefore have a sequence of realizations of filtration levels of the simplicial bar construction

$$(4.7). \quad F_0B(E, M, F) \subset F_1B(E, M, F) \subset \cdots$$

$$\subset F_pB(E, M, F) \subset F_{p+1}B(E, M, F) \subset \cdots$$

where we write  $F_p B$  for  $RF_p B_*$ . It is easy to see the following.

4.8. *Proposition.* *The inclusions*

$$F_p B(E, M, F) \subset F_{p+1} B(E, M, F)$$

*induce pro- $\pi_i$ -equivalences for all  $i \leq p$ .*

Thus these inclusions induce equivalences of functors on the corresponding full sub-categories of  $Ho(\text{pro-Top})$  consisting of pro-spaces of the homotopy type of levelwise CW-complexes of dimension  $\leq p$ .

4.9. *Remarks.* The above filtration 4.7,

$\dots \subset F_p B(E, M, F) \subset F_{p+1} B(E, M, F) \subset \dots$  is equivalent to Steenrod's [St] filtration

$$(4.10). \quad \dots \subset E_p \subset \dots \subset E_{p+1} \subset \dots \subset EG.$$

More precisely, if  $G$  is a topological group, then the filtration 4.7 agrees with Steenrod's filtration 4.10.

We regard the *functor*

$$(4.11). \quad \text{colim}_p Ho(\text{pro-Top}) (-, F_p B(E, M, F))$$

as a *weak homotopy colimit* of the filtration above. For uniformly finite-dimensional pro-spaces of the homotopy type of CW-complexes, this functor behaves as an ordinary homotopy colimit in that the sequence

$$(4.12). \quad \dots \rightarrow Ho(\text{pro-Top}) (X, F_p B(E, M, F)) \rightarrow$$

$$Ho(\text{pro-Top}) (X, F_{p+1} B(E, M, F)) \rightarrow \dots$$

collapses at filtration  $p = \dim(X) + 1$ , but the colimit is not representable. The non-existence of a better homotopy

colimit is closely connected with the failure of Whitehead and Brown theorems on all levelwise CW-objects of pro-Top, see [EH].

Additional properties of the above "filtered" bar construction will be described in Section 6, below.

### 5. Germs of self-equivalences

Let  $X$  be a pro-space. We shall define a monoid  $M(X)$  of germs of self-equivalences of  $X$ , as a monoid-object in pro-Top in which each "element" has a "homotopy inverse." It is the use of this monoid which yields the major simplification in this paper compared with [HW]. We shall use this construction later to relate pro-fibrations with suitably defined principal quasi-fibrations in pro-Top.

First, for any pro-space  $X$  let

$$(5.1). \quad \text{Eq}(X) = \{f: X \rightarrow X \mid \text{Ho}(f) \text{ is an equivalence in } \text{Ho}(\text{pro-Top})\} \subset \text{pro-Top}(X, X).$$

In the case of Top, the set of homotopy equivalences of a space  $X$  inherits a topology from that of the function space  $\text{Map}(X, X)$ . We shall now describe a similar topology on  $\text{Eq}(X)$ . First topologize  $\text{pro-Top}(X, X)$  as the limit of  $\text{MAP}(X, X)$ . Then give the set  $\text{Eq}(X)$  contained in  $\text{pro-Top}(X, X)$  the subspace topology.

5.2. *Definition.* For any pro-space  $X$ , let the monoid of self-equivalences of  $X$ , denoted  $M(X)$ , be the pullback of the diagram

$$\begin{array}{ccc}
 M(X) & \longrightarrow & \text{MAP}(X, X) \\
 \downarrow & & \downarrow \\
 \text{Eq}(X) & \longrightarrow & \text{pro-Top}(X, X),
 \end{array}$$

where  $\text{Eq}(X)$  and  $\text{pro-Top}(X, X)$  are topologized as above and regarded as "constant" objects of  $\text{pro-Top}$ .

It is easy to show that  $M(X)$  is a *group-like monoid-object* in  $\text{pro-Top}$ . The idea now is to relate fibrations with fibre  $F$  with principal fibrations with fibre  $M(V)$ . This uses May's [Ma] generalization of fibrations.

## 6. Categories of fibres in $\text{pro-Top}$

One may define a *principalization functor* in  $\text{pro-Top}$  by using May's theory of  $F$ -fibrations and  $FE$ -fibrations in  $\text{pro-Top}$ . We shall indicate the main ideas of these theories, referring the reader to [Ma], and generally following the exposition in [HW]. We shall then define the principalization functor. Throughout this section we shall assume  $F$  is a fibrant pro-space.

6.1. *Definitions:* [Ma] in the setting of pro-spaces with the exposition following [HW]. A category of fibres in  $\text{pro-Top}$  is a pair  $(F, \mathcal{F})$  consisting of a fibrant pro-space  $F$ , and a subcategory  $\mathcal{F}$  of  $\text{pro-Top}$  such that

(a)  $F \in \text{obj}(\mathcal{F})$ , and

(b)  $\text{obj}(\mathcal{F})$  contains and is closed under products

with one point pro-spaces.

In the case where  $F$  consists of all fibrant spaces pro-homotopy equivalent to  $F$ , and all pro-homotopy equivalences between them, we shall denote the corresponding category of fibres  $(FE, F)$ . We shall also sometimes omit reference to the standard fibre  $F$  in a category of fibres.

An  $F$ -space in pro-Top is a map  $E \rightarrow Y$  for which all fibres (pullbacks over one-point systems mapping into  $Y$ ) are objects of  $F$ . An  $F$ -map in pro-Top consists of a commutative diagram

$$\begin{array}{ccc}
 & f & \\
 D & \longrightarrow & E \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & Y,
 \end{array}$$

in which the restriction of  $f$  to any fibre (pullbacks over one-point system mapping into  $X$ ) is a map in  $F$ .

$F$ -homotopies between  $F$ -maps are defined using analogous diagrams and the cylinder  $D \times I \rightarrow X \times I$ .

$F$ -fibrations are defined using the following  $F$ -covering homotopy property (FCHP). In any commutative diagram of  $F$ -spaces of the following form, in which all horizontal maps except  $h$  represent  $F$ -maps, the indicated filler  $H$  exists and the pair  $(H, h)$  defines an  $F$ -map.

$$\begin{array}{ccccc}
 & & H & & \\
 D \times I & \xrightarrow{\quad \text{---} \quad} & & \xrightarrow{\quad \text{---} \quad} & E \\
 \downarrow & \swarrow & D \times 0 & \searrow & \downarrow \\
 A \times I & \xrightarrow{\quad h \quad} & & \xrightarrow{\quad \text{---} \quad} & B \\
 & \swarrow & A \times 0 & \searrow & \\
 & & & & 
 \end{array}$$



An  $F$ -space over  $B$  is an  $F$ -space  $E \rightarrow B$ , and one may now define  $F$ -maps and  $F$ -homotopies over  $B$  in the evident way.

We shall now recall the relationship [HW] between  $FE$ -fibrations and (Hurewicz) fibrations in  $\text{pro-Top}$ .

6.2. *Proposition.* Let  $\pi: E \rightarrow B$  be a fibration, with  $B$  connected, let  $b \in B$  be a one-point pro-space, and let  $F = p^{-1}(b)$ . Then  $\pi$  is a  $(FE, F)$ -fibration.

Note that  $F$  is fibrant in the above.

6.3. *Proposition.* Let  $\pi: E \rightarrow B$  be a  $(FE, F)$ -fibration, with  $B$  connected. Then  $\Gamma\pi: \Gamma E \rightarrow B$  is a (Hurewicz) fibration  $FE$ -equivalent to  $\pi$ , and, in particular, with fibre pro-homotopy, equivalent to  $F$ . Moreover,  $\Gamma$  maps  $FE$ -equivalences to fibre homotopy equivalences.

We are now ready to define the principalization functor.

6.4. *Principalization.* Let  $F \rightarrow E \rightarrow B$  be a fibration in  $\text{pro-Top}$  with  $B$  connected. Define  $\text{FIBEQ}(F \rightarrow *, E \rightarrow B)$  to be the subobject of  $\text{MAP}(F, E)$  consisting of those (sequences of) maps which map  $F$  via equivalences to fibres (over the corresponding images  $* \rightarrow B$ ).

There is a natural map

$$\text{FIBEQ}(F \rightarrow *, E \rightarrow B) \rightarrow \text{MAP}(*, B) = B$$

6.5. *Lemma [Ma]. With the evident definition of  $\text{MAP}$  in the context of  $FE$ -spaces, denoted  $\text{MAP}_{FE}$ , and  $F \rightarrow E \rightarrow B$  a fibration as above, the natural map*

$$\text{FIBEQ}(F \rightarrow *, E \rightarrow B) \rightarrow B$$

*is  $\text{MAP}_{FE}(F, B)$ .*

6.6. *Proposition. Let  $F \rightarrow E \rightarrow B$  be a fibration in pro-Top with  $B$  connected. Then the natural map*

$$\text{FIBEQ}(F \rightarrow *, E \rightarrow B) \rightarrow B$$

*is a right principal quasifibration with fibre  $\text{Eq}(F, F)$  and a  $(FE, F)$ -fibration.*

The proof follows directly from Lemma 6.5 and adjointness.

We shall generally denote the above principalization functor simply by  $P$ , when there is no chance of confusion. We conclude by stating some results of May in the context of pro-Top and the filtered two-sided bar construction. The proofs are easy modifications of those in [Ma].

6.7. *Proposition. Let  $F$  be a fibrant pro-space, and let  $M = \text{Eq}(F, F)$ . Then the principalization functor  $P$  from the category of  $(FE, F)$ -fibrations to the category of  $(ME, M)$ -fibrations which are right principal  $M$ -quasifibrations is adjoint to the functor  $\times_M F$ . Furthermore, both functors preserve equivalences.*

6.8. *Proposition. Let  $\pi: E \rightarrow B$  be a quasifibration with fibre  $F$ , and let  $M = \text{Eq}(F, F)$ . Then for each filtration level  $p$ ,*

- (a)  $P(\Gamma F_p B(E, M, F))$  is naturally equivalent to  $\Gamma F_p B(E, M, G)$ , and
- (b)  $F_p B(E, M, F) \times_M F$  is naturally equivalent to  $\Gamma F_p B(E, M, M)$ .

6.9. *Proposition.* Let  $M$  be a group-like monoid-object in pro-Top. Let  $\pi: E \rightarrow B$  be a fibration and a principal right  $M$ -quasifibration. Then for each filtration level  $p$ , there is a natural commutative diagram

$$\begin{array}{ccc}
 E & \overset{\longleftarrow}{\dashrightarrow} & F_p B(E, M, M) \\
 \pi \downarrow & & \pi' \downarrow \\
 B & \overset{\longleftarrow}{\dashrightarrow} & F_p B(E, M, *)
 \end{array}$$

in which  $\pi'$  is induced from  $\pi$ , all solid-arrow maps are quasifibrations, and the horizontal maps admit natural sections and are  $(p-1)$ -connected. Analogous results holds for principal left  $M$ -quasifibrations, as well as for non-principal fibrations with fibre  $F, M = \text{Eq}(F, F)$  and with  $F_p B(E, M, M)$  replaced by  $F_p B(E, M, F)$ .

**7. Classification theorem**

One may now use the techniques of May [Ma] and Hastings and Waner [HW] to prove that the above weak colimit classifies pro-fibrations with fiber  $F$  over a pro-space  $B$  provided that both  $F$  and  $B$  are pointed, connected, and finite-dimensional, and have the levelwise homotopy type of CW-complexes. More precisely, we have the following.

7.1. *Theorem.* Let  $F$  be a finite-dimensional, pointed, connected pro-space of the levelwise homotopy type of a CW complex. Then there is a sequence of classifying pro-spaces

$$\cdots \subset F_p B(*, \text{Eq}(F), *) \subset F_{p+1} B(*, \text{Eq}(F), *) \subset \cdots$$

such that the inclusions  $F_p B(*, \text{Eq}(F), *) \subset F_{p+1} B(*, \text{Eq}(F), *)$  are  $\text{pro-}\pi_1$ -isomorphisms for all  $i \leq p$ , and for any finite-dimensional, pointed, connected pro-space  $B$  of the levelwise homotopy type of a CW complex, and any  $p \geq \dim B + 2$ , equivalence classes of fibrations with fibre  $F$  over  $B$  are in bijective correspondence with

$$(7.2). \quad \text{Ho}(\text{pro-Top})(B, F_p B(*, \text{Eq}(F), *)).$$

As usual, the map from the set of homotopy classes of maps (7.2) to the set of fibrations with fibre  $F$  is induced by pulling back the universal fibration with fibre  $F$ ,

$$F_p B(*, \text{Eq}(F), F) \rightarrow F_p B(*, \text{Eq}(F), *).$$

*Proof.* Our proof follows the proof of [HW, Theorem 9.1] based on [Ma], with some required changes. We shall use the following notation to conform with usage in [Ma] and [HW], and to simplify some of the diagrams:

$FE$  shall denote the category of fibres equivalent to  $F$ , above;

$EFE(B)$  shall denote the collection of all  $FE$ -equivalence classes of  $FE$ -fibrations over  $B$ ,

$G$  shall denote  $M(F)$ , the monoid of self-equivalences of  $F$ ;



Since the map  $\Gamma B \leftarrow \Gamma B(P(E), G, *)$  is  $(P-1)$ -connected, the section  $\Gamma s$  is well-defined up to homotopy. This yields the required map  $\phi(\pi): B \rightarrow \Gamma B G$  as the composite

$$B \rightarrow \Gamma B \xrightarrow{\Gamma s} \Gamma B(P(E), G, *) \xrightarrow{\Gamma q} \Gamma B(*, G, *).$$

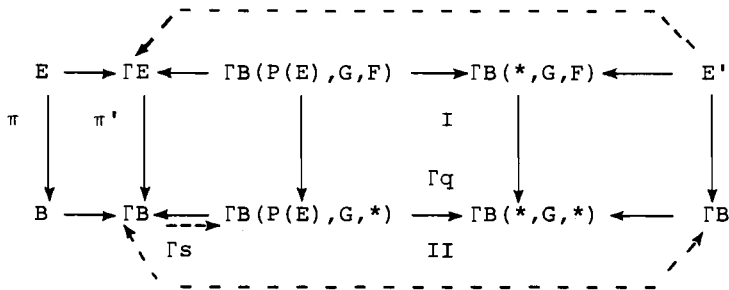
It is easy to see that  $\phi$  is well-defined, and compatible with increasing the filtration level  $p$ .

(b) Definition of a map  $\psi: [B, \Gamma B G] \rightarrow EFE(B)$  (a candidate for  $\phi^{-1}$ )

The map  $\psi$  is defined by pulling back the usual fibration  $\Gamma B(*, G, F) \rightarrow \Gamma B(*, G, *)$ . It is easy to see that  $\psi$  is well-defined, and compatible with increasing the filtration level  $p$ .

(c) Proof that  $\psi\phi = 1$ .

Let  $\pi$  be a  $FE$ -fibration, and form the following commutative diagram of  $FE$ -fibrations by combining diagram 7.3 above with the pullback diagram defining  $\psi$  applied to  $\phi(\pi')$ .



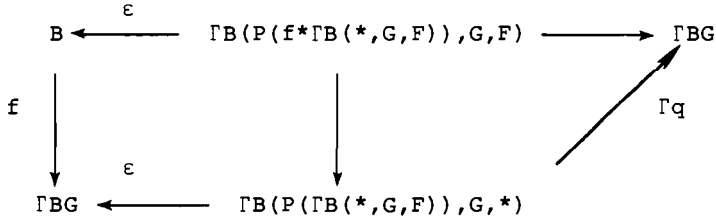
It is easy to see that square I is a pullback square up to homotopy and that square II homotopy commutes with the filler homotopic to the identity. Further, the composite map

$$E' \rightarrow \Gamma B \rightarrow \Gamma \dot{B} \rightarrow \Gamma B(P(E), G, *) \rightarrow \Gamma B(*, G, *)$$

is homotopic to  $\phi(\pi') = \Gamma q \Gamma s$ . Now using the pullback property of square I, we obtain a map  $E' \rightarrow \Gamma B(P(E), G, F)$  such that the composite map  $E' \rightarrow \Gamma B(P(E), G, F) \rightarrow \Gamma E$  is a map of fibrations covering the identity. The claim now follows by the Dold theorem.

(d) Proof that  $\phi\psi = 1$ .

We shall prove that  $\phi\psi$  is an automorphism on  $[B, \Gamma BG]$ . Together with part (c) above, this easily implies that  $\phi\psi$  is the identity. Given a map  $f: B \rightarrow \Gamma BG$ , form the pullback of the usual fibration  $\Gamma B(*, G, F) \rightarrow \Gamma B(*, G, *)$ , and then form the following commutative diagram.



By Proposition 6.9, both maps  $\varepsilon$  are  $(p-1)$ -connected, and hence the sections are well-defined up to homotopy. The composite map  $B \rightarrow \Gamma B(P(f*\Gamma B(*, G, F)), G, F) \rightarrow \Gamma BG$  is  $\phi\psi(f)$  by construction, and is homotopic to  $f$  followed by a sequence of natural maps (natural in  $B$  as well as  $F$  and  $G$ ) which form an automorphism of  $\Gamma BG$ . The claim, and theorem now follow.

7.4. *Remarks.* The repeated use of connectivity, together with the non-existence of homotopy colimits of  $\{B_p\}$ , shows the need for finite-dimensionality here as well as in [HW].

## 8. Shape fibrations

We now extend the above classification to shape fibrations. The first step is to replace shape fibrations by pro-fibrations up to  $Ho(\text{pro-Top})$  - equivalence. This requires the assumption that the fibers  $F$  are finite-dimensional. This replacement [HW] uses the Calder-Hastings [CH] construction of the strong shape category.

Recall from [CH] the following construction, where PL is the category of polyhedra and piece-wise linear (PL) maps.

8.1. *Definition.* Let  $X$  be a compact metric space. Then the strong shape of  $X$  is given by

$$\text{ssh}(X) = (X \downarrow \text{PL}) \rightarrow \text{PL};$$

the category of polyhedra and PL maps under  $X$ .

This construction extends to a functor

$$(8.2) \quad \text{ssh}: \text{CM} \rightarrow \text{pro-PL} \subset \text{pro-Top},$$

where CM denotes the category of compact metric spaces. Furthermore, ssh is coadjoint to the restriction of  $\lim$  to pro-PL. The strong shape category is the quotient category of CM, obtained by inverting maps which become equivalences in pro-homotopy. One may further show that ssh preserves cofibration sequences. In addition, one



can define the strong shape of a shape fibration, and functorially convert shape fibrations into pro-fibrations. We recall the following from [CH].

8.3. *Strong shape of shape fibrations.* Let  $F \rightarrow E \rightarrow B$  be a shape fibration, and consider diagrams of polyhedra and PL maps under  $F \rightarrow E \rightarrow B$ :

$$\begin{array}{ccccc}
 F & \longrightarrow & E & \longrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow \\
 K & \longrightarrow & L & \longrightarrow & M,
 \end{array}$$

in which  $K$  is a sub-polyhedron of  $M$ , and the composite map  $K \rightarrow M$  is null-homotopic. These diagrams and evident morphisms form a category, which we call the strong shape of the shape fibration  $F \rightarrow E \rightarrow B$ . Now replace the map  $\{L_\alpha \rightarrow M_\alpha\}$  by a fibration in pro-Top (here not restricted to towers) to obtain a diagram

$$\begin{array}{ccccc}
 F & \longrightarrow & E & \longrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow \\
 \{K_\alpha\} & \longrightarrow & \{L_\alpha\} & \longrightarrow & \{M_\alpha\} \\
 \downarrow & & \downarrow & & \downarrow \\
 \{K'_\alpha\} & \longrightarrow & \{L'_\alpha\} & \longrightarrow & \{M_\alpha\}.
 \end{array}$$

8.4. *Proposition [CH].* The map  $\{L'_\alpha\} \rightarrow \{M_\alpha\}$  is a fibration in pro-Top, with fibre  $\{K'_\alpha\}$ , and furthermore, the pro-spaces  $\{K'_\alpha\}$ ,  $\{L'_\alpha\}$ , and  $\{M_\alpha\}$  represent the strong shape of  $F$ ,  $E$ , and  $B$ , respectively.

Now, as in [HW], one may obtain a partial classification of shape fibrations with fibre  $F$ .

8.5. *Theorem.* Let  $F$  be a finite-dimensional, pointed, connected pro-space of the levelwise homotopy type of a CW complex. Then there is a sequence of classifying pro-spaces

$\dots \subset F_p B(*, \text{Eq}(F), *) \subset F_{p+1} B(*, \text{Eq}(F), *) \subset \dots$   
such that the inclusions are  $\text{pro-}\pi_i$  - isomorphisms for all  $i \leq p$ , and for any finite-dimensional, pointed, connected pro-space  $X$  of the levelwise homotopy type of a CW complex, and any  $p \geq \dim X$ , equivalence classes of  $F$ -fibrations over  $X$  map injectively to

$$(8.6) \quad \text{Ho}(\text{pro-Top})(\text{ssh}(X), F_p B(*, \text{Eq}(F), *)).$$

If, in addition,  $\text{ssh}(F)$  is a fibrant tower of compact spaces, for example, if  $F$  is a solenoid, infinite product of circles, or compact topological group, the above injection is a bijection.

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Hofstra University

Hempstead, New York 11550