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THE SET OF NONUNIFORM ULTRAFILTERS IS C*-EMBEDDED FOR SINGULAR CARDINALS

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Our notation follows [CN] except that we use κ, rather than α, as the typical cardinal, and we use fewer parentheses. We consider κ to have the discrete topology, and $\beta\kappa$ to be the set of ultrafilters on κ. κ^* is the set of free ultrafilters; for $A \subset \kappa$, set $A^* = \{p \in \kappa^*: A \in p\}$. $\{A^*: A \subset \kappa\}$ is a base for a compact, T_2, topology on $\beta\kappa$. For $p \in \beta\kappa$, set $\|p\| = \min \{|A|: A \in p\}$. Set $U\kappa = \{p \in \kappa^*: \|p\| = \kappa\}$ and $N\kappa = \{p \in \kappa^*: \omega \leq \|p\| < \kappa\}$. We say that a subspace A is C*-embedded in a space X, if every continuous function $f: A \to [0,1]$ can be extended to a continuous $g: X \to [0,1]$. We say that $N\kappa$ is C*-embedded if it is C*-embedded in $\beta\kappa$ (equivalently, in κ^*).

Fine and Gillman [FG] asked whether $N\omega_1$ is C*-embedded; Warren [W] answered, "No". Dow and Merrill [DM] showed (i) if κ is weakly compact, then $N\kappa$ is C*-embedded, (ii) assuming V = L, if κ is uncountable, regular, and not weakly compact then $N\kappa$ is not C*-embedded, (iii) it is consistent, modulo large cardinals, that for uncountable regular κ, $N\kappa$ is C*-embedded. In this paper we consider whether $N\kappa$ is C*-embedded for singular κ. We wish to thank Fred Galvin for calling this question to our attention.

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It is a tempting trap to assert the equation $\cup\{\alpha^*: \delta < \alpha < \kappa\} = N\kappa$; it holds only when κ is regular. If it were true for $\kappa > cf(\kappa) = \omega$, then $N\kappa$ would be an open F_σ, hence C^*-embedded. However, for all $p \in \kappa^*$ and for all $\{A_n: n \in \omega\} \subset p$, there exists $B \in [\kappa]^\omega$ with $B \setminus A_n$ finite for all n, and thus there exists a $q \in \kappa^* \cap \cap\{A_n^*: n \in \omega\}$ with $\|q\| = \omega$. Hence for $\kappa > \omega$, $N\kappa$ is not an F_σ in κ^*.

Let us fix, for the rest of this paper, a singular cardinal κ. Write κ as the disjoint union of small sets: $\kappa = \cup\{L_i: i < cf(\kappa)\}$, where $|L_i| < \kappa$. We make the following definitions.

$$Bd = \cup\{L_i^*: i < cf(\kappa)\}$$

$$\mathcal{T} = \{T \subset \kappa: |T \cap L_i| < \omega \text{ for all } i\}$$

$$Th = \cup\{T^*: T \in \mathcal{T}\}.$$

Note that if $T \in \mathcal{T}$, then $|T| \leq cf(\kappa)$.

Lemma 1. *Bd is a closed subspace of the normal space* $BD \cup \kappa$*; hence Bd is* C^**-embedded in* κ^*.

Proof. Note that $Bd \cup \kappa$ is homeomorphic to the discrete union of the compact T_2 spaces βL_i, $i < cf(\kappa)$, and that κ consists of isolated points. The second sentence follows from Tietze's Extension Theorem.

Lemma 2. $cl_{\kappa^*}Bd = \kappa^* \setminus Th$.

Proof. Suppose that $A \in p \in \kappa^* \setminus Th$. For some i, $B = A \cap L_i$ is infinite. Then $\emptyset \neq B^* \subset A^* \cap Bd$. The reverse inclusion follows since Th is open.

Lemma 3. *Suppose that* $G\varepsilon \subset Th$ *is open in* κ^*. *If there is a* $p' \in cl_{\kappa^*}G\varepsilon \cap U\kappa$, *then there is a* $p \in cl_{\kappa^*}G\varepsilon \cap cl_{\kappa^*}Bd \cap N\kappa$.

Proof. Set $\nu = cf(\kappa)^+$. By induction on $\alpha < \nu$, define $A_\alpha \in p'$ and countable $T_\alpha \in \mathcal{T}$. Set $A_0 = \kappa$; and in general set $A_\alpha = \kappa \setminus \cup \{T_\gamma: \gamma < \alpha\}$. $A_\alpha \in p'$ because p' is uniform. Having defined A_α, we can choose $T_\alpha \subset A_\alpha$, $T_\alpha \in \mathcal{T}$, with $T_\alpha^* \subset A_\alpha^* \cap G\varepsilon$ because $p' \in cl_{\kappa^*}G\varepsilon$. Note that the family $\{T_\alpha: \alpha < \nu\}$ is pairwise disjoint.

Let r be a uniform ultrafilter on ν; for each $\alpha < \nu$, let $q_\alpha \in T_\alpha^*$. We define

$$p = \{D \subset \kappa: \{\alpha \in \nu: D \in q_\alpha\} \in r\}.$$

(Loosely speaking, $p = \int q\, dr$). It is routine to verify that for all $D \subset \kappa$, either $D \in p$ or $\kappa \setminus D \in p$, and that p is closed under intersection because r and the q_α's are. Because $\cup \{T_\alpha: \alpha < \nu\} \in p$, $\|p\| \le \nu$. The family $\{T_\alpha: \alpha < \nu\}$ is pairwise disjoint and r is uniform; hence $\|p\| \ge \nu$. Thus $p \in cl_{\kappa^*}Bd \cap N\kappa$. If $D \in p$, then for some α (in fact for many α's) $(D \cap T_\alpha)^* \subset G\varepsilon$; hence $p \in cl_{\kappa^*}G\varepsilon$.

Theorem 4. *If* κ *is a singular cardinal, then* $N\kappa$ *is* C*-*embedded.*

Proof. We show C*-embedded. Let a continuous function $f: N\kappa \to [0,1]$ be given. Set $f_1 = f|_{Bd}$; by Lemma 1, we can extend f_1 to a continuous $f_2: (Bd \cup \kappa) \to [0,1]$. By the unique extension of properties of $\beta\kappa$, we can extend f_2 to $f_3: \beta\kappa \to [0,1]$. Now f and f_3 agree on $cl_{\kappa^*}Bd$; however f and f_3 may disagree on Th.

Define $g: N\kappa \to [0,1]$ by $g(x) = f(x) - f_3(x)$. Note that $g(x) = 0$ for all $x \notin Th$. Extend g to $g_1: \kappa^* \to [0,1]$ by setting $g_1(y) = 0$ for $y \in U\kappa$. Because $N\kappa$ is open in κ^*, g_1 is continuous on $N\kappa$; we must show that g_1 is continuous on $U\kappa$. Towards a contradiction, assume that for some $p' \in U\kappa$ and $\varepsilon > 0$, p' is not in the interior of $\{x \in \kappa^*: g_1(x) \leq \varepsilon\}$. That is, p' is in the closure of $\{x \in \kappa^*: |g_1(x)| > \varepsilon\} = \{x \in N\kappa: |g(x)| > \varepsilon\} \subset Th$; let's call this set $G\varepsilon$. Now Lemma 2 gives a point $p \in cl_{\kappa^*}G\varepsilon \cap cl_{\kappa^*}Bd \cap N\kappa$. We get $|g(p)| \geq \varepsilon$ and $g(p) = 0$. Contradiction!

Finally, $f_3 + g_1$ is the desired continuous extension of f.

Dow and Merrill [DM] remark that it is known that $N\mu$ is strongly zero dimensional whenever μ is an infinite cardinal and $cf(\mu) < \omega_2$, and they asked for which μ is $N\mu$ strongly zero dimensional. We give a partial answer below.

Corollary 5. *If* κ *is a singular cardinal, then* $N\kappa$ *is strongly zero dimensional and* **2**-*embedded in* $\beta\kappa$.

Proof. We note that $N\kappa$ is dense and C^*-embedded in κ^*; hence $\beta N\kappa = \kappa^*$. Since κ^* is zero dimensional, the result follows from [PW], 4.7 (h) and (f).

Let κ be an infinite cardinal and $\omega < \mu \leq \nu$. Set

$$M(\nu,\mu) = \cup \{A^*: A \subset \nu \text{ and } |A| < \mu\}.$$

Let us suppose that $M(\nu,\mu)$ is not C^*-embedded in ν^*; then $M(\nu,\mu)$ is not C^*-embedded in $\nu \cup M(\nu,\mu)$. Because $M(\nu,\mu)$ is closed in $\nu \cup M(\nu,\mu)$, $\nu \cup M(\nu,\mu)$ is not normal. Theorem 1 of [M] asserts that $M(\nu,\mu)$ is normal iff $\nu \cup M(\nu,\mu)$ is normal. Hence, $M(\nu,\mu)$ is not normal. This suggests the question of whether it is possible for $M(\nu,\mu)$ to be not normal but still C^*-embedded in ν^*. A positive answer follows by coupling Theorem 4 with Proposition 1 of [M], which asserts that $M(\kappa,\kappa)$ is not normal whenever κ is singular.

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