TOPOLOGY PROCEEDINGS

Volume 13, 1988

Pages 365–369

http://topology.auburn.edu/tp/

THE HOMOGENEITY OF SMALL MANIFOLDS

by

Stephen Watson

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

THE HOMOGENEITY OF SMALL MANIFOLDS

Stephen Watson*

1. Introduction

A separable space X is said to be *countable dense homogeneous* if, for any two countable dense subsets A and B in X, there is an autohomeomorphism f of X such that f(A) = B.

A classical result (see [3]) is that any Euclidean space $\mathbb{R}^n$ is countable dense homogeneous.

R. Bennett [1] and, independently, C. Bessaga and A. Pelczynski [2] showed that any manifold of countable weight is countable dense homogeneous (a manifold is any connected topological space for which there is an integer n and an open cover of homeomorphs of $\mathbb{R}^n$). If X is a separable manifold, then $\aleph_0 \leq w(X) \leq 2^{\aleph_0}$. In [4], Steprans and Zhou constructed, by diagonalization, a separable manifold of weight $2^{\aleph_0}$ which is not countable dense homogeneous and observed that the results of [1] and [2] needed only that $w(X) < b$ where b is the least cardinality of an unbounded family in ω^ω (mod finite). In an early version of [4], Steprans and Zhou conjectured that separable manifolds of weight less than continuum might have to be countable dense homogeneous. The purpose of this paper is to construct a separable manifold of

* Supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

weight $\aleph_1$ which is not countable dense homogeneous by means of $\aleph_1$ Cohen reals added to the universe. The use of forcing to define neighborhoods in a manifold is perhaps the most interesting part of the paper.

2. The Construction

Let A be the unit open disc.

Let ∂A be the boundary of A in R^2.

Let D and E be disjoint countable dense subsets of A.

For each $x \in \partial A$, let $L_x : [0,1] \to A \cup \{x\}$ be a continuous injection such that $\mathrm{rng}(L_x \cap E) = \emptyset$ and $L_x(0) = x$ and $L_x(\frac{1}{n}) \in D$. We shall define $M_x : [-1,1] \times [0,1] \to A \cup \{x\}$ to be a continuous mapping such that

1. $M_x \restriction ([-1,1] \times (0,1])$ is an injection
2. $M_x \restriction (\{0\} \times [0,1]) \simeq L_x$ in the canonical way
3. $M_x \restriction ([-1,1] \times \{0\}) \equiv x$

M_x will be defined using $f_x : [0,1] \to [0,1]$ which is a continuous mapping such that $f_x^{-1}(0) = \{0\}$. The counterexample is $X = A \cup ([0,1) \times Y)$ where $Y \subset \partial A$. A is an open subspace of X with the Euclidean topology. $[0,1) \times Y$ as a subspace is the free union of copies of $[0,1)$ with the Euclidean topology.

We need to define how basic open neighborhoods of $[0,1) \times Y$ intersect A.

Let $x \in Y$ and $(r,s) \subset (0,1)$. We declare $((r,s) \times \{x\}) \cup M_x(((r,s) \cup (-s,-r)) \times (0,s - r))$ to be open.

Let $x \in Y$ and $[0,r) \subset [0,1)$. We declare $([0,r) \times \{x\}) \cup M_x((-r,r) \times (0,r))$ to be open

This defines a topology on X which depends on the choice of function f_x.

We shall add $\aleph_1$ Cohen reals to the universe V. Y is any set of cardinality $\aleph_1$ in V. Each d_x, L_x is an element of V but M_x is not an element of V. Any Cohen real added to the universe adds canonically an increasing continuous function $f\colon (0,1] \to (0,1]$ such that if $g\colon (0,1] \to (0,1]$ and $g < f$ then $g \notin V$. List the Cohen reals with index set Y and list the associated increasing functions as $\{f_x\colon x \in Y\}$. Let $P_x\colon [-1,1] \times [0,1] \to A \cup \partial A$ be a homeomorphism such that $P_x \lceil (\{0\} \times [0,1]) \simeq L_x$ (in the canonical way). Actually we need $L_x(1) \in \partial A$ to do this but do not mention it earlier as it is used only to simplify the proof.

Let $\alpha_n \searrow 0$ where $\alpha_0 = 1$. Find a continuous $p\colon [-1,1] \times [0,1] \to [-1,1]$ such that $p(\{0\} \times [0,1]) \equiv 0$ and $p([-1,1] \times [\alpha_{n+1},\alpha_n] \subset [-f_x(\alpha_n),f_x(\alpha_n)]$. Define M_x by $M_x(\alpha,\beta) = P_x(p(\alpha,\beta),\beta)$.

3. Geometric Details

Reading descriptions of manifolds can be difficult because writing out details of evident geometric facts can turn into masses of notation. Why can L_x be defined?

Let $\{O_n\colon n \in \omega\}$ be a sequence of open sets converging to x such that the line segment L between any point in O_n and any point a in O(n+1) "approaches" x. That is, if l and m are points in L and l is closer to a than m then l is closer to x than m.

Choose $L_x(\frac{1}{2n}) \in O_n \cap D$. For each $n \in \omega$, let R be a copy of [0,1] perpendicular to the line segment between $L_x(\frac{1}{2n})$ and $L_x(\frac{1}{2n+2})$. For each $r \in R$, let T(r) be the union of the line segment between $L_x(\frac{1}{2n})$ and r and the line segment between $L_x(\frac{1}{2n+2})$ and r.

$\{T(r): r \in R\}$ is a disjoint family except for the points $L_x(\frac{1}{2n})$ and $L_x(\frac{1}{2n+2})$. At least one of these, say $T(r_0)$ is disjoint from E. Let $r_0 = L_x(\frac{1}{2n+1})$ and let $L_x \upharpoonright ([\frac{1}{2n+2}, \frac{1}{2n}])$ be a one-to-one mapping onto T_{r0}.

4. D cannot be mapped onto E

Let $h: X \to X$ be an autohomeomorphism in the generic extension V[G] by all $\aleph_1$ Cohen reals such that $h(D) = E$. Now $h\upharpoonright D$ is a countable subset of $D \times E$ which determines h and so there is an intermediate generic extension V[H] by countably many Cohen reals such that $h \in V[H]$. Let $x \in Y$ be such that the xth Cohen real is generic over V[H].

Let $y = h^{-1}(x)$. There is a sequence $S \subset D$ in V[H] which converges to y since D is dense. By continuity, $h(S) \subset E$ must converge to x. The problem is that no infinite subset of E in V[H] is even contained in any of the basic open sets about X.

To see this, let $h(S) = \{P_x(\delta_n, r_n) : n \in \omega\}$. Note that $|\delta_n| > 0$ and that this definition takes place in V[H]. If $P_x(\delta_n, r_n)$ lies in any basic open set and $r_n \in [\alpha_{m(n)+1}, \alpha_{m(n)}]$ then $|\delta_n| < f(\alpha_{m(n)})$. Define $g \in V[H]$ by extending $g(r_n) = |\delta_{n+1}|$ to a continuous increasing function from (0,1] to (0,1]. Consider any $\beta \in (0,1]$. Find $n \in \omega$ such that $r_{n+1} < \beta \leq r_n$. We can calculate $g(\beta) \leq g(r_n) = | \delta_{n+1} | < |f(\alpha_{m(n+1)})| < | f(\alpha_{m(n)+1}) | < f(r_{n+1}) < f(\beta)$. We see that $g(\beta) < f(\beta)$ for any β. This contradicts $g \in V[H]$.

References

[1] R. Bennett, *Countable dense homogeneous spaces*, Fund. Math., 74:189-194, 1972.

[2] C. Bessaga and A. Pelczynski, *Selected topics in infinite-dimensional topology*, Polish Scientific Publishers, 1975.

[3] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1948.

[4] J. Steprans and H. Zhou, *Some results on CDH spaces*-I, Topology and its Applications, 28(1988) 147-154.

York University

North York, CANADA