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## ON THE MAPPING CLASS GROUP OF THE CLOSED ORIENTABLE SURFACE OF GENUS TWO

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## ON THE MAPPING CLASS GROUP OF THE CLOSED ORIENTABLE SURFACE OF GENUS TWO

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The surface mapping class groups are involved in different domains of mathematics, especially in 3-manifold topology. In order to study the mapping class groups, choosing a convenient family of generators appears to be very helpful. The first finite set of generators was given by Lickorish [8] as a family of Dehn twists along some simple closed curves in the surface. Later, Humphries [7] found the smallest generating set by Dehn twists, which consists of  $2g + 1$  elements for the closed surface of genus  $g$ .

The Lickorish's Dehn twist generators have been widely studied, since they are so nice in topology. But they seem hard to use directly in algebraic discussions, which are important in various studies in topology. So here, we will give first a family of generators whose algebraic description is straightforward, and whose topological interpretation is still very clear and strongly related to Lickorish generators. Moreover the number of generators is minimal.

In this paper, we will discuss only the case when the genus is two, i.e., the first nontrivial case. In the first section, we define some elementary homeotopy classes, show their topological and algebraic properties, relate

them to Dehn twists, and prove that the mapping class group  $M_2$  of the closed orientable surface  $F_2$  of genus two is generated by two elements from those elementary classes.

In the second section we give another proof that those two classes generate the group  $M_2$ , by giving an algorithm to write an arbitrary mapping class in those specific classes in a unique (not canonical) way, which certainly also solves the word problem for the group  $M_2$ .

In the third section we give a simple presentation of the mapping class group  $M_2$  of two generators and six relators, by using the presentation given by Birman [1]. More precisely, we have

*Theorem 3.2. The mapping class group  $M_2$  of the closed orientable surface of genus two admits a presentation with two generators  $L$  and  $N$ , and six relations:*

$$\begin{aligned} N^6 &= 1, \\ (LN)^5 &= 1, \\ (L\bar{N})^{10} &= 1, \\ L &\leftrightarrow (L\bar{N})^5, \\ L &\leftrightarrow N^3LN^3, \\ L &\leftrightarrow N^2LN^4. \end{aligned}$$

Moreover,  $L = D_0$  and  $N = D_0D_1D_2D_3D_4$ , where  $D_0, D_1, \dots, D_5$  are Dehn twists along the simple closed curves  $\gamma_0, \gamma_1, \dots, \gamma_5$  pictured in Figure 3.1, and any five of them form a Humphries' system of Lickorish generators.

Notationally, we do not distinguish between a homeomorphism and its homeotopy class in the paper.

# 1. The Elementary Mapping Classes on the Surface $F_2$

Let  $F_2 = S^1 \times S^1 \# S^1 \times S^1$  be the closed orientable surface of genus two, which is embedded standardly in  $S^3$  and bounds a handlebody  $H_2$ . Let  $0$  be a chosen point, called the *basepoint* of  $F_2$ , and let  $B = \{a_1, b_1, a_2, b_2\}$  be a family of simple loops based at  $0$ , called the *basecurves* of  $F_2$ , such that  $a_1$  and  $a_2$  are meridian, and  $b_1$  and  $b_2$  are longitudes of the handlebody  $H_2$ , as shown in Figure 1.1. Clearly the basecurves  $B$  generate the fundamental group  $\pi_1(F_2, 0)$  of the surface  $F_2$ . It is well-known that the isotopy class of a self-homeomorphism of a closed surface is uniquely determined by the homotopy classes of the image of the basecurves in the fundamental group of the surface relative to the fixed basepoint. Therefore, it is convenient to denote a homeotopy class  $f$  by

$$f = (B)f = [[(a_1)f], [(b_1)f], [(a_2)f], [(b_2)f]].$$

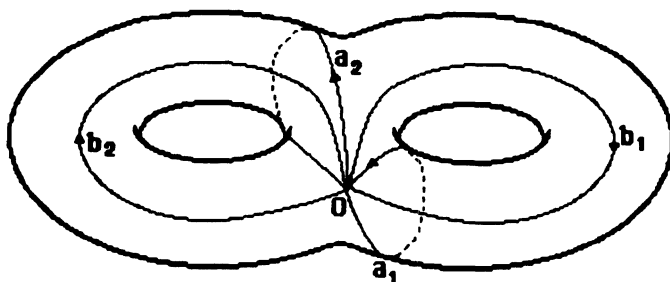


Figure 1.1 Basecurves  $B$  on  $F_2$

Now we will define some *elementary operations* in this way.

0) The *identity*  $I: F_2 \rightarrow F_2$  is given by

$$I = [a_1, b_1, a_2, b_2].$$

1) An orientation-reversing mapping, called *reversion*

$R: F_2 \rightarrow F_2$ , is given by

$$R = [b_1, a_1, b_2, a_2].$$

2) The interchanging handles mapping, called *transport*

$T: F_2 \rightarrow F_2$ , is given by

$$T = [a_2, b_2, a_1, b_1].$$

*Proposition 1.1*

$$(a) \quad R^2 = I,$$

$$(b) \quad T^2 = I,$$

$$(c) \quad TR = RT.$$

It can be easily proved by a direct verification.

3) Homeotopy classes called *linear cuttings*, are obtained in the following way: on the presentation polygon of the surface  $F_2$  relative to the basis  $B$ , we cut some triangle formed by two successive edges  $x$  and  $y$  and glue it back along one of the edges, e.g. along  $x$ , and get a new polygon. If we have a homeomorphism from the old polygon to the new one that maps the basepoint and the basecurves other than  $x$  and  $y$  invariantly, and defines a self-homeomorphism of  $F_2$ , then, it is unique up to isotopy, and we denote it by  $L(x, y; x)$  (Figure 1.2).

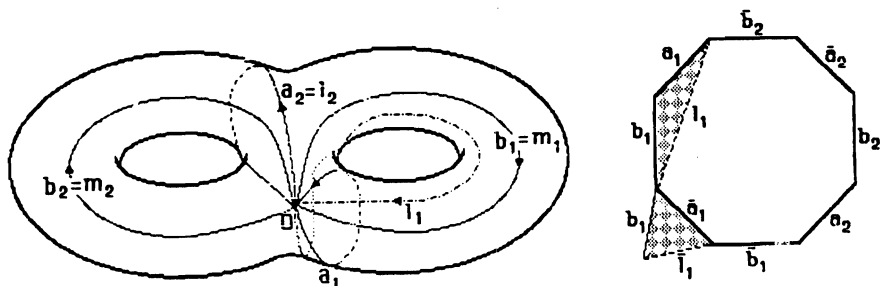


Figure 1.2 Linear cutting L

More explicitly we have

$$L(a_i, b_i; a_i), L(a_i, b_i; b_i), L(b_i, \bar{a}_i; b_i), \\ L(b_i, \bar{a}_i; \bar{a}_i), L(\bar{a}_i, \bar{b}_i; \bar{a}_i), L(\bar{a}_i, \bar{b}_i; \bar{b}_i),$$

where  $i = 1, 2$ .

Among these linear cuttings, we will denote

$$L = L(a_1, b_1; a_1) \text{ and } M = L(b_1, \bar{a}_1; b_1),$$

and call them *longitude cutting* and *meridian cutting* respectively, since they are the Dehn twists along the longitude and meridian circles of the first handle of  $H_2$ .

Their expressions in  $\pi_1(F_2, 0)$  are given by

$$L = [a_1 b_1, b_1, a_2, b_2], \text{ and } M = [a_1, b_1 \bar{a}_1, a_2, b_2].$$

*Proposition 1.2*

$$(a) \quad L(x_2, y_2; y_2) = T \cdot L(x_1, y_1; y_1) \cdot T,$$

for any  $(x, y) \in \{(a, b), (b, a), (b, \bar{a}), (\bar{a}, b), (\bar{a}, \bar{b}), (\bar{b}, \bar{a})\}$ .

$$(b) \quad L(b_1, \bar{a}_1; \bar{a}_1) = RMR = \bar{L},$$

$$L(\bar{a}_1, \bar{b}_1; \bar{b}_1) = RLR = \bar{M},$$

$$L(\bar{a}_1, \bar{b}_1; \bar{a}_1) = R \cdot L(a_1, b_1; b_1) \cdot R \\ = (L(a_1, b_1; b_1))^{-1},$$

$$(c) \quad L(a_1, b_1; b_1) = \bar{R}LRL.$$

These relations may be verified directly. As an immediate consequence, we have

*Corollary 1.3 Every linear cutting is a composition of the homeotopy classes L, T and R.*

4) A homeotopy class called *normal cutting*,

$$N = N(a, \bar{b}; \bar{b}) = N(a_1, \bar{b}_2; \bar{b}_2/a_2, \bar{b}_1; \bar{b}_1),$$

is defined algebraically by

$$N = [\bar{a}_2 b_1, \bar{a}_1, \bar{a}_1 b_2, \bar{a}_2],$$

and topologically by cutting two triangles on the presentation polygon, one between the edges  $a_1$  and  $\bar{b}_2$  and the other between  $a_2$  and  $\bar{b}_1$ , and sewing them along curves  $b_1$  and  $b_2$  respectively, (Figure 1.3).

By a similar discussion as for linear cuttings, we may have another normal cutting  $N(a, \bar{b}; a)$ . But this is nothing new and is just the inverse of  $N$  by the next proposition.

*Proposition 1.4*

$$(a) \quad N^3 = T, \quad N^6 = I,$$

$$(b) \quad TN = NT,$$

$$(c) \quad (RN)^2 = I,$$

$$(d) \quad N(a, \bar{b}; a) = RNR = \bar{N},$$

$$(e) \quad M = \bar{R}L\bar{R} = \bar{N}LN.$$

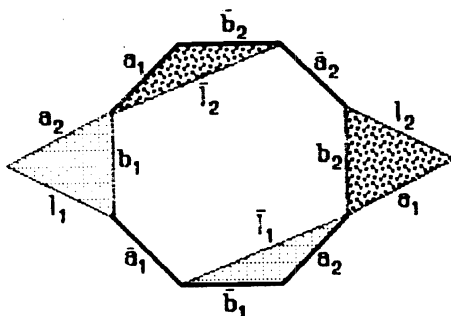


Figure 1.3 Normal cutting N

The proof is again obvious. For example, writing

$$N = [\bar{a}_2 b_1, \bar{a}_1, \bar{a}_1 b_2 \bar{a}_2],$$

we have

$$\begin{aligned} N^2 &= [(\bar{b}_2 a_1) \bar{a}_1, \bar{b}_1 a_2, (\bar{b}_1 a_2) \bar{a}_2, \bar{b}_2 a_1] \\ &= [\bar{b}_2, \bar{b}_1 a_2, \bar{b}_1, \bar{b}_2 a_1], \end{aligned}$$

and

$$\begin{aligned} N^3 &= N^2 \cdot N = [a_2, a_1 (\bar{a}_1 b_2), a_1, a_2 (\bar{a}_2 b_1)] \\ &= [a_2, b_2, a_1, b_1] = T. \end{aligned}$$

Therefore,  $N^3 = T$ , and easily  $N^6 = T^2 = I$ .

*Remark.* From the formulas 1.4.(d) and (e), we have that the reversion  $R$  commutes with the subgroup generated by the classes  $L$  and  $N$  in the homeotopy class group  $\tilde{M}_2$ .

5) The last type of homeotopy classes are called *parallel cuttings*, denoted by  $P(x)$ ,  $x \in \{a, b, \bar{a}, \bar{b}\}$ , defined algebraically by



$$P(a) = [\bar{b}_1, b_1 a_1 \bar{b}_1, \bar{b}_2, b_2 a_2 \bar{b}_2],$$

$$P(\bar{a}) = [\bar{b}_1, \bar{b}_2 a_1 b_1, \bar{b}_2, \bar{b}_1 a_2 b_2],$$

$$P(b) = [a_1 b_1 \bar{a}_2, \bar{a}_1, a_2 b_2 \bar{a}_1, \bar{a}_2],$$

$$P(\bar{b}) = [\bar{a}_1 b_1 a_1, \bar{a}_1, \bar{a}_2 b_2 a_2, \bar{a}_2],$$

and obtained by cutting the presentation polygon in three quadrilaterals (Figure 1.4), such that the center one contains the edges  $x_1$  and  $x_2$ , and gluing them along the curves given by  $x_1$  and  $x_2$ .

Actually, for the convenience of our future discussion, we will call *parallel cutting* the homeotopy class

$$P = LML = MLM = [a_1 b_1 \bar{a}_1, \bar{a}_1, a_2, b_2].$$

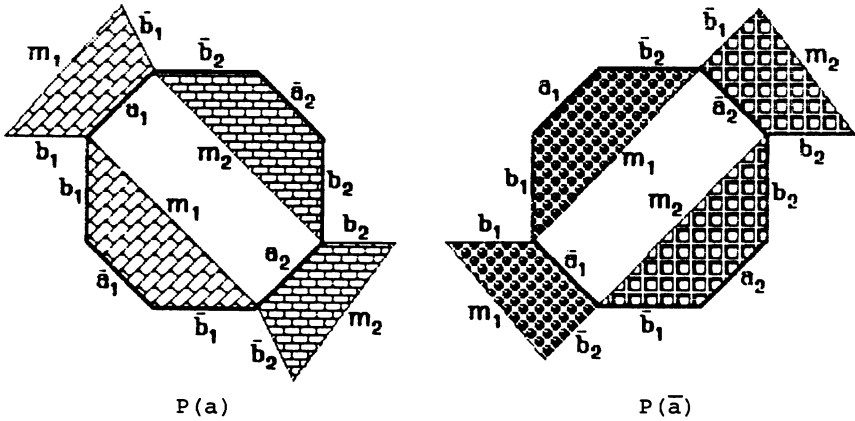


Figure 1.4 Parallel cuttings

*Proposition 1.5*

$$(a) \quad P(a) = \bar{P}T\bar{P}T,$$

$$(b) \quad P(\bar{a}) = \bar{N}L\bar{T}L\bar{T},$$

$$(c) \quad P(b) = R \cdot P(\bar{a}) \cdot R = \bar{N}\bar{M}\bar{T}\bar{M}\bar{T},$$

$$(d) \quad P(\bar{b}) = R \cdot P(a) \cdot R = PTPT,$$

$$(e) \quad RPR = \bar{P}, \text{ since } RLR = \bar{M} \text{ and } RMR = \bar{L}.$$

*Proof.* They can be proved by a direct algebraic verification. However, there is topological interpretation hidden inside. Here we show (a) as an example to illustrate the topological aspect.

As shown in Figure 1.5, first we do the operation

$$O_1 = L(\bar{a}_1, \bar{b}_1; \bar{a}_1) L(\bar{a}_2, \bar{b}_2; \bar{a}_2) = \bar{L}RLR \cdot T\bar{L}RLRT,$$

and then do the second one

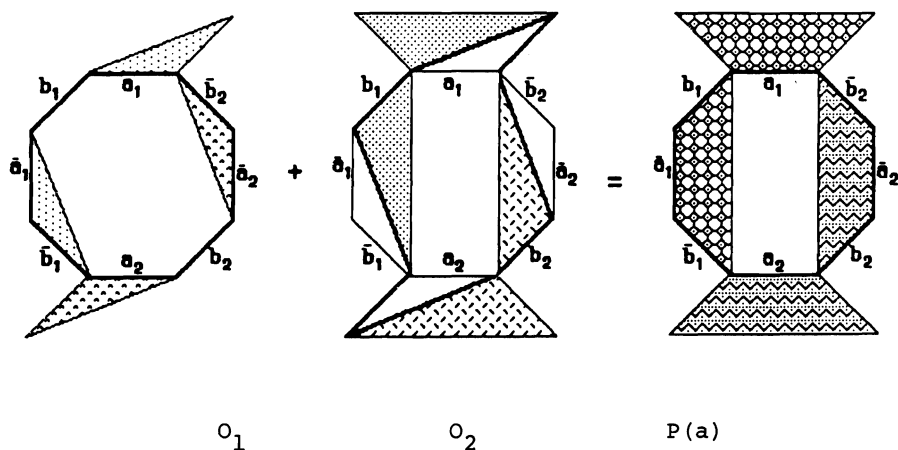
$$O_2 = L(\bar{a}_1, \bar{b}_1; \bar{b}_1) L(\bar{a}_2, \bar{b}_2; \bar{b}_2) = RLR \cdot TRLRT.$$

Then, as shown in the figure, we have

$$\begin{aligned} P(a) &= O_2 \cdot O_1 = L(\bar{a}_1, \bar{b}_1; \bar{a}_1) \cdot L(\bar{a}_2, \bar{b}_2; \bar{a}_2) \cdot L(\bar{a}_1, \bar{b}_1; \bar{b}_1) \cdot \\ &\quad L(\bar{a}_2, \bar{b}_2; \bar{b}_2) \\ &= L(\bar{a}_1, \bar{b}_1; \bar{a}_1) \cdot L(\bar{a}_1, \bar{b}_1; \bar{b}_1) \cdot L(\bar{a}_2, \bar{b}_2; \bar{a}_2) \cdot \\ &\quad L(\bar{a}_2, \bar{b}_2; \bar{b}_2) \\ &= RLR \cdot \bar{L}RLR \cdot TRLRT \cdot T\bar{L}RLRT = (RLR\bar{L}RLRT)^2 \\ &= (RLMLRT)^2 = RPRT RPRT = \bar{P}T\bar{P}T, \end{aligned}$$

since  $RPR = \bar{P}$  and  $RTR = T$  by the formulas (1.1.c), (1.5.e).

All above homeotopy classes are called *elementary operations*, and among them clearly only the reversion  $R$  is orientation-reversing, and all others are orientation-preserving and generated by only the operations  $L$  and  $N$ .

Figure 1.5  $P(a) = O_1 \cdot O_2$ 

In the next part in this section, we relate these elementary operations to Lickorish generators of Dehn twists.

Considering the surface given in Figure 1.6, the Lickorish generators are exactly the five Dehn twists along the simple curves  $a_1$ ,  $a_2$ ,  $b_1$ ,  $b_2$ , and  $c$ , we denote them by  $A_1$ ,  $A_2$ ,  $B_1$ ,  $B_2$ , and  $C$  respectively. Remembering our elementary operations, and we have

$$A_1 = M,$$

$$A_2 = \text{TMT},$$

$$B_1 = L,$$

$$B_2 = \text{TLT}$$

For  $C$ , write the curve  $c$  in the basecurves of  $\mathcal{B}$ , thus

$$c = a_2 \bar{b}_2 \bar{a}_2 b_1$$



*Proof.* Indeed, since  $(LN)^5 = I$  and  $N^6 = I$ , we have

$$\begin{aligned} N &= \overline{N}(LN)^5 N^2 \\ &= \overline{N}LN \cdot L \cdot NL\overline{N} \cdot N^2 \overline{LN}^2 \cdot N^3 \overline{LN}^3 \\ &= B_1 A_1 (A_1 B_1 \overline{C B_1} \overline{A_1}) B_2 A_2. \end{aligned}$$

*Remark.* Later in Section 3 we will give a different correspondence between the Lickorish generators and ours, which is simpler and nicer, and which is conjugate to that given in Proposition 1.7.

## 2. The Algorithm for Writing Homeotopy Classes in the Generators

In this section we give an algorithm to write an arbitrary homeotopy class in the generators  $L$ ,  $N$  and  $R$ , which gives a direct proof of Theorem 1.6.

Given a homeotopy class  $f$ , since exactly one of  $f$  and  $Rf$  is orientation-preserving, we may suppose that  $f$  is a mapping class. In fact, the mapping class group  $M_2$  is a normal subgroup of index 2 of the homeotopy group  $\tilde{M}_2$ . Moreover,

$$RL = \overline{MR}, \quad RM = \overline{LR} \text{ and } RN = \overline{NR}.$$

As in the last section, let

$$f = [l_1, m_1, l_2, m_2]$$

be written in the basis

$$B = \{a_1, b_1, a_2, b_2\}.$$

i.e.  $l_1 = (a_1)f$ ,  $m_1 = (b_1)f$ ,  $l_2 = (a_2)f$  and  $m_2 = (b_2)f$

And suppose all curves intersect transversally. We will denote by  $\#(\gamma_1 \cap \gamma_2)$  the geometric number of the

intersection points other than the basepoint 0 of the curves  $\gamma_1$  and  $\gamma_2$ , and

$$\begin{aligned} \#(\gamma_1 \cap B) &= \#(\gamma_1 \cap a_1) + \#(\gamma_1 \cap b_1) + \#(\gamma_1 \cap a_2) \\ &\quad + \#(\gamma_1 \cap b_2). \end{aligned}$$

Given  $f$  an orientation-preserving self-homeomorphism of the surface  $F_2$ , we will denote the lexicographically ordered multi-index

$$\#(f) = (\#(l_1 \cap B), \#(m_1 \cap B), \#(l_2 \cap B), \#(m_2 \cap B)).$$

Our algorithm is based on the reduction of this multi-index.

Now we start our algorithm.

*Step 1. Given  $f$ , if  $m = \#(l_1 \cap B) \neq 0$ , there is a self-homeomorphism  $h$  which is a composition of elementary operations, such that  $\#((l_1)h^{-1} \cap B) < m$ .*

Suppose that  $l_1 \cap B = \{0, P_1, \dots, P_m\}$ , and denote by  $s_i$ ,  $0 \leq i \leq m$ , the arc between the points  $P_i$  and  $P_{i+1}$  of the curve  $l_1$ , where  $P_0 = P_{m+1} = 0$  by convention. Regarding  $s_i$  as an arc in the presentation polygon with ends in the boundary of the polygon, we will say  $s_i$  of the type  $[x: y]$ , and write  $s_i \in [x: y]$ , if one of its two end points is in the edge  $x$  and the other is in  $y$ , where  $x, y \in \{a_1, b_1, \bar{a}_1, \bar{b}_1, a_2, b_2, \bar{a}_2, \bar{b}_2\}$ . (Figure 2.1). We will denote by  $\#(l_1 \cap [x: y])$  the number of arcs of the type  $[x: y]$ .

Now we consider cases.

*Case 0. There is an arc  $s_i$  of the type  $[x: x]$ , for some edge  $x$ .*

An isotopy of  $f$  decreases the number  $m$ . (Figure 2.2).

Case I.  $s_0$  or  $s_m$  is of one of the types  $[a_k : b_k]$ ,  $[b_k : \bar{a}_k]$  and  $[\bar{a}_k : \bar{b}_k]$ ,  $k = 1$  or  $2$ .

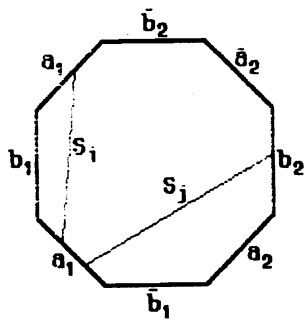


Figure 2.1  $s_i \in [a_1 : \bar{a}_1]$  and  $s_j \in [\bar{a}_1 : b_2]$

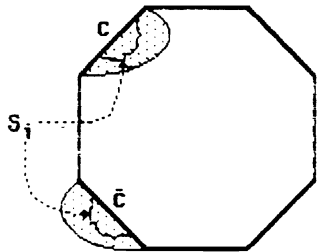


Figure 2.2 Case 0

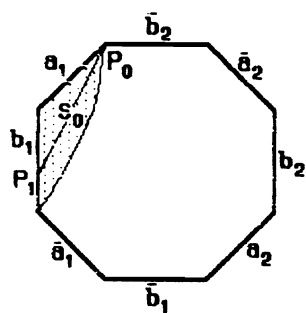


Figure 2.3 Case I

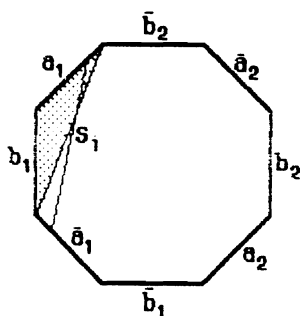


Figure 2.4 Case II

The homeomorphism  $h$  will be chosen to be a suitable linear cutting. For example, suppose  $s_0 \in [a_1: b_1]$  and  $\#(l_1 \cap b_1) > 0$ . Apply  $h = L(a_1, b_1; b_1)$ , (Figure 2.3), and obviously after the cutting and sewing it is easy to see  $\#((l_1)h^{-1} \cap \mathcal{B}) = m - \#(l_1 \cap [a_1: b_1]) < m$ .

*Case II.* There is some  $s_i$  of the type  $[a_k: \bar{a}_k]$  or  $[b_k: \bar{b}_k]$ ,  $k = 1$  or  $2$ .

For example,  $s_i \in [a_1: \bar{a}_1]$ , (Figure 2.4). Evidently,  
 $\#(l_1 \cap b_1) = \#(l_1 \cap [a_1: b_1]) + \#(l_1 \cap [\bar{a}_1: b_1])$ .  
 We have always  $\#(l_1 \cap [a_1: b_1]) \neq \#(l_1 \cap [\bar{a}_1: b_1])$ , since the endpoints of any arc of  $l_1$  are disjoint. If  $\#(l_1 \cap [a_1: b_1]) > \#(l_1 \cap [\bar{a}_1: b_1])$ , let  $h$  be the linear cutting  $L(a_1, b_1; a_1)$ .

Clearly

$$\begin{aligned} \#((l_1)h^{-1} \cap \mathcal{B}) &= m - \#(l_1 \cap a_1) + \#(l_1 \cap h(a_1)) \\ &= m - \#(l_1 \cap a_1) + [\#(l_1 \cap a_1) \\ &\quad + \#(l_1 \cap b_1) - 2\#(l_1 \cap [a_1: b_1])] \end{aligned}$$



$$= m + \#(l_1 \cap [\bar{a}_1: b_1]) - \#(l_1 \cap [a_1: b_1]) < m.$$

The other situations are exactly similar.

Case III. Cases 0, I or II do not occur, and there is some  $s_i \in [a_k: \bar{b}_k]$ ,  $k = 1$  or  $2$ .

Thus,

$$\#(l_1 \cap b_1) = \#(l_1 \cap [a_1: b_1]) + \#(l_1 \cap [b_1: \bar{a}_1]),$$

and

$$\#(l_1 \cap a_1) = \#(l_1 \cap [b_1: \bar{a}_1]) + \#(l_1 \cap [\bar{a}_1: \bar{b}_1]).$$

III-i) When  $\#(l_1 \cap [a_1: b_1]) > \#(l_1 \cap [b_1: \bar{a}_1])$ .

We choose  $h = L(a_1, b_1; b_1)$  and the number  $m$  is reduced. And analogously, if  $\#(l_1 \cap [a_1: b_1]) < \#(l_1 \cap [b_1: \bar{a}_1])$  or  $\#(l_1 \cap [b_1: \bar{a}_1]) \neq \#(l_1 \cap [\bar{a}_1: \bar{b}_1])$ , we also may choose suitable linear cuttings (Figure 2.5).

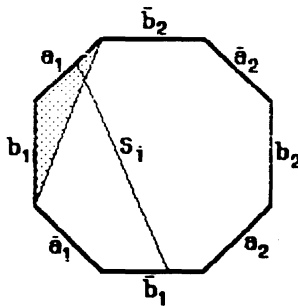


Figure 2.5 Case III-(i)

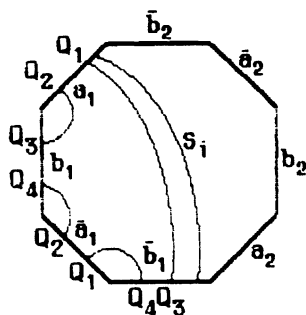


Figure 2.6 Case III-(ii)

III-ii) When  $\#(l_1 \cap [a_1: b_1]) = \#(l_1 \cap [b_1: \bar{a}_1]) = \#(l_1 \cap [\bar{a}_1: \bar{b}_1]) = \lambda$ .

Then  $\#(l_1 \cap a_1) = \#(l_1 \cap b_1) = 2\lambda$ . Denote by  $Q_1, Q_2, Q_3$ , and  $Q_4$  the  $p$ -th and  $(p + 1)$ -th points of  $l_1$  on the basecurves  $a_1$  and  $b_1$  as shown in Figure 2.6. This produces a closed curve which is a proper subset of the simple curve  $l_1$ , showing the impossibility of this case, since it does not contain the basepoint.

Case IV. There is some  $s_i$  of the type  $[a_1: a_2]$ ,  $[b_1: b_2]$ ,  $[\bar{a}_1: \bar{a}_2]$  or  $[\bar{b}_1: \bar{b}_2]$ . We let  $h = P(a), P(b), P(\bar{a})$  or  $P(\bar{b})$ , respectively, and obviously

$$\#((l_1)h^{-1} \cap \mathcal{B}) = m - \#(l_1 \cap [x_1: x_2]) < m,$$

where  $x = a, b, \bar{a}$ , or  $\bar{b}$ . (Figure 2.7).

The remaining cases will be discussed in another way. Consider the starting point  $P_0$  of  $s_0$  in the presentation polygon (Figure 2.8), since the standard operations  $T$  and  $R$  leave the intersection numbers unchanged, it is sufficient to consider the cases  $P_0 = A, B$ , and  $C$ .

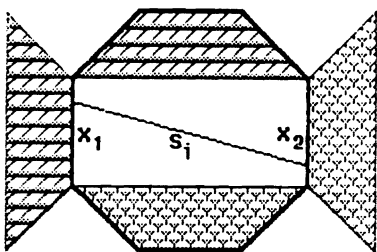


Figure 2.7 Case IV

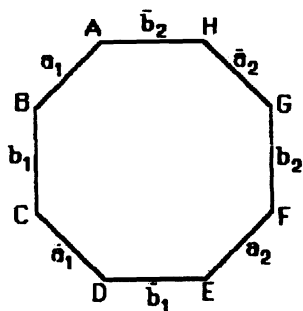


Figure 2.8 Presentation polygon

Case V.  $P_0 = A$ .

V-i) When  $P_1 \in a_1, \bar{b}_2, b_1$  or  $\bar{a}_2$ ,  $s_0$  is of the types in Case I.

V-ii) When  $P_1 \in \bar{a}_1, b_2, \bar{b}_1$ , or  $a_2$ ,  $s_0$  is of the types in Cases II or III.

Case VI.  $P_0 = B$ .

VI-i) When  $P_1 \in a_1, b_1, \bar{a}_1$ , or  $\bar{b}_1$ ,  $s_0$  is of the types from Cases 0-III.

VI-ii) When  $P_1 \in a_2$  or  $b_2$ ,  $s_0$  is of the types in Case IV.

VI-iii) When  $P_1 \in \bar{a}_2$ ,

(a) If  $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) = 0$ , there is no arc crossing the parallel band between  $b_1$  and  $b_2$ . Do  $P(b)$ , creating a situation as in Case I, without changing the intersection number. (Figure 2.9).

(b) If  $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) = 0$ , (Figure 2.10). Then, of course  $\#(l_1 \cap [a_2: b_1]) = 0$ , i.e. there is no arc crossing the band between  $\bar{a}_1$  and  $\bar{a}_2$ , similarly we have Case I after doing  $P(\bar{a})$ .

VI-iv) When  $P_1 \in \bar{b}_2$ ,

(a) If  $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) = 0$ . We do  $P(b)$  as in Case VI-(iii-a), and the situation becomes Case V, (Figure 2.11).

(b) If  $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) \neq 0$  and  $\#(l_1 \cap [\bar{a}_2: b_1]) = 0$ . Analogously, do  $P(\bar{b})$ , obtaining Case I, (Figure 2.12).

(c) If not (a), not (b), and  $\#(l_1 \cap [a_2: b_2]) > \#(l_1 \cap [a_2: \bar{b}_1])$ , let  $h = L(a_2, b_2; b_2)$ , (Figure 2.13), and obviously

$$\begin{aligned} \#((l_1)h^{-1} \cap B) &= m - \#(l_1 \cap b_2) + \#(l_1 \cap h(b_2)) \\ &= m - \#(l_1 \cap [a_2: b_2]) + \\ &\quad \#(l_1 \cap [a_2: \bar{b}_1]) < m. \end{aligned}$$

(d) If not (a), not (b), and  $\#(l_1 \cap [a_2: b_2]) \leq \#(l_1 \cap [a_2: \bar{b}_1])$ . Then we have always  $\#(l_1 \cap [a_1: b_1]) = 0 < 1 \leq \#(l_1 \cap [a_1: \bar{b}_2])$ , let  $h = N(a, \bar{b}; \bar{b})$  be a normal cutting, (Figure 2.14). Clearly

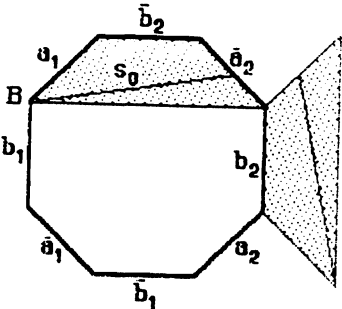


Figure 2.9 Case VI-(iii)-(a)

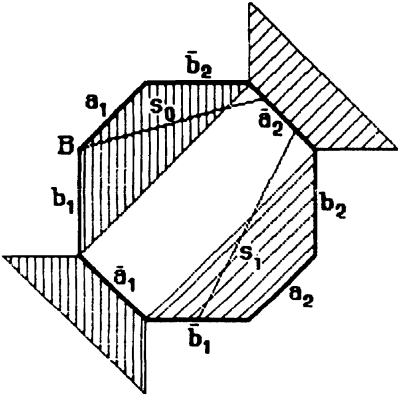


Figure 2.10 Case VI-(iii)-(b)

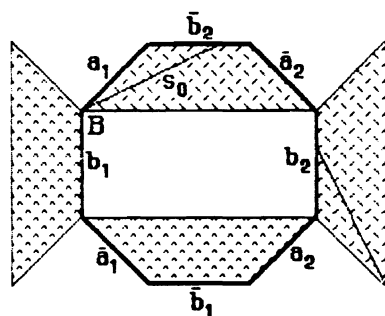


Figure 2.11 Case VI-(iv)-(a)

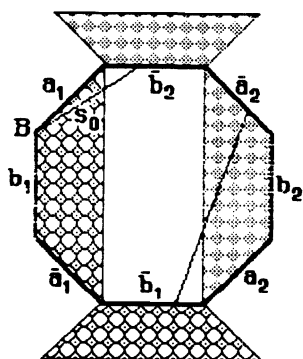


Figure 2.12 Case VI-(iv)-(b)

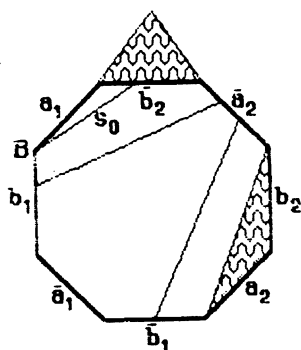


Figure 2.13 Case VI-(iv)-(c)

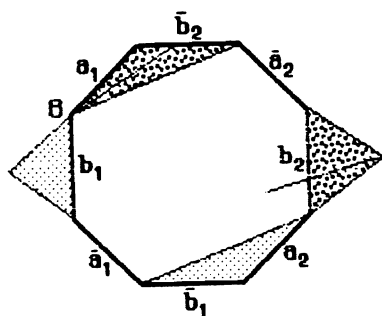


Figure 2.14 Case VI-(iv)-(d)

$$\begin{aligned}
 \#((l_1)^{h^{-1}} \cap \mathcal{B}) &= m - \#(l_1 \cap b_1) - \#(l_1 \cap b_2) + \\
 &\quad \#(l_1 \cap (b_1)h) + \#(l_1 \cap (b_2)h) \\
 &= m - \#(l_1 \cap [a_1: \bar{b}_2]) + \\
 &\quad \#(l_1 \cap [a_2: b_2]) - \\
 &\quad \#(l_1 \cap [a_2: \bar{b}_1]) < m.
 \end{aligned}$$

Case VII.  $P = C$ .

VII-i) When  $P_1 \in a_1, b_1, \bar{a}_1$  or  $\bar{b}_1$ ,  $s_0$  is in Cases O-II.

VII-ii) When  $P_1 \in \bar{a}_2$  or  $b_2$ ,  $s_0$  is in Case IV.

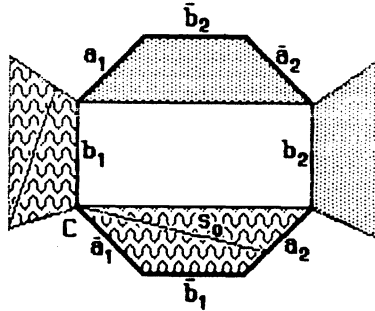


Figure 2.15 Case VII-(iii)

VII-iii) When  $P_1 \in a_2$ , obviously there is no arc crossing the parallel band between  $b_1$  and  $b_2$ . Do the parallel cutting  $P(b)$ , which does not change the intersection number  $m$ , and produces the situation as in Case VI-(iv), (Figure 2.15).

VII-iv) When  $P_1 \in \bar{b}_2$ , apply the reversion  $R$  to obtain Case VII-(iii).

*Step 2. Given  $f$  with  $m = \#(l_1 \cap B) = 0$ , then there exists a self-homeomorphism  $h$  which is a composition of elementary operations, such that  $(l_1)h^{-1} = a_1$ .*

By the first step, we may suppose  $\#(l_1 \cap B) = 0$ , where  $l_1 = (a_1)f$ . Consider the curve  $l_1$  in the presentation polygon of  $F_2$ . We will denote by  $\overline{XY}$  the arc with ends



at the vertices  $X$  and  $Y$  of the presentation polygon, for  $X, Y \in \{A, B, C, D, E, F, G, H\}$ .

- i) If  $l_1 = \widetilde{AB}$ , it is done already.
  - ii) If  $l_1 = \widetilde{AH}$ , let  $h = T \cdot P(a)$ .
  - iii) If  $l_1 = \widetilde{AC}$  or  $\widetilde{AG}$ , we may choose  $h$  to be a linear cutting.
  - iv) If  $l_1 = \widetilde{AD}$  or  $\widetilde{AG}$ , we may choose  $h$  to be a parallel cutting.
  - v) Clearly  $l_1 \neq \widetilde{AE}$ , since  $l_1$  is not null-homologous.
  - vi) If  $l_1 = \widetilde{EX}$ , where  $X = B, C, D, F, G$  and  $H$ , we do first a transport  $T$ , and the case becomes one of the first four cases.
  - vii) If  $l_1 = \widetilde{BC}$ , let  $h = P(a)$ .
  - viii) If  $l_1 = \widetilde{BD}$ , let  $h = L(b_1, \bar{a}_1; \bar{a}_1)$ .
  - ix) If  $l_1 = \widetilde{BF}$ , let  $h = P(b) \cdot L(b_1, \bar{a}_1; \bar{a}_1)$ .
  - x) If  $l_1 = \widetilde{BG}$ , let  $h = P(b)$ .
  - xi) If  $l_1 = \widetilde{BH}$ , let  $h = N$ .
  - xii) If  $l_1 = \widetilde{DX}$ , do first a reversion  $R$ , producing the case of  $l_1 = \widetilde{BY}$ . Denote by  $h'$  the map given by that case, let  $h = P(a) \cdot Rh'R$ .
  - xiii) If  $l_1 = \widetilde{FX}$  or  $\widetilde{HX}$ , apply the transport  $T$ , producing the cases of  $l_1 = \widetilde{BY}$  or  $\widetilde{DY}$ .
  - xiv) If  $l_1 = \widetilde{CG}$ , let  $h = T \cdot P(b) \cdot L(a_1, b_1; a_1)$ .
- This completes Step 2.

From now on we may suppose that  $l_1 = (a_1)f = a_1$ . We will simplify the curve  $m_1 = (b_1)f$  by using the elementary operations, and at the same time leave the curve  $l_1$  unchanged.

*Step 3. Given  $f$  with  $(a_1)f = a_1$ , then there is a self-homeomorphism  $h$  which is a composition of elementary operations, such that  $(a_1)h = a_1$  and  $\#((m_1)h^{-1} \cap B) = 0$ .*

Since  $l_1, m_1, l_2, m_2$  forms a family of basecurves, and  $l_1 = a_1$ , we have

$$\#(m_1 \cap a_1) = 0.$$

So, we have fewer cases. As in the last step, we denote now  $m = \#(m_1 \cap B)$ , and successively  $m_1 \cap B = \{0, P_1, P_2, \dots, P_m\}$ , and  $s_i$  the arc on  $m_1$  between  $P_i$  and  $P_{i+1}$ ,  $i = 0, \dots, m$ , where  $P_0 = P_{m+1} = 0$  by convention.

*Cases 0-IV. There is some  $s_i$  belonging to one of the following types:  $[x_k: x_k]$ ,  $[a_2: b_2]$ ,  $[b_2: \bar{a}_2]$ ,  $[\bar{a}_2: \bar{b}_2]$ ,  $[a_2: \bar{a}_2]$ ,  $[b_k: \bar{b}_k]$ ,  $[a_2: \bar{b}_2]$ ,  $[b_k: \bar{b}_k]$ , where  $x = a, b, \bar{a}$  or  $\bar{b}$  and  $k = 1, 2$ .*

We can do the same operation as in Step 1, which leaves  $l_1 = a_1$  unchanged.

The other cases will be studied by considering the arc  $s_0$ , as we did in Step 1.

*Case V.  $P_0 = A$  or  $E$ .*

Do the same as in Case V of Step 1.

Case VI.  $P_0 = F$ , and not Cases O-IV. We have only the following two possible situations.

- i)  $P_1 \in b_1$ . Then let  $h = P(b)$ .
- ii)  $P_1 \in \bar{b}_1$ s. Then let  $h = N(a, \bar{b}, \bar{b})$ .

Case VII.  $P_0 = G$ , and not Cases O-IV, and

- i)  $P_1 \in b_1$ . Let  $h = P(b)$ .
- ii)  $P_1 \in \bar{b}_1$ , and
  - (a)  $\#(m_1 \cap [a_2: b_2]) < \#(m_1 \cap [a_2: \bar{b}_1])$ . Let  $h = N(a, \bar{b}; \bar{b})$ .
  - (b)  $\#(m_1 \cap [a_2: b_2]) > \#(m_1 \cap [a_2: \bar{b}_1])$ . Let  $h = L(a_2, b_2; b_2)$ .
  - (c)  $\#(m_1 \cap [a_2: b_2]) = \#(m_1 \cap [a_2: \bar{b}_1])$ . Apply  $L(a_2, b_2; b_2)$  to obtain Case IV.

Case VIII.  $P_0 = H$ , and not Cases O-IV, and

- i)  $P_1 \in b_1$ . Apply  $L(a_1, b_1; b_1)$  to obtain Case IV.
- ii)  $P_1 \in \bar{b}_1$ . Proceed as in Case IV.

Case IX.  $P_0 = B$ , and not Cases O-IV, and

- i)  $P_1 \in b_2$ . Let  $h = P(b)$ .
- ii)  $P_1 \in a_2$ . Since none of Cases O-IV occurs, then
 
$$\begin{aligned} \#(m_1 \cap b_2) &= \#(m_1 \cap [b_2: \bar{a}_2]) + \#(m_1 \cap [b_2: a_2]) = \\ &= \#(m_1 \cap [\bar{b}_2: \bar{a}_2]), \end{aligned}$$

and

$$\#(m_1 \cap a_2) = \#(m_1 \cap [b_2: \bar{a}_2]) + \#(m_1 \cap [\bar{b}_2: a_2]) \neq 0$$

for  $P_1 \in a_2$ . Therefore, we can not have

$$\#(m_1 \cap [b_2: \bar{a}_2]) = \#(m_1 \cap [b_2: a_2]) = \#(m_1 \cap [\bar{b}_2: \bar{a}_2]).$$

If either  $\#(m_1 \cap [b_2: ba_2]) \neq \#(m_1 \cap [b_2: a_2])$  or  $\#(m_1 \cap [b_2: ba_2]) \neq \#(m_1 \cap [\bar{b}_2: \bar{a}_2])$ , a suitable linear cutting reduces the intersection number  $m$ .

iii)  $P_1 \in \bar{b}_2$ . The discussion is similar to what we did in Case VI-(iv) of Step 1, in which  $a_1$  was left unchanged.

iv)  $P_1 \in \bar{a}_2$ . Evidently  $\#(m_1 \cap b_2) = \#(m_1 \cap [\bar{a}_2: \bar{b}_2])$ , doing  $L(\bar{a}_2, \bar{b}_2, \bar{b}_2)$  it becomes the above situation (iii).

*Case X.*  $P_0 = D$ , and not Cases O-IV, and

i)  $P_1 \in a_2$ , and

(a)  $\#(m_1 \cap [a_2: b_2]) \neq \#(m_1 \cap [\bar{a}_2: b_2])$ . Do a linear cutting.

(b)  $\#(m_1 \cap [a_2: b_2]) = \#(m_1 \cap [\bar{a}_2: b_2]) = \mu = \frac{1}{2}\#(m_1 \cap b_2)$ . We do first the operation

$h = LN^2LN$  as pictured in Figure 2.16. Clearly

$$\begin{aligned} \#(m_1 \cap h(\beta)) &= \#(m_1 \cap \beta) - \#(m_1 \cap b_1) - \#(m_1 \cap a_2) + \\ &\quad \#(m_1 \cap h(a_1)) + \#(m_1 \cap h(b_2)) \end{aligned}$$

$$= m - \mu_a - \mu_b + \mu_c + \mu_d, \quad (*)$$

since  $h(a_2) = \bar{a}_1$  and  $h(b_1) = b_2$ , where we write

$\mu_a = \#(m_1 \cap a_2)$ ,  $\mu_b = \#(m_1 \cap b_1)$ ,  $\mu_c = \#(m_1 \cap h(a_1))$  and  $\mu_d = \#(m_1 \cap h(b_2))$ . If we suppose  $P_{m+1} = D$  or  $C$ , (other-

wise, we may consider first  $s_m$  instead of  $s_0$ ), and denote

$\mu_1 = \#(m_1 \cap [b_1: a_2])$ ,  $\mu_2 = \#(m_1 \cap [b_1: \bar{a}_2])$  and

$\mu_3 = \#(m_1 \cap [b_1: \bar{b}_2])$ , then

$$\mu_a = \#(m_1 \cap [\bar{b}_1: a_2]) + \mu_1 + \mu$$

$$= \varepsilon + \mu_b + \mu_1 + \mu,$$

$$\mu_b = -\varepsilon' + \mu_1 + \mu_2 + \mu_3,$$

$$\mu_c = \mu + \mu_2 + \mu_3,$$

and 
$$\mu_d = |\mu_1 - \mu_2| + \mu_3,$$

where  $\varepsilon = \#(\{P_1, P_m\} \cap a_2) \in \{1, 2\}$  and  $\varepsilon' = \#(P_{m+1} \cap C) \in \{0, 1\}$ . Therefore, the formula (\*) becomes

$$\begin{aligned} \#(m_1 \cap h(B)) &= m - (\varepsilon - \varepsilon' + 3\mu_1 + \mu_2 - \\ &\quad |\mu_1 - \mu_2|), \end{aligned} \quad (**)$$

which is strictly less than  $m$  except when  $\varepsilon = \varepsilon' = 1$  and  $\mu_1 = 0$ .

(c) When  $\varepsilon = \varepsilon' = 1$  and  $\mu_1 = 0$ , we have  $P_{m+1} = C$  and  $P_m \notin a_2$ . And

(1) if  $P_m \in b_2$  or  $\bar{a}_2$ , let  $h$  be the parallel cutting  $P(b)$  or  $P(\bar{a})$  respectively.

(2) if  $P_m \in \bar{b}_2$ , then  $\mu_1 = \mu_2 = 0$ , (Figure 2.17), and a discussion similar to that in Case III-(ii) of the first step leads to a contradiction.

ii)  $P_1 \in b_2$ , and

(a)  $\#(m_1 \cap [a_2: \bar{b}_1]) > \#(m_1 \cap [a_2: b_2])$ . Let  $h = N(a)$ .

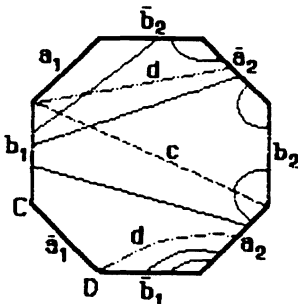


Figure 2.16 Case X-(i)-(b)

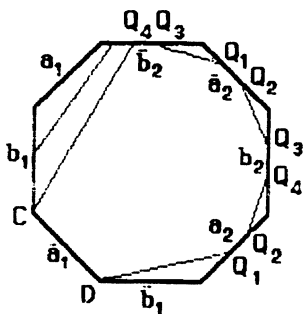


Figure 2.17 Case X-(i)-(c)-(2)

(b)  $\#(m_1 \cap [a_2: \bar{b}_1]) \leq \#(m_1 \cap [a_2: b_2])$ . First let  $h = L(a_2, b_2; b_2)$ , which either reduces the intersection number or produces case (i) above.

iii)  $P_1 \in \bar{a}_2$ , and

(a)  $\#(m_1 \cap [a_2: \bar{b}_1]) > \#(m_1 \cap [a_2: b_2])$ . Let  $h = N(a)$ .

(b)  $\#(m_1 \cap [a_2: \bar{b}_1]) \leq \#(m_1 \cap [a_2: b_2])$ . Let  $h = L(a_2, b_2; b_2)$ .

Case XI.  $P_0 = C$ . Applying  $L(\bar{a}_1, \bar{b}_1; \bar{b}_1)$  yields Case X.

Step 4. Given  $f$  with  $(a_1)f = a_1$  and  $m = \#(m_1 \cap \beta) = 0$ , then, there is a self-homeomorphism  $h$  which is a composition of elementary operations, such that  $(a_1)h = a_1$  and  $(m_1)h^{-1} = b_1$ .

The proof of this step is quite different from the above. It is more topological.

First, we consider two based simple closed curves  $l$  and  $m$ , and we say they are cobasic, if there are other curves  $l'$  and  $m'$  such that the set  $l, m, l', m'$  forms a system of basecurves on the surface  $F_2$ .

*Proposition 2.3.* *The curves  $l$  and  $m$  are cobasic, if and only if*

$$F_2 - \{l, m\} \cong F_2 - \{a_1, b_1\} \cong F_1 - B^2.$$

*Proof.* If  $l$  and  $m$  are cobasic, the formula obviously holds.

If  $F_2 - \{l, m\} \cong F_1 - B^2$ , we consider its boundary circle  $S^1$  which obviously may be written in a word of  $l$  and  $m$ , i.e.

$$S^1 = l^{i_1} m^{j_1} l^{i_2} m^{j_2} \dots l^{i_k} m^{j_k},$$

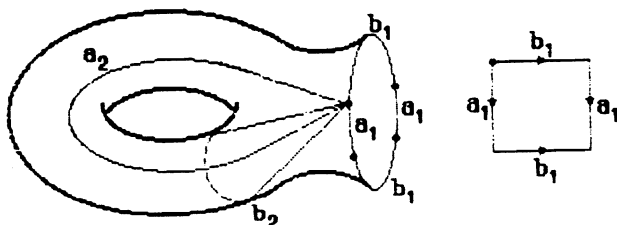


Figure 2.18  $F_2 - \{a_1, b_1\}$

for some integers  $i_1, j_1, i_2, j_2, \dots, i_k, j_k \in \mathbb{Z}$ . Since the orientable surface  $F_2$  is obtained from this surface by gluing along the curves  $l$  and  $m$ , we have that

$$\sum_{p=1}^k |i_p| = \sum_{p=1}^k |j_p| = 2 \text{ and } \sum_{p=1}^k i_p = \sum_{p=1}^k j_p = 0.$$

Thus, the only possibilities  $\mathbb{S}^1 = \text{lm}\overline{\text{lm}}$  and  $\mathbb{S}^1 = \text{lm}\overline{\text{lm}}$  determine the surface  $F_2$ , and in the both cases  $l$  and  $m$  are cobasic. Moreover, for orientation-preserving homeomorphisms, only the first one is possible, and the orientation-reversing operation

$$\text{RP} = [\bar{a}_1, b_1, \bar{a}_1 b_2 a_1, \bar{a}_1 a_2 a_1]$$

interchanges these two situations.

By the previous proposition, this step can be done easily in the following way. It is enough to find elementary operation  $h$  such that  $(a_1)h = a_1$ , and  $(m_1)h = b_1^{+1}$ . We show it by listing all possible cases under the assumption  $\#(m_1 \cap B) = 0$ .

i) First we claim that  $m_1$  can not be one of the following types:  $\widetilde{AA}, \widetilde{AB}, \widetilde{AE}, \widetilde{AF}, \widetilde{AG}, \widetilde{AH}, \widetilde{BB}, \widetilde{BE}, \widetilde{BF}, \widetilde{BG}, \widetilde{BH}, \widetilde{CC}, \widetilde{CD}, \widetilde{CF}, \widetilde{CG}, \widetilde{CH}, \widetilde{DD}, \widetilde{DF}, \widetilde{DG}, \widetilde{DH}, \widetilde{EE}, \widetilde{EF}, \widetilde{EG}, \widetilde{EH}, \widetilde{FF}, \widetilde{FG}, \widetilde{FH}, \widetilde{GG}, \widetilde{GH}$  or  $\widetilde{HH}$ , since the curve  $m_1$  is not null-homologous, is not homotopic to any power of  $a_1$ , and is cobasic with  $a_1$  (i.e.  $F_2 - m_1$  must be homeomorphic to the bounded surface in Figure 2.19).

ii) If  $m_1$  is of the type  $\widetilde{BC}$  or  $\widetilde{DE}$ , a small isotopic deformation of  $F_2$  may turn  $m_1$  into  $b_1$ .

iii) If  $m_1$  is of the type  $\widetilde{AC}, \widetilde{BD}$  or  $\widetilde{CE}$ , we need only one more linear cutting.

iv) If  $m_1$  is of the type  $\widetilde{AD}$ , just do the parallel cutting  $P(b)$ .

*Step 5. Given  $f$  with  $(a_1)f = a_1$  and  $(b_1)f = b_1$ , there is a self-homeomorphism  $h$  which is a composition of*



elementary operations, such that  $(a_1)h = a_1$ ,  $(b_1)h = b_1$  and  $(l_2)h^{-1} = a_2$ .

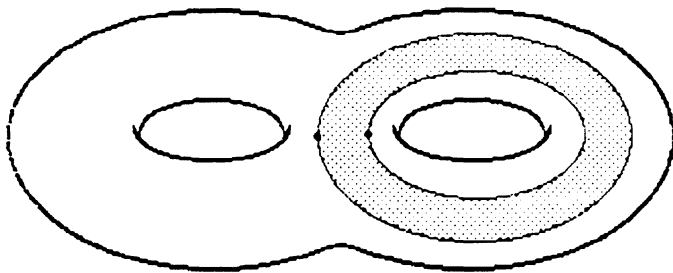


Figure 2.19  $F_2 - m_1$

Denote  $m = \#(l_2 \cap B)$ , where  $l_2 = (a_2)f$ .

Case I.  $m > 0$ .

We do the same thing as in the first step to reduce the number  $m$ .

I-i) If an arc of  $l_2$  in the presentation polygon is of the types in Cases 0-III of the first step, do the same operations as there. Since all possible situations involve only the second handle, the operations leave the basecurves  $a_1$  and  $b_1$  unchanged.

Therefore, from now on we will suppose that Case I-(i) does not occur for any arc of  $l_2$ . We will denote

$$\begin{aligned} \lambda_a &= \#(l_2 \cap a_2) \\ &= \#(l_2 \cap [b_2: a_2]) + \#(\{[B, C, D]: a_2\}) \\ &= \#(l_2 \cap [b_2: \bar{a}_2]) + \#(l_2 \cap [\bar{b}_2: \bar{a}_2]) + \\ &\quad \#(\{[B, C, D]: \bar{a}_2\}) \end{aligned}$$

and

$$\begin{aligned}
 \lambda_b &= \#(l_2 \cap b_2) \\
 &= \#(l_2 \cap [\bar{a}_2: \bar{b}_2]) + \#(\{B, C, D\}: \bar{b}_2) \\
 &= \#(l_2 \cap [a_2: b_2]) + \#(l_2 \cap [\bar{a}_2: b_2]) + \\
 &\quad \#(\{B, C, D\}: b_2),
 \end{aligned}$$

where  $\#(\{B, C, D\}: c_2)$  is the number of arcs of  $l_2$  in the presentation polygon with one endpoint from the set  $\{B, C, D\}$  and the other on the edge  $c_2$ . Obviously

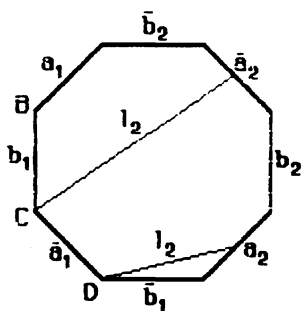
$$\#(\{B, C, D\}: a_2) + \#(\{B, C, D\}: b_2) + \#(\{B, C, D\}: \bar{a}_2) + \#(\{B, C, D\}: \bar{b}_2) \leq 2.$$

The assumption  $m > 0$  implies that  $\lambda_a + \lambda_b > 0$ . Thus, we may suppose  $\lambda_a > 0$  (or  $\lambda_b > 0$  similarly). Then,

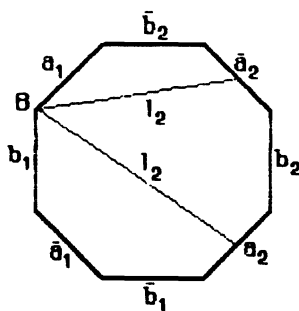
I-ii) if  $\#(l_2 \cap [a_2: b_2]) > \#(\{B, C, D\}: a_2)$  or  $\#(l_2 \cap [\bar{a}_2: \bar{b}_2]) > \#(\{B, C, D\}: \bar{b}_2)$ , a suitable linear cutting on the second handle makes  $m$  smaller.

I-iii) if not (ii) and  $\#(\{B, C, D\}: \bar{b}_2) = 0$ , then  $\lambda_b = 0$ , and this implies that

$$\lambda_a = \#(\{B, C, D\}: a_2) = \#(\{B, C, D\}: \bar{a}_2) = 1.$$



(a)  $P_0 = C$  and  $P_2 = D$



(b)  $P_0 = P_2 = B$

Figure 2.20 Case I-(iii)

(Figure 2.20 shows two of the possible situations.) We will show that this is impossible.

In fact, we consider the presentation annulus of the surface  $F_2$  bounded by the basecurves  $\{a_1, b_1, a_2\}$ , whose one boundary circle is  $a_2$  containing one basepoint and the other is  $a_1 b_1 \bar{a}_1 \bar{b}_1 a_2$  containing five basepoints, (Figure 2.21). Under the given homeomorphism, for the basecurves  $\{a_1, b_1, l_2, m_2\}$ , the presentation annulus of  $F_2$  bounded by the system  $\{a_1, b_1, l_2\}$  also has one boundary circle containing five basepoints and the other containing only one.

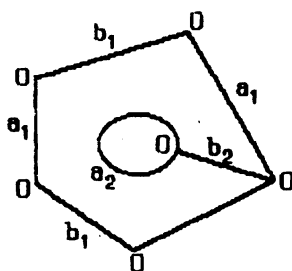


Figure 2.21  $F_2 - \{a_1, b_1, a_2\}$

In the case  $P_0 = C$  and  $P_2 = D$  we have one circle with four basepoints and the other with two (Figure 2.22). The same thing happens for the case  $P_0 = B$  and  $P_2 = C$ . When  $P_0 = B$  and  $P_2 = D$ , both circles have three basepoints. All these cases are impossible. When  $P_0 = P_2 = B$  (or  $C$  or  $D$ , similarly), the circle bounded by five basepoints is  $b_1 \bar{a}_1 \bar{b}_1 a_1 l_2$ , which can not be given by a homeomorphism keeping  $a_1$  and  $b_1$  fixed, (Figure 2.23).

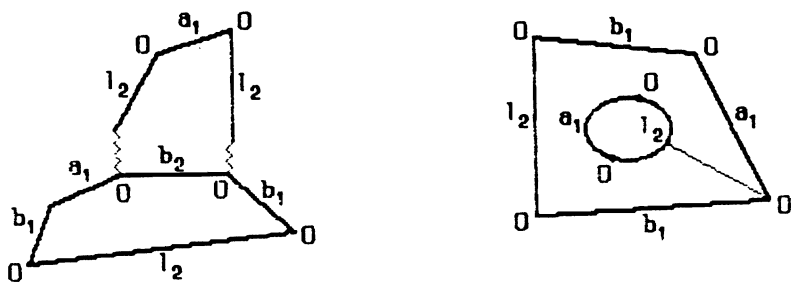


Figure 2.22 Case I-(iii) for  $P_0 = C$  and  $P_2 = D$

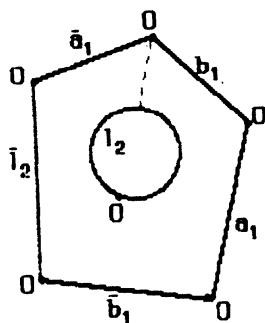


Figure 2.23 Case I-(iii) for  $P_0 = P_2 = B$

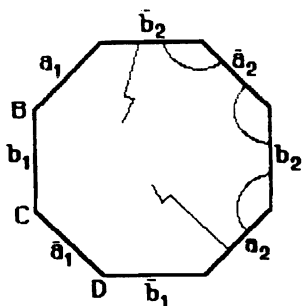


Figure 2.24 Case I-(iv)

I-iv) if  $\#(\{B, C, D\}: \bar{b}_2) = \#(\{B, C, D\}: a_2) = 1$ , then we have  $\#(\{B, C, D\}: \bar{a}_2) = \#(\{B, C, D\}: b_2) = 0$ . The only case is when  $\#(l_2 \cap [a_2: b_2]) \neq \#(l_2 \cap [\bar{a}_2: b_2])$ , which can be simplified by a linear cutting. Indeed, if  $\#(l_2 \cap [a_2: b_2]) = \#(l_2 \cap [\bar{a}_2: b_2]) = \#(l_2 \cap [\bar{a}_2: \bar{b}_2]) (= 1$ , if not (ii)), the curve  $l_2$  is homotopic to a word of  $a_1$  and  $b_1$  (Figure 2.24), which is impossible.

Case II. If  $m = 0$ . An if one endpoint is one of  $B, C$  or  $D$ , the discussion similar to Case I-(iii) shows the impossibility. All remaining cases except  $\widetilde{AE}$ , may be done easily by the linear cuttings.

Step 6. Given  $f$  with  $f(a_1) = a_1$ ,  $f(b_1) = b_1$  and  $f(a_2) = a_2$ , then there exists a self-homeomorphism  $h$  which is a composition of elementary operations, such that  $h(a_1) = a_1$ ,  $h(b_1) = b_1$ ,  $h(a_2) = a_2$  and  $h^{-1}(f(b_2)) = b_2$ .

This is the last step of the algorithm.

i) If  $m = \#(m_2 \cap \mathcal{B}) > 0$ , then one of the cases pictured in Figure 2.25(a) & (b) must occur. Their intersection number  $m$  can be reduced by the linear cuttings  $L(\bar{a}_2, \bar{b}_2; \bar{b}_2)$  and  $L(\bar{a}_2, b_2; b_2)$  respectively.

ii) If  $m = 0$ , at least one endpoint must be  $F, G$  or  $H$ , by considering the homotopy class of  $m_2$ . And it is sufficient to discuss when  $P_1 = H$  (Figure 2.26), since  $F$  is symmetric with  $H$ , and since the linear cutting  $L(\bar{a}_2, \bar{b}_2; \bar{b}_2)$  transfers the case  $P_1 = G$  to that of  $P_1 = H$ .

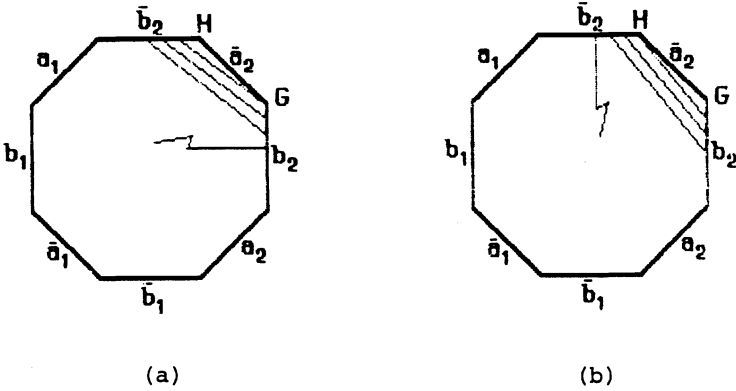


Figure 2.25 Step 6-(i)

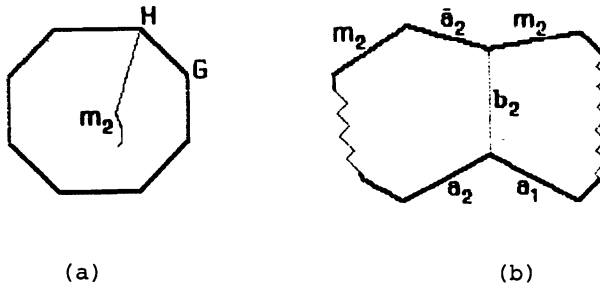


Figure 2.26 Step 6-(ii)

By a discussion as before, listing all possible cases of  $P_0$  and cutting along  $m_2$  and gluing along  $b_2$ , the only cases that may happen are  $P_0 = A$  and  $P_0 = F$ , which can be done either by some isotopic deformation or by the linear cutting  $L(\bar{a}_2, b_2; b_2)$ . This completes our algorithm.

### 3. A Presentation of the Mapping Class Group $M_2$

The presentation of the group  $M_2$  first was given by Birman ([1]) in Lickorish's generators.

*Theorem 3.1. (Birman) The mapping class group  $M_2$  of the closed orientable surface of genus two is presented by five Dehn twists  $D_1, D_2, D_3, D_4$  and  $D_5$  as generators, and following relations:*

- (1.a)  $D_i \leftrightarrow D_j, \text{ for } |i - j| \geq 2;$
- (1.b)  $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}, \text{ for } 1 \leq i \leq 4;$
- (1.c)  $(D_1 D_2 D_3 D_4 D_5)^6 = 1;$
- (1.d)  $(D_1 D_2 D_3 D_4 D_5 D_5 D_4 D_3 D_2 D_1)^2 = 1;$
- (1.e)  $(D_1 D_2 D_3 D_4 D_5 D_5 D_4 D_3 D_2 D_1) \leftrightarrow D_i, \text{ for } i = 1, 2, 3, 4, 5.$

Where  $D_1 = B_1, D_2 = A_1, D_3 = C, D_4 = A_2$  and  $D_5 = B_2$  are Dehn twists along the curves  $b_1, a_1, c, a_2$  and  $b_2$  respectively.

Now we will write a simple presentation in the generators  $L$  and  $N$ .

First we observe that, for any given mapping class  $f$ , we may replace the family of generators  $\{D_i\}$  by the family  $\{fD_i\bar{f}\}$  in Theorem 3.1. i.e., we may suppose that where  $D_1, D_2, D_3, D_4$  and  $D_5$  are Dehn twists along five arbitrary curves  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  and  $\gamma_5$  which may be identified with  $b_1, a_1, c, a_2$  and  $b_2$  by some mapping class  $f$  (Figure 1.6).

Considering  $\Gamma_i = N^i L \bar{N}^i, i = 0, 1, 2, 3, 4, 5$ . They are Dehn twists along the curves  $\gamma_i = N^i(b_1), i = 0, 1, 2, 3, 4, 5$ , which are  $b_1, \bar{a}_1, \bar{b}_1 a_2, b_2, \bar{a}_2$  and  $\bar{b}_2 a_1$  as

pictured in Figure 3.1. By the observation, any five of them may be chosen as a family of generators in Theorem 3.1. Therefore, it is natural to choose that  $D_i = \Gamma_{i-1} = N^{i-1} \bar{L} N^{i-1}$ ,  $i = 1, \dots, 5$ , and then to substitute them in the formulas (1.a) - (1.e). Using this idea, a presentation of  $M_2$  in the generators  $L$  and  $N$  will be nicely given.

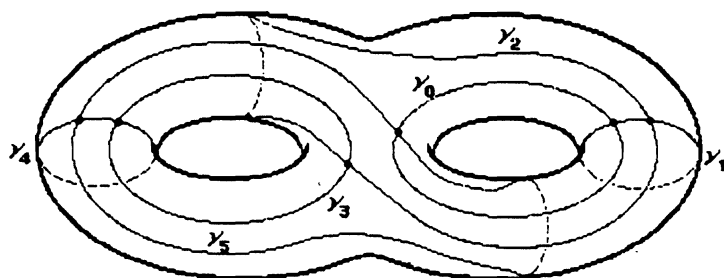


Figure 3.1 Twist curves of  $\Gamma_i$ 's

*Theorem 3.2. The surface mapping class group  $M_2$  is finitely presented by a family of two generators  $L$  and  $N$ , and six relators:*

- (2.a)  $N^6 = 1,$
- (2.b)  $(LN)^5 = 1,$
- (2.c)  $(\bar{L}N)^{10} = 1,$
- (2.d)  $L \leftrightarrow N^2 \bar{L} N^4,$
- (2.e)  $L \leftrightarrow N^3 \bar{L} N^3,$
- (2.f)  $L \leftrightarrow (\bar{L}N)^5.$

The relations in Theorem 3.2 were certainly not easily found. But the proof is just a straightforward verification. As useful facts, we show some of the calculations below.



$$\begin{aligned}
N &= [\bar{a}_2 b_1, \bar{a}_1, \bar{a}_1 b_2, \bar{a}_2], \\
N^2 &= [\bar{b}_2, \bar{b}_1 a_2, \bar{b}_1, \bar{b}_2 a_1], \\
N^3 &= [a_2, b_2, a_1, b_1], \\
N^4 &= [\bar{a}_1 b_2, \bar{a}_2, \bar{a}_2 b_1, \bar{a}_1], \\
N^5 &= [\bar{b}_1, \bar{b}_2 a_1, \bar{b}_2, \bar{b}_1 a_2],
\end{aligned}$$

$$\begin{aligned}
LN &= [\bar{a}_2 b_1 \bar{a}_1, \bar{a}_1, \bar{a}_1 b_2, \bar{a}_2], \\
(LN)^2 &= [\bar{b}_2 a_1 \bar{b}_1 a_2, a_1 \bar{b}_1 a_2, a_1 \bar{b}_1, \bar{b}_2 a_1], \\
(LN)^3 &= [b_1 \bar{a}_1 b_2, \bar{a}_2 b_1 \bar{a}_1 b_2, \bar{a}_2 b_1, b_1 \bar{a}_1], \\
(LN)^4 &= [\bar{b}_1, \bar{b}_2 a_1 \bar{b}_1, \bar{b}_2, \bar{b}_1 a_2],
\end{aligned}$$

$$\begin{aligned}
L\bar{N} &= [\bar{b}_1 \bar{b}_2 a_1, \bar{b}_2 a_1, \bar{b}_2, \bar{b}_1 a_2], \\
(L\bar{N})^2 &= [\bar{a}_1 b_2 \bar{a}_2 b_2 \bar{a}_1, \bar{a}_2 b_2 a_1, \bar{a}_2 b_1, \bar{a}_1], \\
(L\bar{N})^3 &= [\bar{a}_1 b_2 b_1 a_1 \bar{b}_1, b_2 \bar{a}_2 \bar{b}_2 a_1, a_1, \bar{a}_1 b_2 b_1], \\
(L\bar{N})^4 &= [a_2 \bar{b}_2 \bar{b}_1 a_2 b_2 \bar{a}_2, \bar{b}_2 a_1 \bar{b}_1 a_2 b_2 \bar{a}_2, \bar{b}_2 a_1 \bar{b}_1, a_2], \\
(L\bar{N})^5 &= [\bar{b}_2 \bar{a}_2 \bar{a}_1 b_2 a_2, \bar{a}_2 \bar{b}_2 \bar{b}_1 a_2 b_2, \bar{a}_2, \bar{b}_2],
\end{aligned}$$

and  $N^6 = (LN)^5 = (L\bar{N})^{10} = 1.$

We will prove the theorem, after several lemmas.

*Lemma 3.3. The following relations may be obtained by the formulas (2.a) - (2.f):*

- (a)  $L \leftrightarrow N^i \bar{L\bar{N}}^i$ , for  $i = 2, 3, 4$ ,  
 (b)  $L \leftrightarrow N^i \bar{L\bar{N}}^i L\bar{N}^i$ , for  $i = 1, 5$ .

*Proof.* (a) When  $i = 2$ , it is the formula (2.d).

When  $i = 3$ , it is (2.e). And when  $i = 4$ , we have

$$\begin{aligned}
N^4 L\bar{N}^2 \cdot L &= N^4 \cdot L \cdot N^2 L\bar{N}^4 \cdot N^2, && \text{by (2.a),} \\
&= N^4 \cdot N^2 L\bar{N}^4 \cdot L \cdot N^2, && \text{by (2.d),} \\
&= L \cdot N^4 L\bar{N}^2, && \text{by (2.a).}
\end{aligned}$$

(b) This is obtained from (2.b), (2.d) and (2.e).

Indeed,

$$\begin{aligned}\bar{N}LNL\bar{N} &= \bar{N}^2 \overline{LNLNLN}^2, \text{ by (2.b),} \\ &= (N^4 \bar{L}\bar{N}^4)^{-1} (N^3 \bar{L}\bar{N}^3)^{-1} (N^2 \bar{L}\bar{N}^2)^{-1}.\end{aligned}$$

This implies the case  $i = 5$ . The case when  $i = 1$  is equivalent to that  $i = 5$ , since

$$LNL\bar{N}LN = N \cdot \bar{N}LNL\bar{N} \cdot LN.$$

*Lemma 3.4.* The relations (1.a) and (1.b) are consequences of the formulas in Lemma 3.3.

*Proof.* Since  $D_i = N^{i-1} \bar{L}\bar{N}^{i-1}$ , this lemma is evident.

*Theorem 3.5.*

- (a)  $N = D_1 D_2 D_3 D_4 D_5$ ;
- (b)  $D_5 D_4 D_3 D_2 D_1 = \bar{N}(\bar{N}L)^5$ ;
- (c)  $D_1 D_2 D_3 D_4 D_5 D_5 D_4 D_3 D_2 D_1 = (\bar{N}L)^5$ .

*Proof.* The proof is straightforward, since

$$D_1 D_2 D_3 D_4 D_5 = LNLNLNLNL\bar{N}^4 = (LN)^5 \bar{N}^5 = N.$$

Similarly, we can easily prove the formula (b), and the formula (c) is just a product of the formulas (a) and (b).

Conjugating the formula (a) by a power of  $N$ , it follows that,

$$N = \Gamma_i \Gamma_{i+1} \Gamma_{i+2} \Gamma_{i+3} \Gamma_{i+4},$$

for any  $i = 0, 1, \dots, 5$ , where  $\Gamma_{6+j} = \Gamma_j$  by convention.

*Proof of Theorem 3.2.* Since we have Lemma 3.3, Lemma 3.4 and Theorem 3.5, our relations imply those in the Birman's theorem.

In fact, the relations of (1.a) and (1.b) have been shown in Lemma 3.3, and (1.c) is exactly  $N^6 = 1$  by Theorem 3.5(a). Relations (1.d) and (1.e) are equivalent to the other two formulas since we have the formula in Theorem 3.5(c). As a useful fact, we give here two more relations:

*Proposition 3.6.*

$$(a) \quad L^2 = (N\overline{LNLN})^4;$$

$$(b) \quad T \leftrightarrow P^4,$$

where  $T = N^3$  and  $P = L\overline{NLNL}$ , and moreover  $P^4 = (L\overline{NLN})^6$ .

The proof is straightforward.

As a consequence, for the homeotopy group  $\tilde{M}_2$  we have the theorem:

*Theorem 3.7.* The homeotopy group  $\tilde{M}_2$  is finitely presented by three generators: the linear cutting  $L$ , the normal cutting  $N$ , and the reversion  $R$ , and nine relations: six from Theorem 3.2 and three more

$$(3.g) \quad R^2 = I,$$

$$(3.h) \quad NR = R\overline{N},$$

$$(3.i) \quad LR = R\overline{NLN}.$$

An interesting observation is the following.

*Corollary 3.8. Both the mapping class group  $M_2$  and the homeotopy group  $\tilde{M}_2$  are generated by some periodic elements. Actually, the elements  $N$ ,  $LN$ , and  $R$  are periodic of orders 6, 5, and 2 respectively.*

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