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Kathryn F. Porter*

Introduction

In 1965, E. G. Effros [2] published an important result in the theory of transformation groups which has been used extensively in the study of homogeneity, function spaces, and continua theory. For example, G. Ungar [7] used Effros' Theorem to prove, among other things, that: Every 2-homogeneous metric continuum is locally connected. In this paper, the concept of even homogeneity will be introduced and used to extend a form of Effros' Theorem to allow its application to a larger collection of spaces. We first look at some definitions we shall be using.

If $(X,\mathcal{T})$ is a topological space, then $\mathcal{H}(X)$ is the collection of all self-homeomorphisms on X. A subgroup, G, of $\mathcal{H}(X)$, is *transitive* provided that for each $x \in X$, the set $G(x) = X$, where $G(x) = \{y \in X$: there exists $g \in G$ such that $g(x) = y\}$. A space, X, is *homogeneous* if for any $x, y \in X$ there is a homeomorphism, h, such that $h(x) = y$. A topology $\mathcal{T}'$ for G is called RMC [4] for G provided that, for all $g \in G$, the map, $m_g: G \to G$, defined by $m_g(f) = f \circ g$, is continuous.

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1. ULH Spaces

In 1967, G. Ungar introduced the concept of uniform local homogeneity (ULH) [7] in order to generalize L. Ford, Jr.'s idea of strong local homogeneity (SLH) [3]. A completely regular space, $(X,\mathcal{T})$, is called *uniformly locally homogeneous*, (ULH), provided there exists a uniformity, $\mathcal{U}$, for X such that (1) $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$ and (2) for all $x \in X$ and $U \in \mathcal{U}$, there exists an open neighborhood, O, of x such that if $y \in O$ there is some $g \in \mathcal{H}(X)$ with $g(x) = y$ and graph$(g) \subseteq U$.

Note that graph$(g) \subseteq U$ if and only if $g(x) \in U[x]$ for each $x \in X$. Also, when we want to be more specific, we shall say "$(X,\mathcal{T})$ is ULH w.r.t. $\mathcal{U}$" which means that $\mathcal{U}$ satisfies the properties (1) and (2) above.

One of the nice properties of ULH spaces is that finite products of ULH spaces are ULH [7]. We extend this result to arbitrary products.

Theorem 1.1. *Let* Λ *be an index set of arbitrary cardinality, and for all* $\alpha \in \Lambda$, *let* X_α *be a homogeneous* ULH *space. Let* $X = \prod_{\alpha\in\Lambda} X_\alpha$. *Then* X *is* ULH.

Proof. Assume for all $\alpha \in \Lambda$, X_α is ULH w.r.t. $\mathcal{U}_\alpha$. Let $\mathcal{P}$ be the product uniformity on X. $\mathcal{P}$ has as a base the set $B = \{\bigcap_{\alpha\in F} P_\alpha^{-1}(O_\alpha)$: F is some finite subset of Λ and, for all $\alpha \in F$, $O_\alpha \in \mathcal{U}_\alpha\}$ where the function, $P_\alpha: X \times X \to X_\alpha \times X_\alpha$, is defined by $P_\alpha(x,y) =$

$(\pi_\alpha(x), \pi_\alpha(y)) = (x_\alpha, y_\alpha)$. Let $p \in X$ and let $M \in \mathcal{P}$, then there exists a basis element, $B^* = \bigcap_{\alpha \in F} \mathcal{P}_\alpha^{-1}(O_\alpha)$ such that $(p,p) \in B^* \subseteq M$ and F is some finite subset of Λ. Now, for each $\alpha \in \Lambda$, X_α is ULH w.r.t. $\mathcal{U}_\alpha$, therefore, for each $\alpha \in F$, there exists a ULH neighborhood, V_α, in X_α, such that $p_\alpha \in V_\alpha$. Set $V = \bigcap_{\alpha \in F} \pi_\alpha^{-1}(V_\alpha)$, then V is open in X. If $z \in V$, then for each $\alpha \in F$, $z_\alpha \in V_\alpha$. So, for each $\alpha \in F$, there is some $h_\alpha \in \mathcal{H}(X_\alpha)$ such that $h_\alpha(p_\alpha) = z_\alpha$ and graph $(h_\alpha) \subseteq O_\alpha$. For every $\alpha \in (\Lambda \setminus F)$, since X_α is homogeneous, there exists $\tilde{h}_\alpha \in \mathcal{H}(X_\alpha)$ such that $\tilde{h}_\alpha(p_\alpha) = z_\alpha$. Define $\hat{H}$: $X \to X$ by $\hat{H}(s) = \langle \hat{H}_\alpha(s_\alpha) \rangle_{\alpha \in \Lambda}$ where:

$$\hat{H}_\alpha(s_\alpha) = \begin{cases} h_\alpha(s_\alpha) & \text{if } \alpha \in F \\ \tilde{h}_\alpha(s_\alpha) & \text{if } \alpha \in (\Lambda \setminus F) \end{cases}$$

$\hat{H} \in \mathcal{H}(X)$ and $\hat{H}(p) = (z_\alpha)_{\alpha \in \Lambda} = z$. Show graph$(\hat{H}) \subseteq M$: Let $t = (s, \hat{H}(s)) = ((s_\alpha)_{\alpha \in \Lambda}, (\hat{H}_\alpha(s_\alpha))_{\alpha \in \Lambda})$. For all $\alpha \in F$, $(s_\alpha, \hat{H}_\alpha(s_\alpha)) = (s_\alpha, h_\alpha(s_\alpha)) \in \text{graph}(h_\alpha) \subseteq O_\alpha$. Hence, $t \in \bigcap_{\alpha \in F} \mathcal{P}_\alpha^{-1}(O_\alpha) \subseteq M$. Therefore X is ULH w.r.t. $\mathcal{P}$.

2. Even Homogeneity

Let $(X, \mathcal{T})$ be a topological space and let G be a transitive subgroup of $\mathcal{H}(X)$, with a topology $\mathcal{T}'$. We say that X is *evenly homogeneous w.r.t.* $(G, \mathcal{T}')$, provided that for all open sets $O \in \mathcal{T}'$ such that e, the identity map on X, is in O, then there exists an open cover $\hat{\mathcal{V}}$ of X such that if $x, y \in V \in \hat{\mathcal{V}}$ then there exists $h \in O$ with $h(x) = y$. We shall abbreviate this by E.H. w.r.t. $(G, \mathcal{T}')$. The relationship between even homogeneity and ULH is as follows.

Theorem 2.1. *Let* $(X, \mathcal{U})$ *be a uniform space. Then,* X *is* ULH *w.r.t.* $\mathcal{U}$ *if and only if* X *is* E.H. *w.r.t.* $(\mathcal{H}(X), \mathcal{T}_{\hat{\mathcal{U}}})$ *where* $\mathcal{T}_{\hat{\mathcal{U}}}$ *is the topology induced on* $\mathcal{H}(X)$ *by* $\mathcal{U}$.

Proof. Let $(X, \mathcal{U})$ be a uniform space.

($\Rightarrow$) Assume X is ULH w.r.t. $\mathcal{U}$. Let $O \in \mathcal{T}_{\hat{\mathcal{U}}}$ such that $e \in O$. Then there exists $U \in \mathcal{U}$ with $\hat{U}[e] \subset O$, where $\hat{U} = \{(f,g): (f(x), g(x)) \in U \text{ for all } x \in X\}$. Now, there exists $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \subset U$. So, for all $x \in X$, there exists an open neighborhood, O_x, of x such that if $y \in O_x$, there exists $g \in \mathcal{H}(X)$ with $g(x) = y$ and graph(g) $\subset V$. $\{O_x\}_{x \in X}$ is an open cover of X, so if $p,q \in O_x$ for some $x \in X$, there exist $h_1, h_2 \in \mathcal{H}(X)$ with $h_1(x) = p$, $h_2(x) = q$ and for each $z \in X$, $(z, h_1(z))$, $(z, h_2(z)) \in V$. Hence, $h_2 \circ h_1^{-1}(p) = q$ and $(h_1(z), h_2(z)) \in V \circ V \subset U$, for all $z \in X$. Therefore, $(z, h_2 \circ h_1^{-1}(z)) \in U$, for all $z \in X$. Thus, $h_2 \circ h_1^{-1} \in \hat{U}[e] \subset O$. So, X is E.H. w.r.t. $(\mathcal{H}(X), \mathcal{T}_{\hat{\mathcal{U}}})$.

($\Leftarrow$) Assume X is E.H. w.r.t. $(\mathcal{H}(X), \mathcal{T}_{\hat{\mathcal{U}}})$. Let $x \in X$ and $U \in \mathcal{U}$. Then $e \in \hat{U}[e]$, so there exists an open cover, $\{O_\alpha\}_{\alpha \in \Lambda}$, of X such that if $p, q \in O_\alpha$ for some $\alpha \in \Lambda$, there exists $h \in \hat{U}[e]$ with $h(p) = q$. $x \in X$, hence, there is some $\beta \in \Lambda$ such that $x \in O_\beta$. If $y \in O_\beta$, then there exists $h \in \hat{U}[e]$ such that $h(x) = y$. Recall that $h \in \hat{U}[e]$ if and only if graph(h) $\subset U$. Therefore X is ULH w.r.t. $\mathcal{U}$.

As with ULH spaces, we have a result involving arbitrary products of evenly homogeneous spaces.

Theorem 2.2. *Let* Λ *be an index of arbitrary cardinality. For all* $\alpha \in \Lambda$, *let* (X, T_α) *be a topological space and let* G_α *be a transitive subgroup of* $H(X_\alpha)$. *For each* $\alpha \in \Lambda$, *let* T'_α *be a topology for* G_α *such that* X_α *is E.H. w.r.t.* (G_α, T'_α). *Set* $X = \prod_{\alpha\in\Lambda} X_\alpha$ *and* $G = \prod_{\alpha\in\Lambda} G_\alpha$. *Let* T *and* T_P *be the product topologies on* X *and* G *respectively. Then* X *is E.H. w.r.t.* (G, T_P).

Proof. Let $O \in T_P$ with $e \in O$, where $e = \langle e_\alpha \rangle_{\alpha\in\Lambda}$ and e_α is the identity map on X_α. Then there exists a basic open set, $B = \bigcap_{\alpha\in F} \pi_\alpha^{-1}(U_\alpha)$ where F is a finite subset of Λ, U_α is open in G_α for all $\alpha \in F$, and $e \in B \subset O$. Now, for all $\alpha \in \Lambda$, X_α is E.H. w.r.t. (G_α, T'_α). So, for all $\alpha \in F$, there is some open cover, $\hat{W}_\alpha = \{W^\alpha_\beta\}_{\beta\in\Gamma_\alpha}$ such that if $p, q \in W^\alpha_\beta$ for some $\alpha \in F$ and some $\beta \in \Gamma_\alpha$, there exists some $h_{\alpha,\beta} \in U_\alpha$ with $h_{\alpha,\beta}(p) = q$. Then, define $W = \{\bigcap_{\alpha\in F} \pi_\alpha^{-1}(W^\alpha_\beta) : \beta \in \Gamma_\alpha\}$. Note that W is an open cover of X. If $s, t \in \bigcap_{\alpha\in F} \pi_\alpha^{-1}(W^\alpha_\beta)$ then for all $\alpha \in F$, $s_\alpha, t_\alpha \in W^\alpha_\beta$. So, for each $\alpha \in F$, there exists some $h_{\alpha,\beta} \in U_\alpha$ such that $h_{\alpha,\beta}(s_\alpha) = t_\alpha$. Also, since G_α is transitive for each $\alpha \in \Lambda$, we have that for each $\alpha \in (\Lambda \setminus F)$, there is an $h_\alpha \in G_\alpha$ such that $h_\alpha(s_\alpha) = t_\alpha$. Define the function $\hat{H}: X \to X$ by $\hat{H}(x) = \langle g_\alpha(x_\alpha) \rangle_{\alpha\in\Lambda}$ where

$$g_\alpha = \begin{cases} h_{\alpha,\beta} & \text{if } \alpha \in F \\ h_\alpha & \text{if } \alpha \notin F \end{cases}$$

Thus, $\hat{H} \in G$, $\hat{H}(s) = t$, and $\hat{H} \in B \subset O$. Therefore X is E.H. w.r.t. (G, T_P).

3. Evaluation Maps

Let (X,T) be a topological space and let G be a subgroup of $H(X)$ with a topology, T'. For each $x \in X$ we define the *evaluation map*, $E_x: (G,T') \to (X,T)$ by $E_x(g) = g(x)$. Ungar has shown the following about the evaluation map.

Theorem 3.1. [7] *Let* $(X,\mathcal{U})$ *be a homogeneous uniform space and let* $H(X)$ *be given the induced uniform topology,* $T_{\hat{\mathcal{U}}}$. *Then, for each* $x \in X$, *the evaluation map,* E_x, *is open if an only if* X *is ULH w.r.t.* $\mathcal{U}$.

For even homogeneity we have the following results for the evaluation map.

Theorem 3.2. *Let* (X,T) *be a topological space and let* G *be a transitive subgroup of* $H(X)$. *Let* T' *be an RMC topology for* G. *If* X *is E.H. w.r.t.* (G,T') *then for each* $x \in X$, *the evaluation map,* $E_x: (G,T') \to X$, *is open.*

Proof. Assume X is E.H. w.r.t. (G,T'). Let $x \in X$ and let $O \in T'$. Suppose $y \in E_x(O)$, then there is some $h \in O$ such that $h(x) = y$. Then $h \in O$ implies that $e \in Oh^{-1} \in T'$, since T' is RMC. By even homogeneity there exists some open cover, $\{O_\alpha\}_{\alpha\in\Lambda}$, of X such that if $p,q \in O_\alpha$, for some $\alpha \in \Lambda$, then there is some $g \in Oh^{-1}$ with $g(p) = q$. But $y \in X$, so there exists a $\beta \in \Lambda$ such that $y \in O_\beta$ and if $r \in O_\beta$, there is an $f \in Oh^{-1}$ such that $f(y) = r$. Whence, $O_\beta \subset E_x(O)$ since $f \circ h \in O$ and

$f \circ h(x) = f(y) = r$. Therefore, for all $x \in X$, the map $E_x: (G,T') \to X$, is open.

Corollary 3.1. *Let* X *be a topological space and let* G *be a transitive subgroup of* $H(X)$. *Let* T *be an RMC topology for* G *such that* X *is E.H. w.r.t.* (G,T). *Then for all topologies,* T^*, *for* G *with* $T^* \subset T$, *we have that for each* $x \in X$, *the evaluation map,* $E_x: (G,T^*) \to X$, *is open.*

Corollary 3.2. *Let* (X,T) *be a topological space and let* G *be a transitive subgroup of* $H(X)$. *Let* T' *be a topology for* G *such that* (G,T') *is a topological group. Then,* X *is E.H. w.r.t.* (G,T') *if and only if for each* $x \in X$, *the evaluation map,* $E_x: (G,T') \to X$, *is open.*

The following is an example of a topology for a function space for which the evaluation map is only open under extreme circumstances.

Example 3.1. Let (X,T) and $(Y,\tilde{T})$ be topological spaces. For $O \subset X$ and $U \subset Y$, define the set $(O,U) = \{f \in Y^X: f(O) \subset U\}$. Let $S_{oo} = \{(O,U): O \in T \text{ and } U \in \tilde{T}\}$. Then S_{oo} is a subbasis for a topology, T_{oo}, on Y^X, called the *open-open topology* [5].

We shall show that when X is a T_1 space, the evaluation map, $E_x: (H(X),T_{oo}) \to X$, is open for each $x \in X$, if and only if X is discrete: Assume that X is T_1. Let $x \in X$. Then $\{x\}$ is closed so that $(X\setminus\{x\})$ is open. Note that the set $V = ((X\setminus\{x\}), (X\setminus\{x\})) = (\{x\}, \{x\}) \in \mathcal{T}_{oo}$. Thus, $E_x(V) = \{x\}$. Since x was an arbitrary point in X, hence $E_x(V)$ is open if and only if X is discrete.

The reason that the evaluation map, in the case when X is not discrete, fails to be open, is that X is not E.H. w.r.t. $(\mathcal{H}(X), \mathcal{T}_{oo})$: Fix an x in X. We have $(\{x\},\{x\}) \in \mathcal{T}_{oo}$ and $e \in (\{x\},\{x\})$. Suppose that V^* is an open cover of X. Therefore, there exists $V \in V^*$ such that $x \in V$. If $y \in V$, $y \neq x$, there is no $h \in (\{x\},\{x\})$ with $h(x) = y$, since if $h \in (\{x\},\{x\})$ then $h(x) = x$.

We now give an example of a space which is E.H. w.r.t. $(G,\mathcal{T})$, but which is not ULH. This example appeared in Ford's paper [3] as an example of a space which possesses a transitive homeomorphism group with no reasonable topology.

Example 3.2. Let $X = R^2$ and let basic open neighborhoods of a point $(x_0,y_0) \in X$ be of the form: $N_\varepsilon(x_0,y_0) = \{(x,y): x_0 < x < x_0 + \varepsilon \text{ and } y_0 - \varepsilon < y < y_0 + \varepsilon\} \cup \{(x_0,y_0)\}$ where $\varepsilon > 0$. Define $\hat{N}_\varepsilon(x_0,y_0) = N_\varepsilon(x_0,y_0)\setminus \{(x_0,y_0)\}$.

X is T_2 but not regular since the point (0,0) cannot be separated from the set $C = \{(0,y): y \neq 0\}$ which is closed in X. Hence X is not uniformizable and thus not ULH.

Let G be the set of all translations, i.e., $G = \{h: X \to X:$ there exists $a, b \in R$ such that for all $(x,y) \in X$, $h(x,y) = (x + a, y + b)\}$.

Give G the topology, T, whose subbasic open sets are of the form $(\{(p,q)\}, \hat{N}_\varepsilon(s,t)) = \{f \in G: f(p,q) \in \hat{N}_\varepsilon(s,t)\}$, where $(p,q) \in R^2$.

Claim. X is E.H. w.r.t. (G,T): Let O be open in G such that $e \in O$. Then there exists a basic open set, say $B = \bigcap_{i=1}^{n} (\{(x_i,y_i)\}, \hat{N}_{\varepsilon_i}(p_i,q_i))$ with $e \in B \subset O$. Note that, without loss of generality, $(x_i,y_i) \neq (x_j,y_j)$ if $i \neq j$. Also we can choose the ε_i small enough so that the sets $\hat{N}_{\varepsilon_i}(p_i,q_i)$ do not intersect since X is T_2. Now $e \in B$, so we know that for all $i = 1,2,3,\ldots,n$, $(x_i,y_i) \in \hat{N}_{\varepsilon_i}(p_i,q_i)$. For each $i = 1,2,3,\ldots,n$, let $\delta_i = \min\{\varepsilon_i + p_i - x_i, \varepsilon_i - |y_i - q_i|\}$ and let $\varepsilon = \min_{i \in \{1,2,3,\ldots,n\}} \delta_i$. Let $U = \{\hat{N}_\varepsilon(x,y): (x,y) \in X\}$. U is an open cover of X and let $(p,q), (s,t) \in \hat{N}_\varepsilon(\hat{x},\hat{y})$. Then define the function, $h \in G$ by $h(x,y) = (x - s + p, y - t + q)$. $h(s,t) = (p,q)$. So for each $i = 1,2,3,\ldots,n$, $h(x_i,y_i) = (x_i - s + p, y_i - t + q)$. Thus, $|p_i - (x_i - s + p)| \leq |p_i - x_i| + |p - s| < (x_i - p_i) + \varepsilon < \varepsilon_i$ also $|q_i - (y_i - t + q)| \leq |q_i - y_i| + \varepsilon < \varepsilon_i$. Hence, $h \in B \subset O$. Therefore X is E.H. w.r.t. (G,T).

4. Effros' Theorem

The following theorem is the form of Effros' Theorem [2] which is most often used. Recall that the *compact-open topology*, T_{co}, on $H(X)$, has subbasis elements of the form $(C,U) = \{g \in \mathcal{H}(X): g(C) \subset U\}$, where C is a compact subset of X and U is an open set in X. When X is a compact metric space, T_{co}, is the topology on $\mathcal{H}(X)$ induced by the sup metric.

Theorem 4.1. (Effros' Theorem) [1] *Let* (X,T) *be a non-degenerate, compact, homogeneous, metric space. Then for all* $x \in X$, *the evaluation map,* $E_x: (\mathcal{H}(X), T_{co}) \to (X,T)$ *is open.*

We now extend Effros' result to uncountable products of compact, homogeneous, metric spaces, by using the concept of even homogeneity. Note that an uncountable product of metrizable spaces is not metrizable, so that our new result is not covered by Effros' Theorem. First, though, we need the following theorem.

Theorem 4.2. *Let* X *be a topological space and let* G *be a transitive subgroup of* $\mathcal{H}(X)$ *such that* X *is E.H. w.r.t.* (G,T). *Let* H *be a subgroup of* $\mathcal{H}(X)$ *such that* $G \subset H$. *Let* $\hat{T}$ *be a topology on* H *and let* T_S *be the subspace topology on* G *inherited from* $(H,\hat{T})$. *If* $T_S \subset T$ *then* X *is E.H. w.r.t.* $(H,\hat{T})$.

Proof. Let $O \in \tilde{T}$ such that $e \in O$. Then $e \in O \cap G \in T_S \subset T$. Since X is E.H. w.r.t. (G,T), there exists an open cover, $\hat{V}$, of X such that if $p,q \in V$ for some $V \in \hat{V}$, then there is some $h \in O \cap G$ with $h(p) = q$. $O \cap G \subset O$, hence X is E.H. w.r.t. $(H,\hat{T})$.

Finally we are ready to prove the desired theorem.

Theorem **4.3.** *Let* Λ *be an uncountable index set. For all* $\alpha \in \Lambda$*, let* X_α *be a nondegenerate, homogeneous, compact, metric space. Let* $X = \prod_{\alpha\in\Lambda} X_\alpha$ *and give* X *the product topology,* T. *Then for each* $x \in X$*, the evaluation map,* E_x: $(H(X), T_{co}) \to X$ *is open.*

Proof. From Effros' Theorem, we have that for each $\alpha \in \Lambda$ and for all $x_\alpha \in X_\alpha$, the evaluation map, E_{x_α} : $(H(X_\alpha), T_{co}) \to X_\alpha$, is open. We know that $(H(X_\alpha), T_{co})$ is a topological group, since X_α is compact Hausdorff. Hence, Corollary 3.2 gives us that for each $\alpha \in \Lambda$, X_α is E.H. w.r.t. $(H(X_\alpha), T_{co})$. Set $H = \prod_{\alpha\in\Lambda} H(X_\alpha)$. Then by Theorem 2.2, we have that X is E.H. w.r.t. (H, T_p) where T_p is the product topology on H. Let T_S be the subspace topology on H as a subspace of $(H(X), T_{co})$.

Claim. $T_S \subset T_P$ on H: (This Claim is a known result. However, the proof is given here for completeness.) Let B be a subbasic open set in T_S. Without loss of generality, we can choose $B = (C,V) \cap H$ where C is a compact subset of X and V is open in X. Let $f \in B$. Then $f(C) \subset V$ and $f = \langle f_\alpha \rangle_{\alpha\in\Lambda}$. So for all $x \in C$, there exists a basic open set, $O_x = \bigcap_{\alpha\in F_x} \pi_\alpha^{-1}(O_\alpha)$ where F_x is a finite subset of Λ and, for all $\alpha \in F_x$, O_α is open in X_α, and such that $f(x) \in O_x \subset V$. By regularity, for each $x \in C$ and for each $\alpha \in F_x$, there exists some $\varepsilon_{x_\alpha} > 0$ with

$f_\alpha(x_\alpha) \in \mathcal{B}(f_\alpha(x_\alpha),\varepsilon_{x_\alpha}) \subset Cl_{X_\alpha}\mathcal{B}(f_\alpha(x_\alpha),\varepsilon_{x_\alpha}) \subset O_\alpha$. Thus, for all $x \in C$ and for all $\alpha \in F_x$, there exists some $\delta_{x_\alpha} > 0$ such that $x_\alpha \in \mathcal{B}(x_\alpha,\delta_{x_\alpha}) \subset Cl_{X_\alpha}\mathcal{B}(x_\alpha,\delta_{x_\alpha}) \subset f_\alpha^{-1}(\mathcal{B}(f_\alpha(x_\alpha), \varepsilon_{x_\alpha})) \subset f_\alpha^{-1}(Cl_{X_\alpha}\mathcal{B}(f_\alpha(x_\alpha), \varepsilon_{x_\alpha})) \subset f_\alpha^{-1}(O_\alpha)$. Hence, for all $x \in C$ and for all $\alpha \in F_x$, $f_\alpha \in (Cl_{X_\alpha}\mathcal{B}(x_\alpha,\delta_{x_\alpha}), \mathcal{B}(f_\alpha(x_\alpha), \varepsilon_{x_\alpha})) \in \mathcal{T}_{co}$ on X_α.

Now $C \subset \bigcup_{x\in C}\left[\bigcap_{\alpha\in F_x} \pi_\alpha^{-1}(\mathcal{B}(x_\alpha,\delta_{x_\alpha}))\right]$. C is compact, so there exists a finite subcover, say

$$\begin{aligned}
C &\subset \bigcup_{i=1}^{n}\left[\bigcap_{\alpha\in F_{x_i}} \pi_\alpha^{-1}(\mathcal{B}(x_\alpha^i,\delta_{x_\alpha^i}))\right] \\
&\subset \bigcup_{i=1}^{n}\left[\bigcap_{\alpha\in F_{x_i}} \pi_\alpha^{-1}(Cl_{X_\alpha}\mathcal{B}(x_\alpha^i,\delta_{x_\alpha^i}))\right] \\
&\subset \bigcup_{i=1}^{n}\left[\bigcap_{\alpha\in F_{x_i}} \pi_\alpha^{-1}\circ f_\alpha^{-1}(\mathcal{B}(f_\alpha(x_\alpha^i), \varepsilon_{x_\alpha^i}))\right] \\
&\subset \bigcup_{i=1}^{n}\left[\bigcap_{\alpha\in F_{x_i}} \pi_\alpha^{-1}\circ f_\alpha^{-1}(Cl_{X_\alpha}\mathcal{B}(f_\alpha(x_\alpha^i), \varepsilon_{x_\alpha^i}))\right] \\
&\subset \bigcup_{i=1}^{n}\left[\bigcap_{\alpha\in F_{x_i}} \pi_\alpha^{-1}\circ f_\alpha^{-1}(O_\alpha)\right] \\
&= \bigcup_{i=1}^{n}\left[\bigcap_{\alpha\in F_{x_i}} f^{-1}\circ \pi_\alpha^{-1}(O_\alpha)\right]
\end{aligned}$$

$$= \bigcup_{i=1}^{n} f^{-1}\left(\left(\bigcap_{\alpha \in F_{x_i}} \pi_\alpha^{-1}(O_\alpha) \right) \right)$$

$$= \bigcup_{i=1}^{n} f^{-1}(O_{x_i}) \subset f^{-1}(V)$$

Define $L = \bigcap_{\alpha \in F} \pi_\alpha^{-1}(\mathrm{Cl}_{X_\alpha} B(x_\alpha, \delta_{x_\alpha}), B(f_\alpha(x_\alpha), \varepsilon_{x_\alpha}))$ where $F = \bigcup_{i=1}^{n} F_{x_i}$, then $f \in L \in T_P$. If $g \in L$ then for all $\alpha \in F$, $g_\alpha(\mathrm{Cl}_{X_\alpha} B(x_\alpha, \delta_{x_\alpha})) \subset B(f_\alpha(x_\alpha), \varepsilon_{x_\alpha})$. If $t \in C$, there exists some k such that $t \in \bigcap_{\alpha \in F_{x_k}} \pi_\alpha^{-1}(B(x_\alpha^k, \delta_{x_\alpha^k})) \subset \bigcap_{\alpha \in F_{x_k}} \pi_\alpha^{-1}(\mathrm{Cl}_{X_\alpha} B(x_\alpha^k, \delta_{x_\alpha^k}))$. Thus, for all $\alpha \in F_{x_k} \subset F$, $t_\alpha \in \mathrm{Cl}_{X_\alpha}(B(x_\alpha^k, \delta_{x_\alpha^k}))$ which implies that for all $\alpha \in F_{x_k}$, $g_\alpha(t_\alpha) \in B(f_\alpha(x_\alpha^k), \varepsilon_{x_\alpha^k})$. So, $g(t) \in \bigcap_{\alpha \in F_{x_k}} \pi_\alpha^{-1}(B(f_\alpha(x_\alpha^k), \delta_{x_\alpha^k}) \subset O_{x_k} \subset V$. Thus, $g \in (C,V) \cap H$. Hence, $T_S \subset T_P$.

Therefore, by Theorem 4.2, X is E.H. w.r.t. $(H(X), T_{CO})$, and so, Theorem 3.2 implies that $E_x\colon (H(X), T_{CO}) \to X$, is open, for each $x \in X$.

Some spaces which are included in the above case, which were not covered by Effros' Theorem are uncountable products of unit circles, S^1, and uncountable products of [0,1]'s. Note that [0,1] is not homogeneous, but the Hilbert Cube, Q, which is a countable product of [0,1]'s is homogeneous and an uncountable product of [0,1]'s can be written as an uncountable product of Q's.

References

[1] D. P. Bellamy, *Some Topics in Modern Continua Theory*, Continua, Decompositions, Manifolds, Proceedings of Texas Topology Symposium 1980, 1983, pp. 1-26.

[2] E. G. Effros, *Transformation Groups and C*-Algebras*, Ann. of Math., Vol. 81, 1965, pp. 38-55.

[3] L. R. Ford, Jr., *Homeomorphism Groups and Coset Spaces*, Trans. Amer. Math. Soc., Vol. 77, 1954, pp. 490-497.

[4] K. F. Porter, *Set-Set Topologies and Semitopological Groups*, Inter. J. Math. and Math. Sci., Vol. 12 No. 4 (1989), pp. 669-672.

[5] K. F. Porter, *The Open-Open Topology for Function Spaces*, preprint.

[6] K. F. Porter, *Evaluation Maps on Groups of Self-homeomorphisms*, Ph.D. Dissertation, 1987, University of Delaware.

[7] G. S. Ungar, *Local Homogeneity*, Duke Math. J., Vol. 34, 1967, pp. 693-700.

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