TOPOLOGY PROCEEDINGS

Volume 13, 1988

Pages 371–384

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

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1. Preliminaries.

Any space X in which no nonvoid open subset is meager is called a *Baire* space. In this paper, we will consider only T_2 Baire spaces. Given a space X, we will use $\mathcal{O}(X)$ to denote the set of all open subsets of X. If $\mathcal{F}(X)$ is any family of subsets of X, $\mathcal{F}^+(X)$ will denote $\mathcal{F}(X) \setminus \{\emptyset\}$. Thus the expression $U \in \mathcal{O}^+(X)$ indicates that U is a nonvoid open subset of X. We will denote the collection of meager subsets of X by $\mathcal{M}(X)$. For any $A \subset X$, M(A) will denote the set of meager points of A, i.e., the set of all $x \in X$ such that for some open neighborhood U of x, $U \cap A$ is meager. M(A) is an open subset of X and $A \cap M(A)$ is meager in X.

* This paper was submitted while I was a graduate student at the University of Delaware. I am grateful to Professor John C. Oxtoby of Bryn Mawr who graciously read the first version of this paper and offered some helpful comments. In particular, I would like to mention that the original version of Lemma 2.2 was incorrect. I also wish to thank the referee for his many helpful comments, corrections, and considerable patience; he will see his influence. I trust Professor Oxtoby will see the influence of his work.

We will denote $X\setminus M(A)$, the set of nonmeager points of A, by D(A). D(A) is locally second category in X. ($S \subset X$ is *locally second category* if for all $U \in \mathcal{O}^+(X)$, $U \cap S \neq \emptyset$ implies that $U \cap S$ is second category.) Note that A is locally second category if an only if $A \subset D(A)$. A set $A \subset X$ has the *Baire property* if there exists an open set $O \subset X$ such that the symmetric difference, $A \Delta O$, is meager. We will use $\mathcal{B}(X)$ to denote the σ-algebra of all subsets of X having the Baire property. By a *system (relative to X)* we will mean any subset $\mathcal{S}$ of the power set, P(X), of X such that if $S \in \mathcal{S}$ then $X\setminus S \in \mathcal{S}$. If every set in a family of sets has a particular property then we will say that the family has that property. For example, if for all $S \in \mathcal{S}$, $cl(S) = X$, then we will say that $\mathcal{S}$ is a dense family.

Any set which is either meager or residual or else fails to have the Baire property is *strongly transitive*. When applied to a system, this clearly agrees with the original definition of strong transitivity found in [2], namely, given a system $\mathcal{S}$ and any element $S \in \mathcal{S}$ having the Baire property, either S or $X\setminus S$ is meager. Hence, if $\mathcal{S}$ is not strongly transitive, there is an $S \in \mathcal{S} \cap \mathcal{B}(X)$ such that both S and $X\setminus S$ are nonmeager. In this case, there must be disjoint, nonvoid open sets U and V with $cl(U) \cup cl(V) = X$ such that both $S \Delta U$ and $(X\setminus S) \Delta V$ are meager in X.

A *partition* of X is any pairwise disjoint collection of nonvoid subsets of X whose union equals X. The notation "$X \twoheadrightarrow P$," read "X is partitioned into P," will be used to indicate that P is a partition of X. The *system generated by a partition* P of X, denoted by $\langle P \rangle$, is precisely all unions of members of P. When it is said that P is *strongly transitive* (respectively, *dense, etcetera*), it means that $\langle P \rangle$ is strongly transitive (respectively, dense, etcetera). By a *$\star$-partition*, denoted by $X \twoheadrightarrow \star\text{-}P$, we will mean a dense, meager partition. One example of a *$\star$-partition* is the Vitali partition of the unit interval I (see [4]). Another example is the partition of a metric indecomposable continuum into its composants. (The *composant* of a point $x \in X$ is the union of all the proper subcontinua of X which contain x.) Let C denote the class of metric indecomposable continua. For all $X \in C$, $\tilde{X}$ will denote the system generated by the composants of X.

2. Category Preserving Maps.

Let X and Y be spaces. A map $f\colon X \to Y$ is *category preserving* if for any set $A \subset Y$, $f^{-1}(A)$ is meager if and only if A is meager. If instead, for all $A \subset X$, $f(A)$ is meager if only if A is meager, then f will be called *strongly category preserving*, abbreviated by *s.c.p.* To facilitate discussion, the following definitions will be employed: f will be called

B1 if for every meager set $M \subset X$, $f(M)$ is meager in Y,

B2 if for every meager set $M \subset Y$, $f^{-1}(M)$ is meager in X, B3 if for every nonmeager set $S \subset X$, $f(S)$ is nonmeager in Y,

B4 if for every nonmeager set $S \subset Y$, $f^{-1}(S)$ is nonmeager in X,

B5 if for every residual set $R \subset X$, $f(R)$ is residual in Y,

B6 if for every residual set $R \subset Y$, $f^{-1}(R)$ is residual in X.

Lemma 2.1. *For any map* $f: X \to Y$ *the following are valid.*

(i) f *is* B2 $\Leftrightarrow$ f *is* B3 $\Leftrightarrow$ f *is* B6.

(ii) *If* f *is onto then* f *is* B1 $\Rightarrow$ f *is* B4 $\Leftrightarrow$ f *is* B5 $\Leftrightarrow$ *for every meager* $M \subset X$ *with* $M = f^{-1}(f(M))$, $f(M)$ *is meager. The converse of the first implication fails.*

(iii) f *is* s.c.p. $\Leftrightarrow$ f *is* B1 *and* B2 $\Leftrightarrow$ f *is* B1 *and* B3 $\Leftrightarrow$ f *is* B1 *and* B6. *Moreover, no other pair of properties* B1 - B6 *is equivalent to* s.c.p.

Proof.

(i) Suppose f is B2. Let S be second category in X. If $f(S)$ were meager then $f^{-1}(f(S))$ would be meager, a contradiction since $S \subset f^{-1}(f(S))$. Thus f is B2 $\Rightarrow$ f is B3. Now suppose f is B3 and let R be a residual set in Y. If $f^{-1}(R)$ were not residual in Y, then $X \setminus f^{-1}(R)$ would be nonmeager and $f(X \setminus f^{-1}(R))$ would have to be nonmeager. So f is B3 $\Rightarrow$ f is B6. Finally suppose f is B6 and let M be any meager subset of Y. Then $f^{-1}(M) = X \setminus f^{-1}(Y \setminus M)$ is meager.

(ii) Suppose f is B1 and let $S \subset Y$ be nonmeager. As $S = f(f^{-1}(S))$, $f^{-1}(S)$ can not be meager. So f is B4. Now assume f is B4 and let $R \subset X$ be residual. Then $f^{-1}(Y \setminus f(R)) \subset X \setminus R$ and must be meager. Hence $Y \setminus f(R)$ must be meager and $f(R)$ must be residual. Thus f is B5. Now let $M = f^{-1}(f(M))$ be meager in X. Then $f(M \cup (X \setminus M)) = f(M) \cup f(X \setminus M)$ where $f(M) \cap f(X \setminus M) = \emptyset$. But $f(X \setminus M)$ is residual so $f(M)$ must be meager.

Finally, suppose that for every meager $M \subset X$ with $M = f^{-1}(f(M))$, $f(M)$ is meager. Let $S \subset Y$ be nonmeager. Since $f^{-1}(S) = f^{-1}(f(f^{-1}(S)))$, $f^{-1}(S)$ can not be meager, so f is B4.

To see that the converse of the first implication fails, consider the projection, $\pi_1\colon I \times I \to I$, of $I \times I$ onto the first coordinate. Then the set $M = \{(x,0) | x \in I\}$ is meager in $I \times I$, but $\pi_1(M) = I$. (Note that this map is B2 since if M is meager in I, $\pi_1^{-1}(M) = M \times I$ is meager in $I \times I$.)

(iii) By definition, f is s.c.p. if an only if f is B1 and B3. The map $f\colon [-1,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } x \geq 0 \end{cases}$$

is clearly B1 but not B2. Statement (iii) now follows.

A function $f\colon X \to Y$ is *substantially open*, or *subopen*, if for every $U \in \mathcal{O}^+(X)$, $f(U)$ is a second category set with the Baire property. f is *nearly closed* if for every closed set $F \subset X$, $f(F)$ has the Baire property. Note that open (respectively closed) maps are subopen (respectively nearly closed).

Lemma 2.2. *Suppose* $f\colon X \to Y$ *is a continuous, subopen surjection. Then* f *is* B2. *If, in addition, the image of every* G_δ *has the Baire property, then* f *satisfies* B5.

Proof. Since every meager set lies in a meager F_σ, one sees that to decide whether the forward or inverse image of every meager set is meager, it is enough to determine what happens to the closed meager sets. Now suppose that A is a closed subset of Y. If $\operatorname{int}(f^{-1}(A))$ is nonvoid then $f(f^{-1}(A))$ must be second category. Hence f is B2.

Now suppose the image of every G_δ has the Baire property. It remains to show that f is also B5. To this end let G be a dense open subset of X. Since the continuity of f, f(G) is dense in Y. Suppose that $\operatorname{cl}(G') \neq Y$. Then there is a nonvoid open set $V \subset Y \setminus G'$. From this it follows that $f^{-1}(V) \cap G$ must be nonvoid and open, whence $f(f^{-1}(V) \cap G)$ can not be meager. But $f(f^{-1}(V) \cap G) \subset P$ so $\operatorname{cl}(G') = Y$. Now let $\{G_i\}_{i=1}^{\infty}$ be a collection of dense open subsets of X. Suppose the set $A = [\cap f(G_i)] \setminus [f(\cap G_i)]$ is not meager. Since A has the Baire property, there is a $V \in \mathcal{O}^+(Y)$ such that $A \,\Delta\, V \in \mathcal{M}(X)$. Since $V \setminus A$ is meager and contains $V \cap f(\cap G_i)$, it follows that $f^{-1}(V \setminus A)$ is meager. But this is a contradiction since $f^{-1}(V \setminus A) \subset f^{-1}(V) \cap f^{-1}(f(\cap G_i))$, which is the intersection of an open set and a residual set. Hence the image of a dense G_δ is residual, and as every residual set contains a dense G_δ, f must be B5.

Lemma 2.3. *Let* X *be a regular, second countable space and suppose* $f: X \to Y$ *is either subopen or nearly closed. Then there is a closed set* $H \subset X$ *such that for every open set* $V \subset X$ *meeting* H, $f(V \cap H)$ *is a nonmeager Baire set, and for every open set* $U \subset X$ *missing* H, $f(U)$ *is meager.* (H may be empty; see Corollary 1.)

Proof. If f is subopen, take H = X. Suppose f is nearly closed but not subopen. Set

$$G = \cup\{U \subset X \mid U \text{ is open and } f(U) \text{ is meager}\}$$

and notice that for all $x \in G$ there exists an open set $U_x \in G$ such that f(U) is meager. Thus $G = \cup_{x\in X} U_x$. Because X is second countable, there is a countable subcollection of the U_x, say $\{U_i\}$ such that $G = \cup U_i$. It follows that $f(G) = \cup f(G_i)$ is a countable union of meager sets and therefore meager itself. Now take $H = X\setminus G$ and let U be any nonvoid open set meeting H. One may write $U = \cup_{i=1}^{\infty} F_i$ where each F_i is a closed set whose interior meets H. Then $f(U) = \cup_{i=1}^{\infty} f(F_i)$, which has the Baire property. Finally, to see that f(U) is nonmeager, observe that $f(U) = f(U \cap H) \cup f(U \cap G)$. Then if f(U) is meager, $U \subset G$ in which case $U \cap H$ must be empty.

Corollary 1. *Let* X, Y, *and* f *be as in* Lemma 2.3. *Then* f(X) *is nonmeager in* Y *if and only if* $H \neq \emptyset$.

Proof. This follows immediately from the construction of H.

Corollary 2. *Let* X, Y, f, *and* H *be as in Lemma* 2.3. *Then if* $H \neq \emptyset$ *and* f *is* B1, $\mathrm{int}(H) \neq \emptyset$ *and* $f|H$ *is* s.c.p.

Proof. Assume the hypotheses. Let $U \in \mathcal{O}(X)$ and suppose $U \cap H \neq \emptyset$. Then $f([\mathrm{cl}(U)] \cap H)$ is nonmeager by the lemma. Now $[\mathrm{cl}(U)] \cap H = G \cup N$ where $G = \mathrm{int}([\mathrm{cl}(U)] \cap H)$ and $N = ([\mathrm{cl}(U)] \cap H)\setminus G$. But N is meager and as f is B1, f(N) is meager. Hence $f(G) \neq \emptyset$ which means $G \neq \emptyset$, and as $G \subset H$, $\mathrm{int}(H) \neq \emptyset$. By the construction of H, $f|H$ is subopen and hence B2 by Lemma 2.2, and clearly $f|H$ is B1. Thus $f|H$ is s.c.p.

If $f(\mathcal{M}(X)) \subset \mathcal{B}(Y)$, and f is subopen, then f is nearly closed. Using the previous lemma one deduces that if $f(\mathcal{M}(X)) \subset \mathcal{B}(Y)$, and X is a second countable, T_3 space, then f is nearly closed. Naturally, if f is B1 then $f(\mathcal{M}(X)) \subset \mathcal{B}(Y)$. It is also clear that a one-to-one category preserving map must be s.c.p. In general, however, sufficient conditions for a map to be B1 are difficult to obtain. Even a two-to-one category preserving map need not be B1. For instance, the map $f: I \times I \to I \times I$ defined by

$$f(x,y) = \begin{cases} (x,y) & \text{if } y > 0 \\ g(x) & \text{if } y = 0, \end{cases}$$

where g is a bijection between the unit interval and the unit square, is not B1 since the set $\{(x,y)\,|\,y = 0\}$ is meager but has nonmeager image.

Lemma 2.4. *Suppose* $f: X \to Y$ *is a nearly closed, open, continuous, finite-to-one map. Then* f *is* s.c.p.

Proof. f is B2 by Lemma 2.2. It remains to show that f is B1. To this end, let $N = \mathrm{cl}(N) \in M(X)$ and suppose $f(N)$ is nonmeager. Since N is closed and f is nearly closed, $f(N)$ has the Baire property, and thus, there exists a nonvoid open set $V \subset Y$ such that $f(N) \Delta V$ is meager. Hence $f(N) \cap V$ is comeager in V. For $n = 1, 2, 3, \ldots$, set

$$Z_n = (y \in f(N) \cap V | \mathrm{card}(f^{-1}(y)) = n).$$

As $\cup Z_n = f(N) \cap V$, there must be a least n, say n_0, such that Z_{n_0} is of the second category. Set $V' = \mathrm{int}(D(Z_{n_0}))$ and note that $V' \subset \mathrm{int}(\mathrm{cl}(V))$. Since the intersection of a comeager subset of an open set and a second category subset of the same open set must be nonvoid, there exists a $y \in V' \cap f(N) \cap Z_{n_0}$. Write $f^{-1}(y) = \{x_1, \ldots, x_{n_0}\}$ and let $U_1, \ldots, U_{n_0}$ be pairwise disjoint, open neighborhoods of $x_1, \ldots, x_{n_0}$ respectively, where for each i, $U_i \subset f^{-1}(V')$. For $i = 1, 2, \ldots, n_0$, set

$$N_i = [N - (\bigcup_{j \neq i} U_j)] \cap f^{-1}(V').$$

Clearly $\cup f(N_i)$ is comeager in V' and hence $V' \cap (\cap M(f(N_i))) = \emptyset$. It follows that $V' \subset \cup D(f(N_i))$. Hence, for some i, $y \in f(N_i) \cap D(f(N_i))$. Without loss of generality, suppose $y \in f(N_1) \cap D(f(N_1))$. Because $y \in \cap f(U_i)$, the set

$$W = [\cap f(U_i)] \cap [\mathrm{int}(D(f(N_1)))]$$

is nonvoid. For $i = 1,\ldots,n_0$, set $U_i' = f^{-1}(W) \cap U_i$ and observe that $f(U_i') = W$. Now, $f(U_1' - N)$ is nonmeager in W and so is $f(N_1)$, the first because it is a dense open subset of U_1' and f is continuous and open, the latter by choice. Therefore, there must be a point $y' \in f(N_1) \cap f(U_1' - N) \cap Z_{n_0}$. But N_1 misses $\cup_{j\neq 1} U_j$ and so, *a fortiori*, $N_1 \cap (\cup_{j\neq 1} U_j') = \emptyset$. It follows that $f^{-1}(y')$ must meet each U_i' in addition to meeting N_1, contradicting the assumption that $\operatorname{card}(f^{-1}(y')) = n_0$.

Corollary. If X *is second countable and "open" is replaced by "subopen," then there is a dense* G_δ, G, *in* X *such that* $f|G$ *is* s.c.p.

Proof. Let $\{U_i\}$ be a countable open basis for X. Since f is subopen, we may write for all i, $f(U_i) = V_i \Delta M_i$ where V_i is open and M_i is meager. Now each M_i lies in a meager F_σ, say M_i'. Set $M = \cup M_i'$ and note that M is a meager F_σ itself. Since f is B2 (Lemma 2.2) and continuous, $f^{-1}(M)$ is a meager F_σ in X. Set $G = X - f^{-1}(M)$. Then G is a dense G_δ. Since $G = f^{-1}(f(G))$, $f(U_i \cap G) = f(U_i) \cap f(G) = V_i - M$ is open relative to $f(G) = f(X) - M$. Thus $f|G\colon G \to f(G)$, is open. Obviously $f|G$ remains continuous, nearly closed and finite-to-one.

3. An Application.

Let B_0 denote the simplest Knaster continuum, a description of which may be found in [2] or in [3], v.II, p. 204. For any set $A \subset X \in \mathcal{C}$, set $\operatorname{cps}(A) = \cup_{x\in A}\operatorname{cps}(x)$, i.e,

$$\mathrm{cps}(A) = \cup\{C \subset X \mid C \text{ is a composant of } X \text{ and } C \cap A \neq \emptyset\}.$$

Let Γ denote the Cantor ternary set. In [2], Kuratowski shows that if $C \subset B_0$ is a union of composants then $C \cap \Gamma$ is meager if and only if C is meager in B_0. Using this, he proves that $\tilde{B}_0$ is strongly transitive.

Lemma 3.1. *Let* S *be a union of nondegenerate arcs in* B_0. *Then* S *is meager in* B_0 *if and only if* cps(S) *is meager in* B_0.

Proof. (if) This is trivial since $S \subset \mathrm{cps}(S)$.

(only if) Suppose cps(S) is second category. For $n = 0, 1, 2, \ldots$, let Γ_n denote the set of points in $\Gamma \cap D_n$ such that there is an arc in $S \cap D_n$ of positive length. Clearly, $\mathrm{cps}(S) = \mathrm{cps}(\cup\Gamma_n)$ so $\cup\Gamma_n$ is nonmeager in Γ. Therefore, there is an n such that Γ_n is nonmeager in Γ. Now let S_1 denote the union of arcs in $S \cap D_n$ with length ≥ 1. For $k = 2, 3, 4, \ldots$, let

$$S_k = \cup\{A \subset S \cap D_n \mid A \text{ is an arc and } \frac{1}{k} \leq \text{length}(A) \; \frac{1}{k-1}\}$$

and define $\Gamma_n(k)$ to be the points in Γ which meet the same semicircles of D_n as S_k. For some k, $\Gamma_n(k)$ is nonmeager and it follows that the set of semicircles in D_n meeting S_k is nonmeager (again, see [2]). Divide D_n into sectors in such a way that the outer component in each sector has length $\leq \frac{1}{3^k}$. (See Figure 1.)

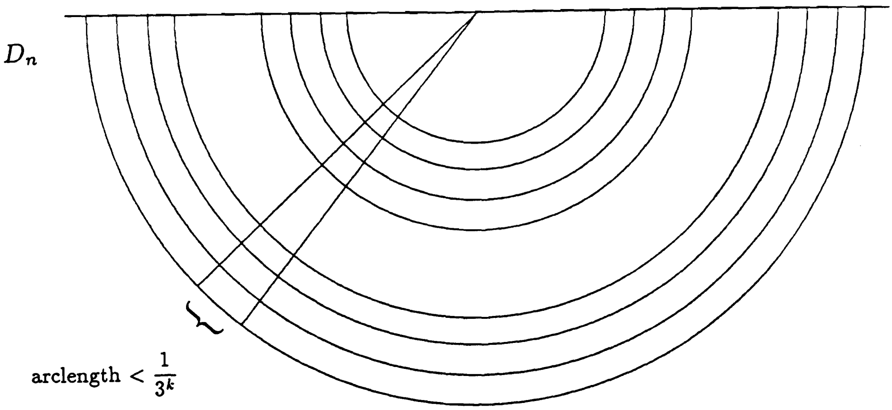

Figure 1.

Note that for each arc R in $S_k \cap D_n$, there is some sector such that R meets both radial boundary lines of that sector. Since the union of the set of semicircles meeting S_k in D_n is nonmeager, there is some sector such that $S_k \cap D_n$ is nonmeager in that sector, and hence, also in B_0.

Let

X -o>*- $\mathcal{P}$ and let Y -o>*- $\mathcal{Q}$. A surjective function f: X $\to$ Y will be called $(\mathcal{P},\mathcal{Q})$-irreducible if for every Q $\in \mathcal{Q}$, $f^{-1}(Q) \in \langle \mathcal{P} \rangle$. When $\mathcal{P}$ and $\mathcal{Q}$ are understood, we may speak simply of f being *irreducible*. The surjection f: $(X,\mathcal{P}) \to (Y,\mathcal{Q})$ is *strictly irreducible* if for every Q $\in \mathcal{Q}$, $f^{-1}(Q) \in \mathcal{P}$.. (The definition of irreducible may be extended in a natural way to arbitrary collections of sets. To illustrate, notice that f is $(\mathcal{M}(X),\mathcal{M}(Y))$-irreducible if and only if it is B2. As another example, f is a continuous map if and only if it is $(\mathcal{O}(X),\mathcal{O}(Y))$-irreducible.)

Corollary. Let $X \in \mathcal{C}$ *and suppose* $\ell: (X,\tilde{X}) \to (B_0,\tilde{B}_0)$ *is a continuous light, irreducible surjection. Then* ℓ *is subopen.*

Proof. Let $H = cl(U)$, $U \in \mathcal{O}^+(X)$. Then $\ell(H)$ is a union of nondegenerate arcs meeting each composant of B_0 and is therefore nonmeager.

Theorem 3.2. *Let* X *be a regular, second countable space and let* $f: (X,\mathcal{P}) \to (Y,\mathcal{Q})$ *be a strictly irreducible, subopen,* s.c.p. *map. Suppose* $\mathcal{Q}$ *is strongly transitive. Then* $\mathcal{P}$ *is strongly transitive.*

Proof. Suppose, by way of contradiction, that there exists a $B \in \langle \mathcal{P} \rangle \cap \mathcal{B}(X)$ for which $M(B) \neq \emptyset \neq D(B)$. Now $M(B)$ is open, and for the case in point, we have $int(D(B)) \neq \emptyset$. Thus both $f|M(B)$ and $f|D(B)$ are subopen. By Corollary 2 of Lemma 2.3, there exist closed sets $H_1 \subset M(B)$ and $H_2 \subset D(B)$ such that both $f|H_1$ and $f|H_2$ are s.c.p. Now $f(H_1) = f(H_1 \cap B) \cup f(H_1 \cap (X\setminus B))$ and as $f(H_1 \cap B)$ is meager, $f(H_1 \cap (X\setminus B))$ must be nonmeager. On the other hand $f(H_2 \cap B)$ is nonmeager. Hence both $f(B)$ and $f(X\setminus B)$ are second category which is impossible since $\mathcal{Q}$ is strongly transitive.

Corollary. Let $X \in \mathcal{C}$. *Suppose* $f: X \to B_0$ *is a continuous, finite-to-one map such that for each composant* $C \subset B_0$, $f^{-1}(C)$ *is a composant of* X. *Then* $\tilde{X}$ *is strongly transitive.*

Proof. Since f is a light map, it is subopen. It is Bl by Lemma 2.4. The corollary now follows from the theorem.

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