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In [6] J. De Groot introduced the notion of (countably) subcompact spaces and proved a remarkable result that a metric space X is subcompact if and only if X is Čech-complete. Aarts and Lutzer [1] proved that a Moore space X is subcompact if X is Čech-complete, and asked whether every Čech-complete space is countably subcompact. Recently, M. C. Rayburn [8] remarked that it is not even known whether every paracompact, Čech-complete space is countably subcompact.

In this paper, we investigate relations between Čech-completeness and countably subcompactness. In Section 2, as a generalization of the above result obtained by Aarts and Lutzer, we show that a quasi-Moore space X is subcompact if X is Čech-complete, and moreover, that a peripherally compact Moore space X is Čech-complete, if X is countably $\mathcal{B}$-subcompact for a special subcollection $\mathcal{B}$ of the regular open sets. In Section 3, we prove that if X is $\mathcal{B}$-subcompact for an open base $\mathcal{B}$, and X has a normal sequence of $\mathcal{B}$-covers, then the metric space $\tilde{X}$ associated with this normal sequence is complete. From this result, if X is a paracompact, countably $\mathcal{B}$-subcompact space, then the metric space $\tilde{X}$ associated with any normal sequence of $\mathcal{B}$-covers is complete, and hence, we show that a countably subcompact, paracompact p-space is Čech-complete. In Section 4, we give some examples.

All spaces considered in this paper are assumed to be completely regular Hausdorff. N denotes the set of positive integers, and βX denotes the Stone-Čech compactification of a space X. The notions and terminologies can be found in [4].

1. Preliminaries

Let $\mathcal{B}$ be an open base of a space X. A non-empty subset $\mathcal{F}$ of $\mathcal{B}$ is called *regular $\mathcal{B}$-filter base* if every finite intersection of elements of $\mathcal{F}$ contains an element of $\mathcal{F}$ whose closure is contained in this intersection [6]. A regular $\mathcal{B}$-filter base $\mathcal{F}$ is countable, then $\mathcal{F}$ is called *countably regular $\mathcal{B}$-filter base*, especially, $\{U_n \in \mathcal{B} | n \in N, cl_X U_{n+1} \subset U_n\}$ is called a *regular $\mathcal{B}$-sequence* A space X is called *$\mathcal{B}$-subcompact* (resp. *countably $\mathcal{B}$-subcompact*) if every regular $\mathcal{B}$-filter base (resp. countably regular $\mathcal{B}$-filter base) has the non-empty intersection. Every $\mathcal{B}$-subcompact space is countably $\mathcal{B}$-subcompact. A space X is *subcompact* (resp. *countably subcompact*) if there exists an open base $\mathcal{B}$ such that X is $\mathcal{B}$-subcompact (resp. countably $\mathcal{B}$-subcompact) [6].

Now we mention a few facts about subcompactness. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are open base of X and $\mathcal{B}_1 \subset \mathcal{B}_2$, then $\mathcal{B}_2$-subcompactness (resp. countably $\mathcal{B}_2$-subcompactness) of X implies $\mathcal{B}_1$-subcompactness (resp. countably

$\mathcal{B}_1$-subcompactness). A space X is $\mathcal{B}$-subcompact (resp. countably $\mathcal{B}$-subcompact) relative to any open base $\mathcal{B}$ of X iff X is compact (resp. pseudo-compact) [6, 8]. If X is locally compact and $\mathcal{B} = \{U | U$ is open and $cl_X U$ is compact$\}$, then X is clearly $\mathcal{B}$-subcompact. Therefore if X is a locally compact and not compact (resp. not pseudo-compact), then there exist two open bases $\mathcal{B}_1$ and $\mathcal{B}_2$ such that X is $\mathcal{B}_1$-subcompact (resp. countably $\mathcal{B}_1$-subcompact) but not $\mathcal{B}_2$-subcompact (resp. not countably $\mathcal{B}_2$-subcompact). If X is pseudo-compact and not compact, then there exists an open base $\mathcal{B}$ such that X is countably $\mathcal{B}$-subcompact but not $\mathcal{B}$-subcompact.

Let $\mathcal{T}$ be a collection of subsets of X. We recall that $\mathcal{T}$ have the "*descending chain condition*" provided that every sequence of elements of $\mathcal{T}$ which consists of properly decreasing sets, is finite [6]. We need the following lemmas.

Lemma 1.1. *Let* $\mathcal{B}$ *be an open base of a space* X. *Then* X *is countably* $\mathcal{B}$-*subcompact if an only if every regular* $\mathcal{B}$-*sequence has the non-empty intersection.*

Proof. The "only if" part is obvious, we we prove the "if" part. Let $\mathcal{F}$ be a countably regular $\mathcal{B}$-filter base and let $\mathcal{F} = \{F_n | n \in N\}$. Put $U_1 = F_1$ and put $n_1 = \min\{K | cl_X F_k \subset U_1\}$. There exists a $U_2 \in \mathcal{F}$ such that $U_2 \subset F_1 \cap F_2 \cap \ldots \cap F_n$. Put $n_2 = \min\{k | cl_X F_k \subset U_2\}$, then $n_1 < n_2$. Repeating this process, we obtain a regular

$\mathcal{B}$-sequence $\{U_n \mid n \in N\}$ such that $\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$. Therefore X is countably $\mathcal{B}$-subcompact.

The proof of the following is similar to one of the set theoretic lemma of Groot [6].

Lemma 1.2. *Every collection* $\mathcal{F}$ *of subsets of* X *has a subcollection* $\mathcal{F}'$ *satisfying the descending chain condition such that*

$$\cup\{F \mid F \in \mathcal{F}\} = \cup\{F' \mid F' \in \mathcal{F}'\}.$$

2. Čech-completeness and Countably Subcompactness of a Moore Space

A *development* (resp. *quasi-development* [2]) for a space X is a sequence $\{\mathcal{U}_n \mid n \in N\}$ of open covers (resp. families of open sets) such that for every point $x \in X$, $\{st(x, \mathcal{U}_n) \mid n \in N\}$ is a base at x, where $st(x, \mathcal{U}_n) = \cup\{U \mid U \in \mathcal{U}_n, x \in U\}$. In this paper, a *Moore space* (resp. *quasi-Moore space*) is a Tychonoff space which has a development (resp. quasi-development).

As relations between Čech-completeness and subcompactness, the following are known: (1) Čech-completeness $\Leftrightarrow$ subcompactness $\Leftrightarrow$ countably subcompactness for a metric space [6], and (2) Čech-completeness $\Rightarrow$ subcompactness $\Leftrightarrow$ countably subcompactness for a Moore space [1]. We first give a generalization of (2).

Theorem 2.1. *Suppose* X *is a quasi-Moore space. Then:*

(1) *if* X *is Čech-complete, then* X *is subcompact.*

(2) X *is subcompact if and only if* X *is countably subcompact.*

Proof. (1) Let $(\mathcal{W}_n)$ be a quasi-development of X and $\{G_n | n \in N\}$ a family of open subsets of βX such that $X = \bigcap_{n=1}^{\infty} G_n$. For each $x \in X$ and $n \in N$, there exists an open subset $W_n(x)$ of βX such that $x \in W_n(x) \subset cl_{\beta X} W_n(x) \subset G_n$. Let $\mathcal{U}_n(x) = \{U \cap W_n(x) | x \in U \text{ and } U \in \mathcal{W}_n\}$. Put $\mathcal{U}_n = \{V \in \mathcal{U}_n(x) | x \in X\}$, and $\mathcal{B} = \{V \in \mathcal{U}_n | n \in N\}$. It is evident that $\{\mathcal{U}_n | n \in N\}$ is a quasi-development and $\mathcal{B}$ is an open base of X. By Lemma 1.2 we may assume that each $\mathcal{U}_n$ satisfies the descending chain condition. Let $\mathcal{F}$ be a regular $\mathcal{B}$-filter base and $\{U_n \in \mathcal{F} | n \in N\}$ a regular $\mathcal{B}$-sequence. Now we may suppose that $U_n \in \mathcal{U}_n$, because each $\mathcal{U}_n$ satisfies the descending chain condition, and hence we may assume $U_n \in \mathcal{U}_n$ for each $n \in N$. Therefore $\phi \neq \bigcap\{cl_{\beta X} F | F \in \mathcal{F}\} \subset \bigcap_{n=1}^{\infty} cl_{\beta X} U_n \subset \bigcap_{n=1}^{\infty} G_n = X$, and hence X is $\mathcal{B}$-subcompact.

(2) It suffices to show that countable $\mathcal{B}$-subcompactness implies $\mathcal{B}$-compactness. Suppose that X is countably $\mathcal{B}$-subcompact for some open base $\mathcal{B}$. Without loss of generality, we can take a quasi-development $\{\mathcal{U}_n | n \in N\}$ with $\mathcal{U}_n \subset \mathcal{B}$ such that every $\mathcal{U}_n$ satisfies the descending chain condition. Let $\mathcal{F}$ be a regular $\mathcal{B}$-filter base and $\bigcap \mathcal{F} = \phi$. Then there is no minimal element in $\mathcal{F}$, and hence $\mathcal{F}$ contains a regular $\mathcal{B}$-sequence $\{U_n | n \in N\}$. Now we may assume that $U_n \in \mathcal{U}_n$, because each $\mathcal{U}_n$ satisfies the

descending chain condition. Since X is countably $\mathcal{B}$-subcompact, we take a point $x \in \bigcap_{n=1}^{\infty} U_n$. Then $\{st(x, \mathcal{U}_n) \mid n \in N\}$ is a base at x, and $\mathcal{W} = \{U_n \mid n \in N\}$ is also a base at x. Thus $\{x\} = \cap\{U_n \mid U_n \in \mathcal{W}\}$. On the other hand, since $\cap\mathcal{F} = \phi$, there is a $U_a \in \mathcal{F}$ such that $x \notin U_a$. Take $U_b \in \mathcal{F}$ such that $cl_X U_b \subset U_a$. Since $x \notin cl_X U_b$, there exists $U_j \in \mathcal{W}$ such that $U_j \cap U_b = \emptyset$. This is a contradiction, and hence X is $\mathcal{B}$-subcompact.

Let $\mathcal{B}$ be an open base in a space X. We use the notion of *π-bases* in the sense of Nagami-Kodama [7], and then we shall say that $\mathcal{B}$ is a *weak π-base* if for any $U_1, U_2, \ldots, U_n \in \mathcal{B}$, following three conditions are satisfied,

(1) $X - cl_X U_i \in \mathcal{B}$ for $i = 1, 2, \ldots, n$.

(2) $\bigcap_{i=1}^{\infty} U_i \in \mathcal{B}$.

(3) $U_i - cl_X U_j \in \mathcal{B}$ for $i, j = 1, 2, \ldots, n$.

We recall that a space X is *peripherally compact* if for each $x \in X$ and any neighborhood V of x, there exists an open neighborhood U of x such that $U \subset V$ and the boundary ∂U of U is compact. If X is peripherally compact, then it is easy to see that $\mathcal{B}_{ro}(X) = \{U \mid U$ is regular open and ∂U is compact$\}$ is a weak π-base.

Theorem 2.2. *Suppose* X *is a peripherally compact, Moore space. Then* X *is Čech-complete if* X *is countably* $\mathcal{B}$*-subcompact for some weak π-base* $\mathcal{B} \subset \mathcal{B}_{ro}(X)$.

Proof. Let $\{\mathcal{U}_n \mid n \in N\}$ be a development of X. We may assume that $\mathcal{U}_n \subset \mathcal{B}$ for each $n \in N$. Put $G_i = \cup\{\text{int}_{\beta X} \text{cl}_{\beta X} W \mid W \in \mathcal{U}_i\}$, and take $p \in \bigcap_{i=1}^{\infty} G_i - X$. For each $i \in N$, there is a $W_i \in \mathcal{U}_i$ such that $p \in \text{int}_{\beta X} \text{cl}_{\beta X} W_i$. Since ∂W_1 is compact, there are $V_1, \ldots, V_n \in \mathcal{B}$ such that $\partial W_1 \subset \bigcup_{i=1}^{n} V_i$ and $p \notin \text{cl}_{\beta X} V_i$ for each $i = 1, \ldots, n$. Put $W_1' = W_1 - \bigcup_{i=1}^{n} \text{cl}_X V_i$. Since $\mathcal{B}$ is a weak π-base, $W_1' \in \mathcal{B}$ and $p \in \text{int}_{\beta X} \text{cl}_{\beta X} W_1'$. Put $U_1 = W_1$ and $U_2 = W_2 \cap W_1'$. Then $U_2 \in \mathcal{B}$, $p \in \text{int}_{\beta X} \text{cl}_{\beta X} U_2$ and $\text{cl}_X U_2 \subset U_1$.

Repeating this process, we have a $\mathcal{B}$-regular sequence $\{U_n \mid n \in N\}$ such that $U_n \subset W_n$ and $p \in \text{int}_{\beta X} \text{cl}_{\beta X} U_n$ for each $n \in N$. From the $\mathcal{B}$-countably subcompactness of X, there exists a point $x \in \bigcap_{n=1}^{\infty} U_n$. Since $x \in \bigcap_{n=1}^{\infty} W_n$ and $\{W_n \mid n \in N\}$ is a neighborhood base at x, $\{U_n \mid n \in N\}$ is also a neighborhood base at x. Since X is a first countable space, $\{\text{int}_{\beta X} \text{cl}_{\beta X} U_n \mid n \in N\}$ is a neighborhood base at x in βX, and then $\bigcap_{n=1}^{\infty} \text{int}_{\beta X} \text{cl}_{\beta X} U_n = \{x\}$. But $p \neq x$ and $p \in \bigcap_{n=1}^{\infty} \text{int}_{\beta X} \text{cl}_{\beta X} U_n$. This is a contradiction.

Therefore $X = \bigcap_{i=1}^{\infty} G_i$, and hence X is Čech-complete.

3. Čech-completeness and countably subcompactness of a paracompact p-space

A sequence $\{\mathcal{U}_n \mid n \in N\}$ of open covers of X is said to be a *normal sequence* if $\mathcal{U}_{n+1}^* < \mathcal{U}_n$ for every $n \in N$, where $\mathcal{U}_{n+1}{}^* = \{st(U,\mathcal{U}_{n+1}) \mid U \in \mathcal{U}_{n+1}\}$. For an open base $\mathcal{B}$ of X, we call $\{\mathcal{U}_n \mid n \in N\}$ a *normal sequence of $\mathcal{B}$-covers* if $\mathcal{U}_n \subset \mathcal{B}$ $(n \in N)$. As is well-known, for a normal sequence $\{\mathcal{U}_n \mid n \in N\}$ of open covers of a space X, there exists a pseudo-metric function ρ on $X \times X$ such that

(1) $st(x, \mathcal{U}_n) \subset B(x, 1/2^{n-2})$,

(2) $B(x, 1/2^n) \subset st(x, \mathcal{U}_n)$,

where $B(x, 1/2^n) = \{y \in X \mid \rho(x,y) < 1/2^n\}$ (cf.[7]). Moreover, by the equivalence relation "$x \sim y$ iff $\rho(x,y) = 0$, equivalently $y \in \bigcap_{n=1}^{\infty} st(x, \mathcal{U}_n)$" $(\tilde{X},\tilde{\rho})$ is a metric space, and $\varphi: X \to \tilde{X}$ defined by $\varphi(x) = \tilde{x} = \bigcap_{n=1}^{\infty} st(x, \mathcal{U}_n)$ is a continuous mapping of X onto $\tilde{X}$, where

$$\tilde{X} = \{\tilde{x} \mid x \in X, \bigcap_{n=1}^{\infty} st(x, \mathcal{U}_n) = \tilde{x}\} \text{ and } \tilde{\rho}(\tilde{x},\tilde{y}) = \rho(x,y).$$

$(\tilde{X},\tilde{\rho})$ is called a *metric space associated with* $\{\mathcal{U}_n \mid n \in N\}$. It is obvious that the conditions (1) and (2) above imply the following.

(1*) $\varphi(st(x, \mathcal{U}_n)) \subset \tilde{B}(\varphi(x), 1/2^{n-2})$

(2*) $\varphi^{-1}(\tilde{B}(\tilde{x}, 1/2^n)) \subset st(x, \mathcal{U}_n)$.

Theorem 3.1. *Let* X *be* $\mathcal{B}$-*subcompact for an open base* $\mathcal{B}$. *If* X *has a normal sequence of* $\mathcal{B}$-*covers* $\{\mathcal{U}_n \mid n \in N\}$, *then the metric space* $(\tilde{X},\tilde{\rho})$ *associated with* $\{\mathcal{U}_n \mid n \in N\}$ *is complete.*

Proof. We shall use the notations above.

Let us take a Cauchy sequence $\{\tilde{x}_n \mid n \in N\}$ in $\tilde{X}$, and put $T_n = \{x_i \mid i \geq n\}$. For every $n \geq 2$, there is a

$k(n) \in N$ such that if $i, j \geq k(n)$ then $\tilde{\rho}(\tilde{x}_i, \tilde{x}_j) < 1/2^n$. Thus by (2*),

$$T_{k(n)} \subset \varphi^{-1}(\tilde{B}(\tilde{x}_{k(n)}, 1/2^n)) \subset st(x_{k(n)}, \mathcal{U}_n) \subset cl_X\, st(x_{k(n)}, \mathcal{U}_n) \subset st^2(x_{k(n)}, \mathcal{U}_n).$$

Since $x_i \in st(x_{k(n)}, \mathcal{U}_n)$ for each $x_i \in T_{k(n)}$,

$$st(x_i, \mathcal{U}_n) \subset cl_X\, st(x_i, \mathcal{U}_n) \subset st^2(x_{k(n)}, \mathcal{U}_n)$$

There is a $U_{n-1} \in \mathcal{U}_{n-1}$ such that $st^2(x_{k(n)}, \mathcal{U}_n) \subset U_{n-1}$, because $\mathcal{U}_n^* < \mathcal{U}_{n-1}$. Therefore every U in $\mathcal{U}_n$ which contains x_i for any $i \geq k(n)$ satisfies that $cl_X U \subset U_{n-1}$.

Similarly we show that there is a $U_n \in \mathcal{U}_n$, and every V in $\mathcal{U}_{n+1}$ which contains x_i for any $i \geq k(n+1) \geq k(n)$ satisfies $cl_X V \subset U_n$. Thus we have a regular $\mathcal{B}$-sequence $\{U_n \mid n \in N\}$, and then $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$. Take $x \in \bigcap_{n=1}^{\infty} U_n$ and let $\varphi(x) = \tilde{x}$. Since $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} cl_X U_n$, for every neighborhood W of x and $n \in N$, $W \cap U_n \neq \emptyset$. Taking this W as V_n which is contained in $\mathcal{U}_n$ and $x \in V_n$ for each $n \in N$, $U_n \subset st^2(x, \mathcal{U}_n)$. Therefore $U_n \subset st(x, \mathcal{U}_{n-1})$. From (1*) $\varphi(st(x, \mathcal{U}_{n-1})) \subset \tilde{B}(\varphi(x), 1/2^{n-3})$. Since there is $m_0 \in N$ such that $m \geq m_0$ implies $x_m \in U_{n-1}$, we have $\tilde{\rho}(\tilde{x}, \tilde{x}_m) < 1/2^{n-3}$ for every $m \geq m_0$. This shows that a Cauchy sequence $\{\tilde{x}_n \mid n \in N\}$ converges to $\tilde{x}$, and then $\tilde{X}$ is complete.

From this theorem, we have the following.

Theorem 3.2. *If* X *is a paracompact countably* $\mathcal{B}$-*subcompact space, then for any normal sequence of* $\mathcal{B}$-*covers*

$\{\mathcal{U}_n \mid n \in N\}$, *a metric space associated with* $\{\mathcal{U}_n \mid n \in N\}$ *is complete.*

We recall that a space X is a *paracompact p-space* if there exists a normal sequence $\{\mathcal{U}_n \mid n \in N\}$ of open covers of X such that $\{st(x, \mathcal{U}_n) \mid x \in X, n \in N\}$ has a strict p-structure, that is, $K(x) = \bigcap_{n=1}^{\infty} st(x, \mathcal{U}_n)$ is compact for every $x \in X$ and for any open set $U \supset K(x)$ there exists an $n \in N$ with $st(x, \mathcal{U}_n) \subset U$ (cf.[3]).

Lemma 3.3. *Let* $\{\mathcal{U}_n \mid n \in N\}$ *and* $\{\mathcal{W}_n \mid n \in N\}$ *be two normal sequences of open covers of* X *with* $\mathcal{W}_n{}^* < \mathcal{U}_n$ $(n \in N)$. *If* $\{st(x, \mathcal{U}_n) \mid x \in X, n \in N\}$ *has a strict* p-*structure, so has* $\{st(x, \mathcal{W}_n) \mid x \in X, n \in N\}$.

Proof. Since $\{\mathcal{W}_n \mid n \in N\}$ is a normal sequence of covers, it is easily showed that $Q(x) = \bigcap_{n=1}^{\infty} st(x, \mathcal{W}_n) = \bigcap_{n=1}^{\infty} cl_X(st(x, \mathcal{W}_n))$, and then $Q(x)$ is compact by compactness of $K(x)$. Moreover, if $x_n \in st(x, \mathcal{W}_n)$ for every $n \in N$, then by the assumption, the sequence $\{x_n \mid n \in N\}$ has a cluster point in $Q(x)$. Thus $\{st(x, \mathcal{W}_n) \mid x \in X, n \in N\}$ has a strict p-structure.

Theorem 3.4. *A countably subcompact, paracompact* p-*space is* Čech-*complete.*

Proof. Let X be a countably $\mathcal{B}$-subcompact, paracompact p-space where $\mathcal{B}$ is an open base of X. From 3.3 there is a normal sequence of $\mathcal{B}$-covers $\{\mathcal{U}_n \mid n \in N\}$ such that

$\{st(x, \mathcal{U}_n) \mid x \in X, n \in N\}$ is a strict p-structure. We consider $\varphi: X \to \tilde{X}$. $\varphi^{-1}(\tilde{x}) = \bigcap_{n=1}^{\infty} st(x, \mathcal{U}_n)$ $(=K(x))$ is compact. To see that φ is a closed mapping, we suppose that F is closed in X and $\tilde{x} \notin \varphi(F)$. Since $K(x) \cap F = \emptyset$, there exists an $n \in N$ such that $st(x, \mathcal{U}_n) \cap F = \emptyset$, and hence $\varphi^{-1}(\tilde{B}(\tilde{x}, 1/2^n)) \cap F = \emptyset$ by (2*). Thus $\tilde{B}(\tilde{x}, 1/2^n)) \cap \varphi(F) = \emptyset$. This implies $\tilde{x} \notin cl_{\tilde{X}}\ \varphi(F)$, i.e., $\varphi(F)$ is closed, so φ is closed. Thus X is Čech-complete by Theorem 3.2 and Frolik's theorem [5]; a paracompact space X is Čech-complete iff X admits a perfect mapping ont a complete metric space.

4. Examples

In this section, we give examples related to Theorems 3.1 and 3.5.

Example 4.1. The Sorgenfrey line S is paracompact but not Čech-complete (for example see [4, p. 270] or [1, 2.4.6]). On the other hand, S is $\mathcal{B}$-subcompact where $\mathcal{B}$ is an open base of S which consists of bounded, half-open intervals and has the property that if I_1 and I_2 are distinct members of $\mathcal{B}$, then $\sup I_1 \neq \sup I_2$ [1; 5.1.4]. Thus, Theorem 3.5 is not true if we drop the assumption "p-ness" of space.

Next, let $\mathcal{U}_n = \{[x, 1/2^n) \mid x \in S\}$ $(n \in N)$. Then $\{\mathcal{U}_n \mid n \in N\}$ is a normal sequence of open covers of S, but $\mathcal{U}_n \not\subset \mathcal{B}$ for any $n \in N$.

Let $(\tilde{X}, \tilde{\rho})$ be a metric space associated with $\{\mathcal{U}_n | n \in N\}$. Then $\tilde{X}$ is the set of real numbers. Moreover, $\tilde{\rho}(x,y) = 1$ if $|x - y| \geq 1$ and $\tilde{\rho}(x,y) = |x - y|$ if $|x - y| < 1$. Thus $(\tilde{X}, \tilde{\rho})$ is complete. This shows that there exists a normal sequence of open covers $\{\mathcal{U}_n | n \in N\}$, $\mathcal{U}_n \not\subset \mathcal{B}$, such that the metric space associated with $\{\mathcal{U}_n | n \in N\}$ is complete even if S is paracompact $\mathcal{B}$-subcompact.

Example 4.2. Let Q be the set of all rational numbers, and let $P = Q \cap [0, 1]$, where [0, 1] be a closed unit interval in the Euclidean space R. For every $r \in P$, put $X_r = [0, 1]$. Let X be the topological sum of $X_r (r \in P)$, and let φ be a mapping from X onto P such that $\varphi(x) = r$ if $x \in X_r$. It is obvious that X is σ-compact and locally compact. Thus X is Čech-complete and subcompact.

Let $\mathcal{W}_n = \{(p - 1/2^n, p + 1/2^n) \cap P | p \in P\}$ for any $n \in N$, and let $\mathcal{U}_n = \varphi^{-1}(\mathcal{W}_n)$. Since $\{\mathcal{W}_n | n \in N\}$ is a normal sequence of open covers of a subspace P in R, $\{\mathcal{U}_n | n \in N\}$ is a normal sequence of open covers of X. Let $(\tilde{X}, \tilde{\rho})$ be a metric space associated with $\{\mathcal{U}_n | n \in N\}$. Since $(\tilde{X}, \tilde{\rho}) = (Q,d)$, where d is usual metric, $(\tilde{X}, \tilde{\rho})$ is not complete. This shows that it does not always follow that the completeness of the metric space associated with a normal sequence $\{\mathcal{U}_n | n \in N\}$ is complete if we drop the assumption $\mathcal{U}_n \subset \mathcal{B}$ in Theorem 3.1 even if a space X is σ-compact, locally compact.

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