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HOMEOMORPHISMS OF A SOLID HANDLEBODY AND HEEGAARD SPLITTINGS OF THE 3-SPHERE S^3

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Let H_g be a solid handlebody of genus g , with boundary $\partial H_g = F_g$. Let $B = \{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$ be a fixed system of basecurves based at a common basepoint O , such that the a_i 's are meridian circles of H_g . Let M_g denote the mapping class group of the closed orientable surface F_g . And let K_g denote the subgroup of M_g consisting of mapping classes induced by some homeomorphism of the handlebody H_g . An element of K_g will be called an *extendible mapping class*.

The subgroup K_g plays a very important role in Heegaard splitting of 3-manifolds (Cf. [1] & [8]). In this paper, we describe this subgroup explicitly by giving a finite set of generators in the first two sections. Comparing to Suzuki's generators [7], not only is the number of generators one less, but also the expressions in the generators of the mapping class group M_g are quite easy. In the third section, all Heegaard splittings of the 3-sphere S^3 are explicitly given, this was asked in Hempel's book ([3] p. 164).

1. Some Extendible Mapping Classes

First we are going to give some extendible mapping classes, show they generate the group K_g , then reduce

the number by using the technique given in the papers [4] and [5].

Recall that the mapping class group M_g is generated by three elements: *the linear cutting L, the normal cutting N and the transport T*. Algebraically, they are given by

$$L = [a_1 b_1, a_2, b_2, \dots, a_g, b_g],$$

$$N = [x \bar{a}_2 b_1, \bar{a}_1, \bar{a}_1 x b_2, \bar{a}_2, a_3, b_3, \dots, a_g, b_g],$$

where $x = [a_1, b_1][a_2, b_2]$, and

$$T = [a_g, b_g, a_1, b_1, \dots, a_{g-1}, b_{g-1}].$$

We also denote by $M = \bar{N}LN$ *the meridian cutting*, $P = LML = MLM$ *the parallel cutting*, $Q = TPT = N^3 \bar{P} N^3$ *the parallel cutting of the second handle*, $c_i = [a_i, b_i]$ *the waist of the i-th handle*, and $x = c_1 c_2$ *the waist of the first two handles*.

Now we list some elementary extendible mapping classes.

- 1) *The meridian cutting M*, given by

$$M = [a_1, b_1 \bar{a}_1, a_2, b_2, \dots, a_g, b_g].$$

- 2) *The transport T*.

- 3) *The handle rotation ϕ* , (Figure 1.1), given by

$$\phi = [c_1 \bar{a}_1, \bar{b}_1 \bar{c}_1, a_2, b_2, \dots, a_g, b_g],$$

is obtained by a 180° -rotation of the first handle along its waist circle c_1 .

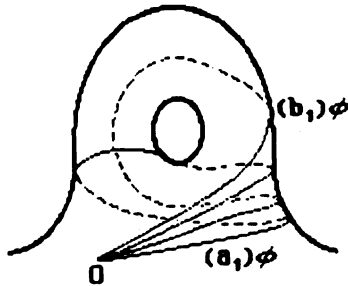


Figure 1.1 Handle rotation

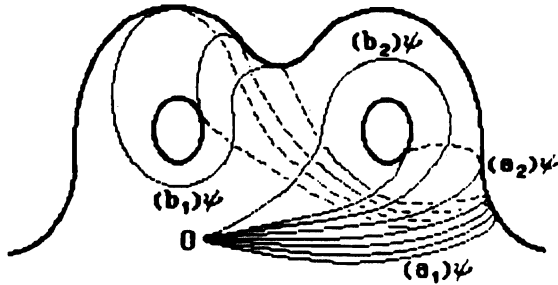
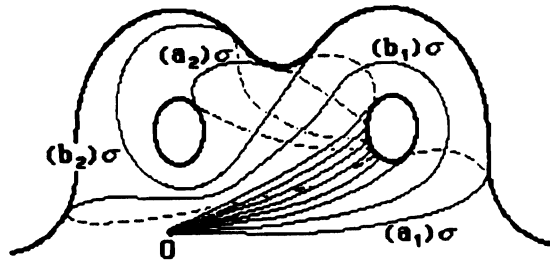
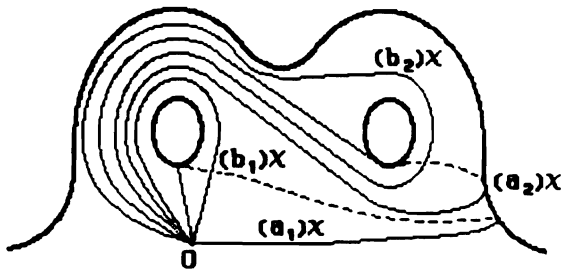
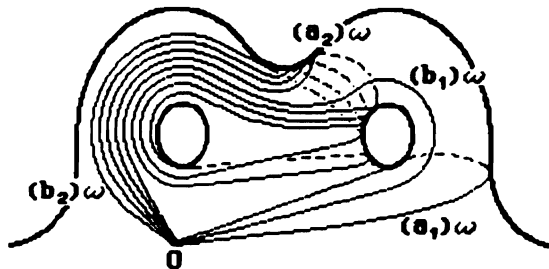


Figure 1.2 Handle switching

4) The *handle switching* ψ , (Figure 1.2), given by
$$\psi = [c_1 a_2 \bar{c}_1, c_1 b_2 \bar{c}_1, a_1, b_1, a_3, b_3, \dots],$$
 is obtained by moving the second handle around the first handle into the position in front of the first one.

5) The *handle rounding* σ , (Figure 1.3), given by
$$\sigma = [a_1, b_1 \bar{a}_1 \bar{b}_1 c_2 b_1 a_1, \bar{a}_1 c_1 a_2 \bar{c}_1 a_1, \bar{a}_1 c_1 b_2 \bar{c}_1 a_1, a_3, b_3, \dots],$$
 is obtained by moving one foot of the first handle around the second one.

Figure 1.3 Handle rounding σ Figure 1.4 Handle crossing χ Figure 1.5 One-foot sliding ω

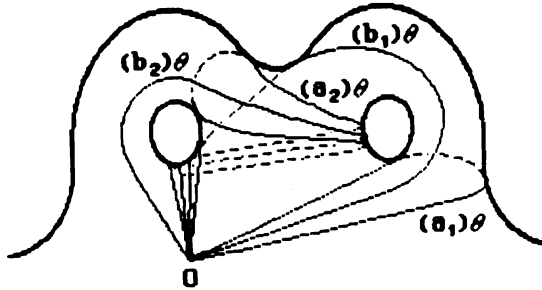


Figure 1.6 Handle knotting

6) The *handle crossing* χ , (Figure 1.4), given by

$$\chi = [c_1 a_2, b_2, b_2 a_1 \bar{b}_2, b_2 b_1 \bar{b}_2, a_3, b_3, \dots],$$

is obtained by sliding the whole first handle along the longitude circle b_2 of the second handle.

7) The *one-foot sliding* ω , (Figure 1.5), given by

$$\omega = [a_1, b_2 b_1, b_2 \bar{c}_1 a_1 \bar{a}_2 \bar{a}_1 c_1 a_2 b_2 \bar{a}_1 c_1 \bar{b}_2, b_2 \bar{c}_1 a_1 b_2 \bar{a}_1 c_1 \bar{b}_2, a_3, b_3, \dots],$$

is obtained by sliding one foot of the first handle along the longitude circle b_2 of the second handle.

8) The *one-foot knotting* θ , (Figure 1.6), given by

$$\theta = [a_1, \bar{a}_2 b_1, \bar{a}_2 \bar{c}_1 a_1 a_2 \bar{a}_1 c_1 a_2, b_2 \bar{a}_1 c_1 a_2, a_3, b_3, \dots],$$

is obtained by moving one foot of the first handle along the meridian circle a_2 of the second handle.

9) The *handle replacing* η , (Figure 1.7), given by

$$\eta = [\bar{a}_1 c_1 a_2, \bar{a}_2 \bar{c}_1 \bar{b}_1 a_2, \bar{a}_2 \bar{c}_1 \bar{b}_1 a_2 b_1 c_1 a_2, b_2 b_1 c_1 a_2, a_3, b_3, \dots],$$

is obtained by replacing the first handle with the cylinder between the first and the second handles.

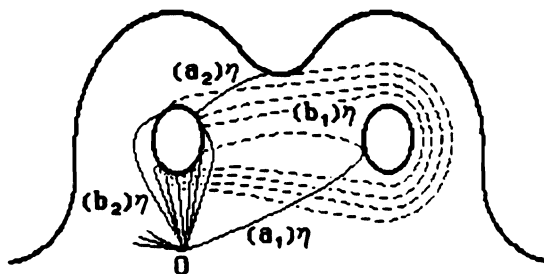
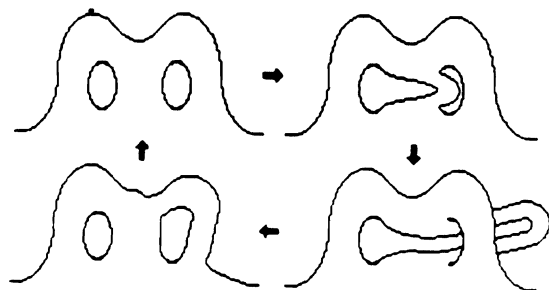


Figure 1.7 Handle replacing η

Remark. By definition, among all elementary extendible mapping classes, the operations T , ϕ , ψ , σ , χ , ω and η can be obtained by an isotopy deformation of \mathbf{S}^3 (i.e., obtained by moving the handlebody \mathbf{H}_g inside of \mathbf{S}^3 without cutting it open). And the operation θ is a combination of ω and meridian twists, which can be obtained in the following way: pass the left foot of the first handle along the longitude \bar{B}_2 of the second handle in the anti-clockwise way, twist the second handle along its meridian,

pull back the foot of the first handle along the new longitude, and adjust the longitude b_1 by a twist along the meridian of the first handle. Precisely,

$$\theta = \omega \cdot \overline{TMT} \cdot \overline{\omega} \cdot M.$$

Theorem 1.1. In the mapping class group M_g , we have the following expressions:

- (i) $\phi = P^2 = (LM)^4,$
- (ii) $\psi = \overline{P}^4 N^3 = N^3 \overline{Q}^4,$
- (iii) $\sigma = \overline{P}(\overline{LN})^5 (\overline{LN})^5 P \overline{Q}^4 = (\overline{PQN})^2 \overline{Q}^4,$
- (iv) $\chi = \overline{L}(\overline{NL})^5 (\overline{NL})^5 N^3 = (QNP)^2 \overline{N}^3,$
- (v) $\omega = Q^3 \overline{NQ}^2 P,$
- (vi) $\theta = Q^2 \overline{NQ} P = \overline{Q} \omega Q,$
- (vii) $\eta = \overline{PQ}^2 \overline{NQ} = \overline{P} \theta \overline{P}.$

Proof. The expressions are found by using the algorithm given in [4] and [5], which certainly was not easy. After the formulas have been discovered, the proof is just an immediate verification.

For example, for (iii), we know that,

$$(\overline{LN})^5 = [x\overline{a}_1, \overline{b}_1 x, \overline{a}_1 \overline{b}_1 c_2 \overline{a}_2 b_1 a_1, \overline{a}_1 \overline{b}_1 \overline{b}_2 \overline{c}_2 b_1 a_1, a_3, b_3, \dots],$$

thus $(\overline{NL})^5 = [\overline{a}_1 x, x\overline{b}_1, \overline{x a}_1 \overline{b}_1 c_2 \overline{a}_2 b_1 a_1 x, \overline{x a}_1 \overline{b}_1 \overline{b}_2 \overline{c}_2 b_1 a_1 x, a_3, b_3, \dots],$

and $(\overline{LN})^5 = [\overline{a}_1 c_1, x\overline{b}_1, \overline{x a}_1 x \overline{a}_2 \overline{c}_1 a_1 x, \overline{x a}_1 c_1 \overline{b}_2 \overline{x a}_1 x, a_3, b_3, \dots],$

then $(\overline{LN})^5 (\overline{LN})^5 = [x\overline{c}_1 a_1, b_1, b_1 x a_2 x \overline{b}_1, b_1 x b_2 x \overline{b}_1, a_3, b_3, \dots].$

Because $P = [c_1 b_1, \bar{a}_1, a_2, b_2, \dots],$
 and $\bar{Q}^4 = [a_1, b_1, \bar{c}_2 a_2 c_2, \bar{c}_2 b_2 c_2, a_3, b_3, \dots],$

$$\begin{aligned} \bar{P}(\bar{LN})^5(LN)^5 P \bar{Q}^4 &= \bar{P}[x b_1, \bar{a}_1, \bar{a}_1 c_1 a_2 \bar{c}_1 a_1, \bar{a}_1 c_1 b_2 \bar{c}_1 a_1, \\ &\quad a_3, b_3, \dots] \\ &= [a_1, \bar{a}_1 x b_1 a_1, \bar{a}_1 c_1 a_2 \bar{c}_1 a_1, \bar{a}_1 c_1 b_2 \bar{c}_1 a_1, a_3, b_3, \dots] \\ &= \sigma. \end{aligned}$$

Also we have,

$$\begin{aligned} \bar{PQN} &= [a_1, \bar{a}_1 x \bar{a}_2 b_1 a_1, \bar{a}_1 c_1 c_2 \bar{a}_2 \bar{c}_1 a_1, \bar{b}_2 \bar{x} a_1, a_3, b_3, \dots] \\ \text{and } (\bar{PQN})^2 &= [a_1, \bar{a}_1 x b_1 a_1, \bar{a}_1 x a_2 \bar{x} a_1, \bar{a}_1 x b_2 \bar{x} a_1, a_3, b_3, \dots], \end{aligned}$$

Clearly, $\sigma = \bar{Q}^4 (\bar{PQN})^2 = (\bar{PQN})^2 \bar{Q}^4.$ And similarly we can prove the other formulas.

2. Generators of the Subgroup K_g

In this section, we are going to prove that,

Theorem 2.1. The extendible mapping class subgroup K_g of the surface mapping class group M_g is generated by five elements:

$$M, T, N^3, P^2, \text{ and } PN^2P,$$

and also by the five elements:

$$T, N^3, \bar{N}LN, NLN^2\bar{L}, \text{ and } L\bar{N}L^2NL.$$

Regard the handlebody H_g as the down-semispace of the Euclidean space E^3 with g pairs of holes on its boundary F_g identified (Figure 2.1). Instead of the basecurves $B = \{a_i, b_i\}_{1 \leq i \leq g},$ we will study the basearcs $B = \{p_i, q_i, r_i\}_{1 \leq i \leq g},$ where as joining the oriented arcs, we have

$$a_i \sim p_i r_i \bar{p}_i, \quad b_i \sim q_i \bar{p}_i,$$

for $i = 1, 2, \dots, g$.

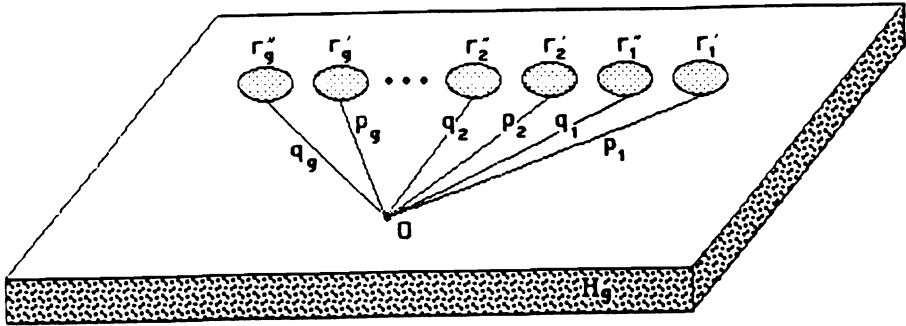


Figure 2.1

The disk holes, denoted by D_i 's, will be chosen as the meridian disks, which form a cutting system of the handlebody H_g . Their boundary circles, r_i 's, are the fixed meridian circles. In the plane in Figure 2.1, the disks D_i and the circles r_i are split in two. We will denote by $D_i^!$ and $D_i^!$ the two copies of D_i , denote by $r_i^! = \partial D_i^!$ and $r_i^! = \partial D_i^!$ the two copies of r_i , and call them the cutting disks and cutting circles respectively. Moreover, we also suppose that $r_i^!$ contains an endpoint of p_i and $r_i^!$ contains one of q_i .

We call this new description the planar representation of F_g . Using it, a mapping class of the surface F_g may be drawn easily in the plane. For example, the mapping classes ϕ , ψ and θ are drawn in Figure 2.2, and it is quite easy to understand how they move the feet of handles.

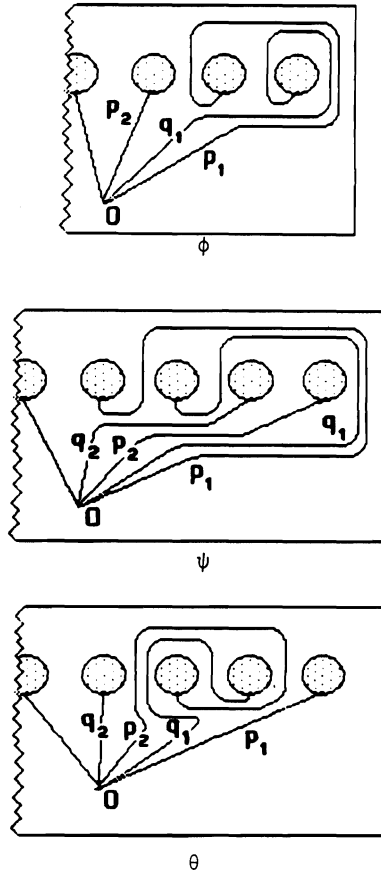


Figure 2.2

Using ϕ , ψ , θ and T , we construct some more elementary movings of handle-feet. A family of mapping classes, called the i -th foot knotting θ_i and the i -th foot knotting θ'_i , is defined by moving the foot r''_i of the first handle along the meridian circle a_i and the meridian circle $b_i \bar{a}_i \bar{b}_i$, i.e. r'_i and r''_i , respectively, (see Figure 2.3). Therefore, $\theta'_i = \phi_i \theta_i \bar{\phi}_i$, where $\phi_i = T^{i-1} \phi \bar{T}^{i-1}$, for $i = 2, \dots, g$. Precisely, we have

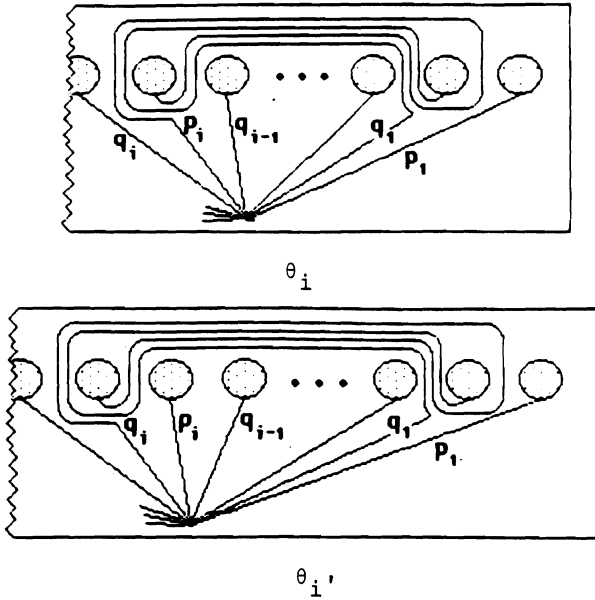


Figure 2.3

Proposition 2.2. The i -th foot knotting and \bar{I} -th foot knotting are generated by the mapping classes T , ϕ , ψ , and θ . Furthermore, they have the following expressions:

$$\theta_i = T^{i-2} (\overline{\psi T})^{i-2} \theta (T\psi)^{i-2} \overline{T}^{i-2},$$

and
$$\theta_i' = T^{i-1} \overline{\phi T} (\overline{\psi T})^{i-2} \theta (T\psi)^{i-2} T \phi \overline{T}^{i-1}.$$

Proof. By Figure 2.3,

$$\begin{aligned} \theta_i = [& a_1, c_2 \dots c_{i-1} \bar{a}_i \bar{c}_{i-1} \dots \bar{c}_2 b_1, a_2, b_2, \dots, a_{i-1}, \\ & b_{i-1}, \bar{c}_{i-1} \dots \bar{c}_2 \bar{a}_1 c_1 c_2 \dots c_{i-1} a_i, \bar{c}_{i-1} \dots \\ & \bar{c}_2 \bar{c}_1 a_1 c_2 \dots c_{i-1}, b_i \bar{c}_{i-1} \dots \bar{c}_2 \bar{a}_1 c_1 c_2 \dots c_{i-1}, \\ & a_{i+1}, b_{i+1}, \dots]. \end{aligned}$$

$$\begin{aligned} \theta_i' = [& a_1, c_2 \dots c_{i-1} c_i \bar{a}_i \bar{c}_{i-1} \dots \bar{c}_2 b_1, a_2, b_2, \dots, \\ & a_{i-1}, b_{i-1}, a_i, \bar{a}_i \bar{c}_{i-1} \dots \bar{c}_2 \bar{c}_1 a_1 c_2 \dots c_{i-1} a_i b_i, \\ & a_{i+1}, b_{i+1}, \dots]. \end{aligned}$$

Then, a direct calculation implies the proposition. For example, let us compute θ_i . Let $\psi_j = T_{j-1}\psi\bar{T}^{j-1}$ and $z = c_2c_3\dots c_{i-1}$, then

$$\begin{aligned}\psi_2\dots\psi_{i-1} &= [a_1, b_1, za_i\bar{z}, zb_i\bar{z}, a_2, b_2, \dots, a_{i-1}, b_{i-1}, \\ &\quad a_{i+1}, b_{i+1}, \dots], \\ \theta\psi_2\dots\psi_{i-1} &= [a_1, z\bar{a}_i\bar{z}b_1, z\bar{a}_i\bar{z}c_1a_1za_i\bar{z}a_1c_1z\bar{a}_i\bar{z}, \\ &\quad zb_i\bar{z}a_1c_1za_i\bar{z}, a_2, b_2, \dots, a_{i-1}, b_{i-1}, a_{i+1}, \\ &\quad b_{i+1}, \dots],\end{aligned}$$

and

$$\begin{aligned}\bar{\psi}_{i-1}\dots\bar{\psi}_2\theta\psi_2\dots\psi_{i-1} &= [a_1, z\bar{a}_i\bar{z}b_1, a_2, b_2, \dots, a_{i-1}, b_{i-1}, \\ &\quad \bar{a}_i\bar{z}c_1a_1za_i\bar{z}a_1c_1z\bar{a}_i\bar{z}, b_i\bar{z}a_1c_1za_2, a_2, b_2, \\ &\quad a_{i+1}, b_{i+1}, \dots].\end{aligned}$$

Now we want to start proving that the elementary extendible mapping classes generate the group K_g .

Let f be an extendible mapping class, i.e. an element of K_g . The idea is to find another extendible mapping class g generated by our generators, such that either gf or fg becomes "simpler" than f . The process will be repeated until the identity map is obtained.

Lemma 2.3. Let α be an oriented simple arc on the surface F_g from the basepoint O to the endpoint Q of q_1 at r_1^n , which does not intersect any of the meridian circles r_i for all i , and does not intersect any of the arcs p_j and q_j for $j \geq s$. Then, there exists a self-homeomorphism g whose homeotopy class is generated by the classes T , M , ϕ , ψ , and θ , such that $(q_1)g = \alpha$, $(r_i)g = r_i$, for any $i \geq 1$.

Furthermore, $(p_j)_g = p_j$ if $\alpha \cap p_j = \{O\}$, and $(q_j)_g = q_j$ if $\alpha \cap q_j = \{O\}$, for any $j \geq 1$.

Proof. Suppose the arc α intersects q_1 transversally, and denote by k the number of points of the intersection $\alpha \cap q_1$ other than O and Q . When $k = 0$, the union of these two curves becomes a simple closed curve $\gamma = \overline{\alpha q_1}$.

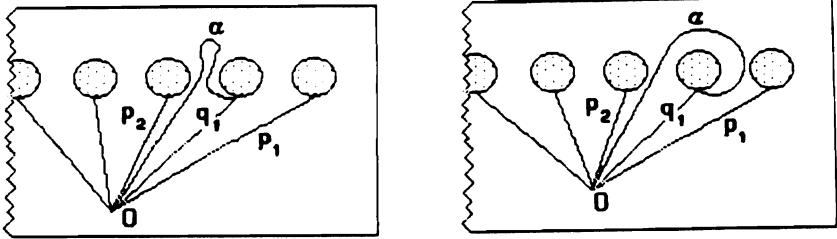


Figure 2.4

If γ does not intersect any of the arcs p_i and q_i other than q_1 , we may let g either be an isotopy if γ does not separate the circles r_1' and r_1'' , or a meridian twist from M^{+1} if the disk area bounded by γ includes r_1'' (Figure 2.4). Otherwise, let P be an intersection point closest to Q along α . If $P \in p_1$ we may use the mapping class ϕ to remove it, and if $P \in p_i$ or q_i , for some $i \geq 2$, we may use the mapping class θ_i' or θ_i given in Proposition 2.2 to remove it. Actually, g will be the mapping which moves the cutting circle r_1'' along the curve γ , its explicit expression in mappings θ_i 's, θ_i' 's and ϕ may be easily found from the intersection set $\gamma \cap (\cup(p_i \cup q_i))$ along the curve γ . This clearly leaves the unintersected p_i ' and q_i 's unchanged.

Suppose $k \geq 1$, and let P be the intersection point of α and q_1 closest to the point O along α . Let $\beta = \alpha \mid_{OP} q_1 \mid_{PQ}$. After an isotopy deformation, we have the intersection numbers $k(q_1, \beta) = 0$ and $k(\beta, \alpha) \leq k - 1$. And clearly β does not intersect other p_i 's and q_i 's more than α does, since we have

$$\beta \cap (U(p_i \cup q_i)) = \alpha \mid_{OP} \cap (U(p_i \cup q_i)) \subset \alpha \cap (U(p_i \cup q_i)),$$

(Figure 2.5). By induction, we have g_1 and g_2 generated by T, M, ϕ, ψ, θ and η , such that $(q_1)g_1 = \beta$ and $(\beta)g_2 = \alpha$. Then, take $g = g_1g_2$.

Lemma 2.4. Let f be an arbitrary self-homeomorphism of the handlebody H_g , such that $(r_i)f = r_i$, for all i . Then, the homeotopy class of f is generated by the classes T, M, ϕ, ψ , and θ .

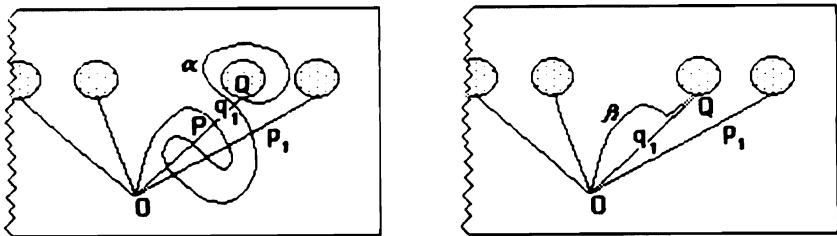


Figure 2.5

Proof. This is a direct consequence of Proposition 2.2 and Lemma 2.3. Inductively, suppose we have $(p_i)f = p_i$ and $(q_i)f = q_i$ for $i \leq s - 1$, for some s .

Rotate the handles until (p_s, q_s) is in the first position, switch p_s and q_s by ϕ , simplify p_s by using Lemma 2.3, then switch back to simplify q_s in the same way, and finally rotate it back. Again by Lemma 2.3, all p_i 's and q_i 's for $i \leq s - 1$ are unchanged.

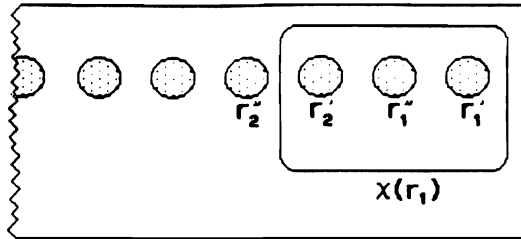


Figure 2.6

$$(r_2)\chi = r_1, (r_i)\chi = r_i \text{ for } i \geq 3$$

By Lemma 2.4, from now on, it is sufficient to study the image of the cutting system r_i 's of an extendible class. Thus, we first discuss some extendible mapping classes which change the cutting system. For example, the images of the cutting system of the mapping classes χ , ω and η are drawn in Figures 2.6-8.

Lemma 2.5. Let γ be an oriented simple closed curve on the surface F_g , which does not intersect any of the meridian circles r_i , and whose homology class in $H_1(F_g, \mathbb{Z})$ relative to the meridian circle r_i is nontrivial (i.e., γ separates r_1' and r_1'' in two sides in the planar representation). Then, there exists a self-homeomorphism g whose homeotopy class is generated by the classes T , M , ϕ , ψ , θ and η , such that $(\gamma)g = r_1$, and $(r_i)g = r_i$, for any $i \geq 2$.

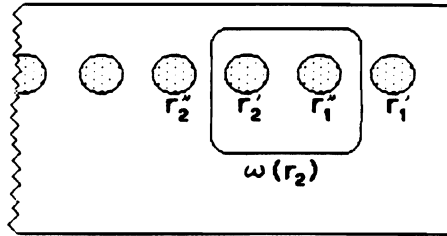


Figure 2.7

$$(r_i)\omega = r_i \text{ for } i \neq 2$$

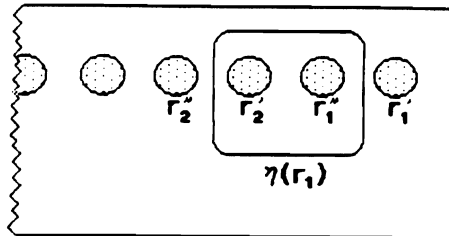


Figure 2.8

$$(r_i)\eta = r_i \text{ for } i \geq 2$$

Proof. Denote by k the number of cutting circles in the disk area Δ bounded by γ in the planar representation of F_g . The lemma will be proved by induction on k .

For $k = 1$, the cutting circle in Δ must be either r_1' or r_1'' . If γ is oriented in the same way as this cutting circle, we may let g be an isotopy deformation, which deforms γ into r_1 . If γ is oriented in the opposite way, follow the isotopy by the operation ϕ , which reverses the orientation of r_1 .

For $k = 2$, by some handle switchings and rotations, i.e. a mapping class generated by ϕ , ψ and T , we may

suppose that the two cutting circles in Δ are r_1'' and r_2' . Connecting the point $P = p_2 \cap r_2'$ and the point $Q = q_1 \cap r_1''$ by a simple arc δ in Δ which intersects neither q_1 nor p_2 (Figure 2.9).

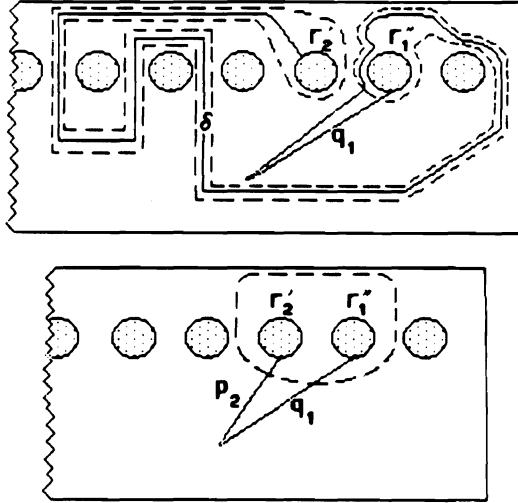


Figure 2.9

If δ does not intersect any other p_i 's and q_i 's, the disk Δ is isotopic to a neighborhood of $r_1'' \cup \delta \cup r_2'$ whose boundary is exactly the circle $(r_1)_{\eta}$ as shown in Figure 2.9. Thus, the lemma is done.

If δ does intersect some p_i 's or q_i 's, we may simplify the intersection by the method we did in Lemma 2.3. Actually, letting $\alpha = p_2\delta$, apply Lemma 2.3 to reduce to the previous case.

For $k \geq 3$, by some handle switchings and rotations, i.e. a mapping class generated by ϕ , ψ and T , we may suppose again that the cutting circles r_1'' and r_2' are in the

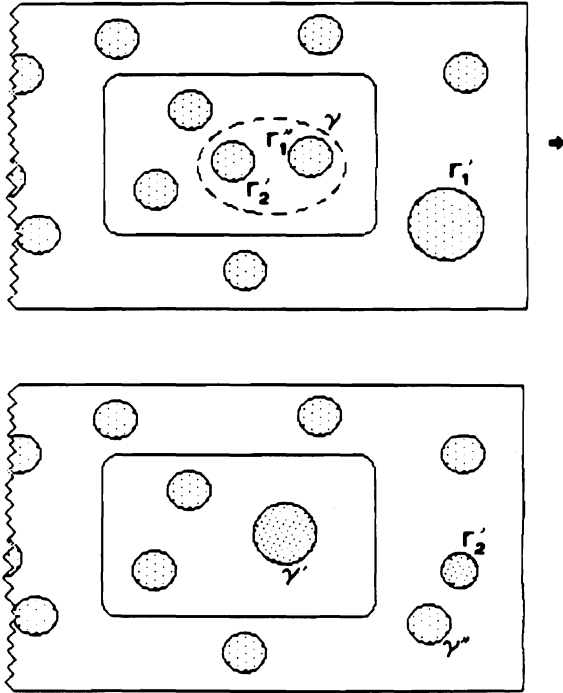


Figure 2.10

domain Δ . Connecting the point $P = p_2 \cap r_2'$ and the point $Q = q_1 \cap r_1''$ by a simple arc δ in Δ which intersects neither q_1 nor p_2 (Figure 2.10), we may choose a disk neighborhood Δ' of $r_1'' \cup \delta r_2'$ contained in the interior of Δ but including no other cutting circles. Applying the case of $k = 2$ to the disk Δ' , the original Δ will be reduced to the case of $k - 1$.

Applying Lemma 2.5 repeatedly, we have the following immediate consequence.

Lemma 2.6. Let f be an arbitrary self-homeomorphism of the handlebody H_g , with the property that, $(r_i)f \cap r_j = \emptyset$, for all $i, j = 1, 2, \dots, g$. Then, there exists another self-homeomorphism g whose homeotopy class is generated by the classes T, ϕ, ψ, θ and η , such that $(r_i)g = (r_i)f$, for $i = 1, 2, \dots, g$.

Lemma 2.7. Let f be an arbitrary self-homeomorphism of the handlebody H_g , then there exists another self-homeomorphism g whose homeotopy class is generated by the classes T, ϕ, ψ, θ and η , such that

$$\left(\bigcup_{i=1}^g (r_i)f \right) \cap \left(\bigcup_{i=1}^g (r_i)g \right) = \emptyset.$$

i.e. none of the circles $(r_i)\bar{g}f$'s intersects a meridian circle of r_i 's.

Proof. Denote by k_i , for $i = 1, 2, \dots, g$, and k the numbers of intersection points given by

$$k_i = \#((r_i)f \cap \left(\bigcup_{j=1}^g r_j \right)) \text{ and}$$

$$k = \sum_{j=1}^g k_i = \# \left(\left(\bigcup_{i=1}^g r_i \right) g f \cap \left(\bigcup_{j=1}^g r_j \right) \right).$$

For $k = 0$, take g to be the identity.

For $k \geq 1$, we may suppose $k_1 \neq 0$, i.e. $(r_1)f \cap (\cup_j r_j) \neq \emptyset$. Consider the meridian disks D_i bounded by the r_i in the solid handlebody H_g , which have nonempty intersection with the disk $(D_1)f$. By an isotopy deformation, we can suppose the set $(D_1)f \cap (\cup_j D_j)$ is a collection of disjoint

arcs in $(D_1)f$. Thus, there is a disk component of $(D_1)f - (\cup_j D_j)$ whose boundary circle is formed exactly by one arc α from $(r_1)f$ and one arc β from $(D_1)f \cap D_s$ for some s (Figure 2.11). In the planar representation of H_g , the disk D_s and the arc β have two copies D'_s, D''_s , and β' and β'' for each of them, and one of the arcs β' and β'' , e.g. β' , together with the arc α forms a simple closed curve.

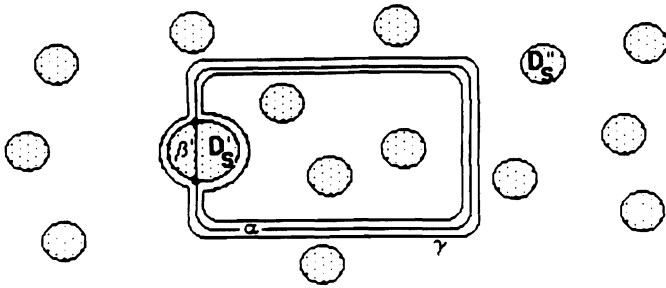


Figure 2.11

Consider the two boundary circles of an annular neighborhood of $D'_s \cup \alpha$ in the representation plane, there is one and only one of them, denoted by γ , separating D'_s and D''_s in two parts. By Lemma 2.5, we may replace r_s by γ without changing other r_i 's by composing some mapping classes generated by M, T, ϕ, ψ, θ and η . Since $\#(\gamma \cap (r_1)f) \leq \#(r_s \cap (r_1)f) - 2$, and $\#(\gamma \cap (r_j)f) \leq \#(r_s \cap (r_j)f)$, for $j \geq 2$, the number k has been reduced by at least two. This completes our lemma.

From Lemmas 2.4, 2.6 and 2.7, we conclude that,

Theorem 2.8. The subgroup K_g is generated by M, T, ϕ, ψ, θ and η .

Proof of Theorem 2.1. All we need is to give the relations between the generators claimed in Theorem 2.1 and the mapping classes M, T, ϕ, ψ, θ and η . By Theorem 1.1 and using some relations from the paper [5], we have the following equations:

$$M = \overline{NLN},$$

$$p^2 = M \cdot \overline{LNL}^2 NL \cdot M,$$

$$\phi = p^2,$$

$$\psi = p^4 N^3,$$

$$\theta = TP^2 \overline{T} \cdot (PQN)^{-1} \cdot p^2,$$

$$\eta = TP^2 \overline{T} \cdot (QNP)^{-1},$$

$$PQN = \overline{N}^3 \cdot PN^2 P \cdot \overline{P}^2 \cdot PNP N,$$

$$QNP = N^3 p^2 \cdot (PN^2 P)^{-1} \cdot p^2,$$

$$\begin{aligned} PN^2 P &= \overline{LNLNLN}^2 \overline{LNLNL} = \overline{LNLN}^3 \overline{LN}^2 LNLNL = \overline{LNLN}^3 \overline{LN}^3 \overline{LNLN} = \\ &= \overline{LN}^2 \overline{LN}^2 \overline{LN} = \overline{LN}^2 \overline{LN}^4 LN = (\overline{NLN}^2 \overline{L})^{-1} \cdot N^6 M, \end{aligned}$$

and

$$\begin{aligned} PNP N &= M \cdot \overline{LNLNLNLNLN}^2 = M \cdot \overline{LN}^2 \overline{LNLNL}^2 LN^2 \cdot N^6 = \\ &M \cdot \overline{LN}^2 \overline{LN} \cdot N^3 \cdot NLN^2 \overline{L}. \end{aligned}$$

By Theorem 2.8 and by the above formulas, Theorem 2.1 is obvious.

Remark. The topological explanation of the generators of K_g is very clear. M is the 360° -twist along the meridian circle a_1 , p^2 and N^3 are the 180° -twists along the circles $[a_1, b_1]$ and $[a_1, b_1][a_2, b_2]$ respectively, T rotates

the handles, and $\bar{N}^3PN^2P = \bar{N}^3LN^3 \cdot \overline{NLN} \cdot M$ is a composition of Dehn twists along the curves $a_1, b_1\bar{a}_1\bar{b}_1a_2b_2\bar{a}_2$ and b_2 , and is also obtained by sliding one foot of the first handle around the longitude b_2 of the second handle (Figure 2.12).

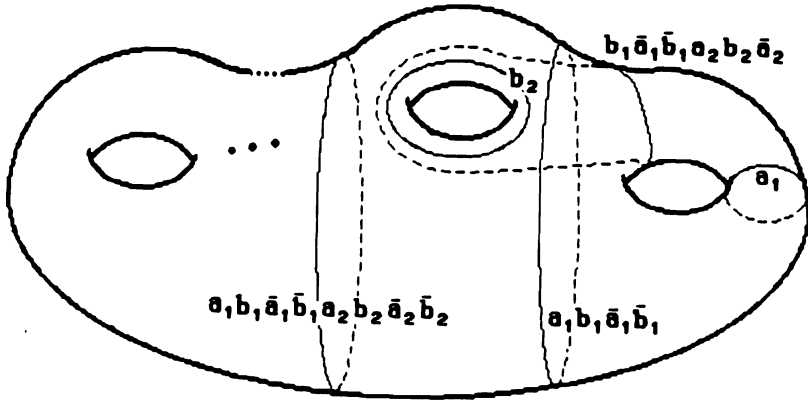


Figure 2.12

3. Heegaard Splitting of the 3-Sphere S^3

Let F_g be the closed orientable surface of genus g embedded unknottedly in S^3 and bounding two handlebodies H_g and H'_g . Let $\mathcal{B} = \{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$ be a system of *basecurves* on the surface F_g based at a basepoint O , such that a_i 's are meridians of the handlebody H_g , and b_i 's are meridians of the handlebody H'_g . Let K_g and K'_g denote the subgroups of the group M_g formed by the mapping classes which can extend to the solid handlebodies H_g and H'_g

respectively. For any mapping class f of M_g , we will denote by

$$M_f = H'_g \cup_f H_g$$

the closed 3-manifold associated by f , formed by identifying each point X of $\partial H'_g$ with the point $(X)f$ of ∂H_g . It is easy to see,

Proposition 3.1. For any mapping classes $f \in M_g$, $h \in K_g$ and $h' \in K'_g$,

$$M_f = M_{h'fh}$$

In particular, Waldhausen ([8]) proved that,

Theorem 3.2. Any genus- g Heegaard splitting of the 3-sphere is an element of the semiproduct of subgroups, $K'_g \cdot K_g$.

We obtained a specific description of K_g in the last section, now we need one for K'_g . In fact, if φ is a homeotopy class induced by a homeomorphism from the handlebody H'_g onto the handlebody H_g , then $K'_g = \varphi K_g \bar{\varphi}$. We will call such a homeotopy class a transfer operation. For example,

Examples 3.3.

(1) the reversion map R is a transfer operation, since

$(a_i)R = b_{g-i+2 \pmod{g}}$, and $(b_i)R = a_{g-i+2 \pmod{g}}$, for any $i = 1, 2, \dots, g$.

(2) the homeotopy class $\pi = (PT)^{\mathcal{G}\bar{T}\mathcal{G}} = P_1P_2\dots P_g$ is a transfer operation, where $P_i = T^{i-1}P\bar{T}^{i-1}$, since

$$(a_i)\pi = a_i b_i \bar{a}_i, \text{ and } (b_i)\pi = \bar{a}_i,$$

for any $i = 1, 2, \dots, g$.

Using the homeotopy class π , we have,

Proposition 3.4. The subgroup $K'_g = \pi K_g \bar{\pi}$ is generated by the mapping classes T, N^3, P^2, PN^2P and L .

Proof. The proposition is an obvious consequence of the following formulas:

$$\pi M \bar{\pi} = P M \bar{P} = L P \bar{P} = L,$$

$$\pi T \bar{\pi} = P_1 P_2 \dots P_g T P_g \dots \bar{P}_2 \bar{P}_1 = P_1 P_2 \dots P_g \bar{P}_1 \bar{P}_g \dots \bar{P}_3 \bar{P}_2 T = T,$$

$$\pi N^3 \bar{\pi} = P_1 P_2 N^3 \bar{P}_1 \bar{P}_2 = P N^3 \bar{P} N^3 \bar{P} N^3 = N^3,$$

$$\pi P^2 \bar{\pi} = P_1 \cdot P^2 \cdot \bar{P}_1 = P^2,$$

$$\begin{aligned} \text{and } \pi P N^2 P \bar{\pi} &= P_1 P_2 \cdot P N^2 P \cdot \bar{P}_2 \bar{P}_1 = P^2 \cdot P_2 N^2 \bar{P}_2 \\ &= P^2 N^3 \cdot P N^2 P \cdot \bar{P}^2 \bar{N}^3. \end{aligned}$$

Denote by N the subgroup of M_g generated by the elements T, P^2, N^3 and PN^2P , which obviously is a subgroup of $K'_g \cap K_g$. Using a result of Powell [6] that the subgroup $K'_g \cap K_g$ is generated by T, N^3, P^2, ω and η , we have the following consequence:

Corollary 3.5.

$$N = K'_g \cap K_g.$$

Theorem 3.6. The associated 3-manifold M_f of a mapping class f is the 3-sphere S^3 if and only if f is an element of the set

$$\langle L, N \rangle \cdot \langle N, M \rangle.$$

Before we end this section, we discuss some more relations among the mapping classes in those subgroups.

Let $L_i = T^{i-1}L\bar{T}^{i-1}$, and $M_i = T^{i-1}M\bar{T}^{i-1}$, for $i = 1, 2, \dots, g$. Let L and M denote the abelian subgroups of rank g generated by the L_i 's and M_i 's respectively.

Proposition 3.7. For any $i = 1, 2, \dots, g$,

- (a) $L_i T = T L_{i-1}$, $M_i T = T M_{i-1}$,
- (b) $L_i P^2 = P^2 L_i$, $M_i P^2 = P^2 M_i$,
- (c) $L_i N^3 = N^3 L_i$, $M_i N^3 = N^3 M_i$, for $i \neq 1, 2$,
 $L_1 N^3 = N^3 L_2$, $M_1 N^3 = N^3 M_2$,
 $L_2 N^3 = N^3 L_1$, $M_2 N^3 = N^3 M_1$,
- (d) $L_i P N^2 P = P N^2 P L_i$, $M_i P N^2 P = P N^2 P M_i$, for $i \neq 1, 2$,
 and $L_1 P N^2 P = P N^2 P L_2$, $M_2 P N^2 P = P N^2 P M_1$.

Proof. Since

$$T = [a_g, b_g, a_1, b_1, \dots, a_{g-1}, b_{g-1}],$$

$$N^3 = [x a_2 \bar{x}, x b_2 \bar{x}, a_1, b_1, \dots, a_3, b_3, \dots, a_g, b_g],$$

$$P^2 = [c_1 \bar{a}_1, \bar{b}_1 \bar{c}_1, a_2, b_2, \dots, a_g, b_g],$$

and $P N^2 P = [x \bar{b}_2 \bar{x} a_1 a_2 b_2 \bar{x}, x b_2 \bar{x}, a_1, \bar{b}_2 \bar{c}_2 b_1, a_3, b_3, \dots, a_g, b_g],$

the proposition is clear.

Proposition 3.8. $L \cap N = 1$, and $M \cap N = 1$.

Proof. Consider the image of L and N in Siegel's modular group ([2]). For any element $f \in N$, f leaves the subspace \mathbb{Z}^g generated by the a^i 's in $H_1(F_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ invariant, by looking at the expressions in the proof of the last proposition. But the only element of L having this property is the identity. Therefore $L \cap N = 1$. And analogously, $M \cap N = 1$.

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