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by

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1. Introduction

For a topological space (X,t), $C(X)$ denotes the semigroup of all continuous selfmaps where composition is the binary operation. M-spaces and S*-spaces are known classes of topological spaces which have the property that if two spaces (X,t) and $Y,s)$ are in one of the classes, then (X,t) is homeomorphic to (Y,s) if and only if $C(X)$ is isomorphic to $C(Y)$. M*-spaces and S**-spaces have a similar property. Magill [3] and [4] showed that every 0-dimensional Hausdorff space and every completely regular Hausdorff space containing at least two distinct points which are connected by an arc is an S*-space. Thus, these form a rather large class of spaces.

Let $N(X)$ denote the semigroup of all near selfmaps on a nearness space (X,μ). If h is a near-homeomorphism from (X,μ) onto (Y,ν), $f \to h \circ f \circ h^{-1}$ is an isomorphism from $N(X)$ onto $N(Y)$. For what kinds of nearness spaces is it true that if $N(X)$ is isomorphic to $N(Y)$, then (X,μ) is near-homeomorphic to (Y,ν)? Near M-spaces, near M*-spaces, near S*-spaces and near S**-spaces are defined. All have the desired property, and each one is related to the class of topological spaces with the corresponding name.

2. Preliminaries

For an introduction to nearness spaces, we refer the reader to [2]. A nearness space can be defined by giving any one of several different structures on a set. In what follows, it seems most convenient to specify the uniform covers of a nearness space.

Definition 1. Let X be a set and μ a collection of covers of X, called uniform covers, which satisfy the following:

(N1) $A \in \mu$ and A refines $\mathcal{B}$ implies $\mathcal{B} \in \mu$.

(N2) $\{X\} \in \mu$.

(N3) If $A \in \mu$ and $\mathcal{B} \in \mu$, then $A \wedge \mathcal{B} \in \mu$, where $A \wedge \mathcal{B} = \{A \cap B | A \in A \text{ and } B \in \mathcal{B}\}$.

(N4) If $A \in \mu$, then int $A \in \mu$, where int $A = \{\text{int } A | A \in A\}$ and int $A = \{x \in X | \{A, X \backslash x\} \in \mu\}$.

Then, (X,μ) is called a *nearness space*.

It is easily verified that if (X,μ) is a nearness space, then the operator int is an interior operator on X and thus defines a topology on X. This topology is denoted by t_μ. It is a symmetric topology. That is, if $x \in \text{cl}\{y\}$ then $y \in \text{cl}\{x\}$.

Conversely, if (X,t) is any symmetric topological space, then there exist several compatible nearness structures on X. For example, $\mu_0 = \{\mathcal{U} \subset P(X) | \cup \{U^0 | U \in \mathcal{U}\} = X\}$, where U^0 indicates the interior of U with respect to t.

Definition 2. A *base* for a nearness structure μ on X is a subset $\mu_0 \subset \mu$ such that for each $\mathcal{U} \in \mu$, there exists a $\mathcal{U}_0 \in \mu_0$ which refines $\mathcal{U}$.

Definition 3. A nearness space (X,μ) is said to be *topological* if every $\mathcal{U} \subset P(X)$ with $\cup \operatorname{int} \mathcal{U} = X$ is in μ.

Notice that for a topological nearness space, $\mathcal{U} \in \mu$ if and only if $\operatorname{int} \mathcal{U}$ covers X. Thus, if t is a topology on X, there is only one topological nearness structure on X compatible with t. It is the set of all covers which are refined by some open cover.

Definition 4. An N_1 *nearness space* (X,μ) is one which satisfies the condition $x \neq y$ implies $\{X\backslash x, X\backslash y\} \in \mu$.

Definition 5. If (X,μ) and (Y,ν) are nearness structures and $f: X \to Y$ is a mapping, then f is called a *near map* if the inverse image under f of every uniform cover of Y is a uniform cover of X.

The following two propositions are well-known.

Proposition 1. *If* $f: (X,\mu) \to (Y,\nu)$ *is a near map then* f *is continuous with respect to the underlying topologies* t_μ *and* t_ν.

The converse of Proposition 1 is not true in general. However, the following partial converse holds.

Proposition 2. *If* (X,μ) *and* (Y,ν) *are nearness spaces,* (X,μ) *is topological, and* $f: X \to Y$ *is continuous with respect to the underlying topologies, then* f *is a near map.*

Definition 6. A 1-1, onto near map whose inverse is also a near map is called a *near-homeomorphism.*

Let $F(X)$ denote the semigroup of all functions from a set X into itself. If $x \in X$, let $\bar{x}$ be the constant function, $\bar{x}(a) = x$ for all $a \in X$. Let $Z(X)$ denote the collection of all constant functions in $F(X)$. Any subsemigroup of $F(X)$ which contains $Z(X)$ will be called an alpha-semigroup.

If (X,t) is a topological space we will let $C(X,t)$ (or just $C(X)$, if the topology under consideration is obvious) denote the collection of all continuous selfmaps on (X,t). If (X,μ) is a nearness space, we will let $N(X,\mu)$ (or just $N(X)$) denote the collection of all near selfmaps on (X,μ). It is easy to show that both $C(X,t)$ and $N(X,\mu)$ are alpha-semigroups.

Suppose ϕ is an isomorphism of an alpha-semigroup $\alpha(X)$ onto an alpha-semigroup $\alpha(Y)$, and let ϕ^* be the restriction of ϕ to $Z(X)$. Hicks and Haddock [3] showed that if $f \in \alpha(X)$, then $f \in Z(X)$ if an only if $f \circ g = f$ for every $g \in \alpha(X)$. Thus, ϕ^* maps $Z(X)$ onto $Z(Y)$. Define a mapping $x^*: X \to Z(X)$ by $x^*(a) = \bar{a}$ for all $a \in X$, and a mapping $y^*: Y \to Z(Y)$ by $y^*(b) = \bar{b}$ for all $b \in Y$. Then

let $h: X \to Y$ be the mapping $(y^*)^{-1} \circ \phi^* \circ x^*$.

$$\begin{array}{ccc} \alpha(X) & \xrightarrow{\phi} & \alpha(Y) \\ Z(X) & \xrightarrow{\phi^*} & Z(Y) \\ {\scriptstyle x^*}\uparrow & & \uparrow{\scriptstyle y^*} \\ X & \xrightarrow{h} & Y \end{array}$$

Hicks and Haddock [3] proved the following lemma.

Lemma 1. h (as defined above) *maps* X 1-1 *and onto* Y.

For $f \in F(X)$, let $H(f) = \{x \in X \mid f(x) = x\}$. The following lemma is due to Magill [4].

Lemma 2. (i) $h(H(f)) = H(\phi(f))$ *for every* $f \in \alpha(x)$, *and*

(ii) $h^{-1}(H(g)) = H(\phi^{-1}(g))$ *for every* $g \in \alpha(Y)$.

Also, Rothmann [7] proved that:

Lemma 3. (i) $h(f^{-1}(x)) = (\phi(f))^{-1}(h(x))$ *for all* $f \in \alpha(X)$ *and* $x \in X$

(ii) $h^{-1}(g^{-1}(y)) = (\phi^{-1}(g))^{-1}(h^{-1}(y))$ *for all* $g \in \alpha(Y)$ *and* $y \in Y$.

3. Near M-spaces

In [3], Hicks and Haddock prove the following theorem.

Theorem 1. *Suppose* X *and* Y *are topological spaces and there exist alpha-semigroups,* $\alpha(X)$ *and* $\alpha(Y)$ *such that* $\{H(F) \mid f \in \alpha(X)\}$ *and* $\{H(f) \mid f \in \alpha(Y)\}$ *form bases for the closed set of* X *and* Y *respectively. If* ϕ *is an isomorphism from* $\alpha(X)$ *onto* $\alpha(Y)$, *then* $h = (y^*)^{-1} \circ \phi^* \circ x^*$ *is a homeomorphism from* X *onto* Y.

They then define a topological space X to be an M-space if $\{H(F) \mid f \in C(X)\}$ is a base for the closed sets of X.

Let X be a set and let $\alpha(X)$ be an alpha-semigroup on X. For $F \subset \alpha(X)$, define $U(F) = \{X \setminus H(f) \mid f \in F\}$.

Theorem 2. *If* (X,μ) *is a nearness structure for* X *for which there exists an alpha-semigroup* $\alpha(X)$ *such that* $\{U(F) \mid F \subset \alpha(X)$ *and* $U(F)$ *covers* $X\}$ *is a base for* μ, *then* $\{H(f) \mid f \in \alpha(X)\}$ *forms a base for the closed sets in* (X,t_μ).

Proof. We show that $\{X \setminus H(f) \mid f \in \alpha(X)\}$ is a base for t_μ. We first show that $X \setminus H(f) \in t_\mu$ for each $f \in \alpha(X)$. Let $x \in X \setminus H(f)$ and let $F = \{f, \overline{x}\}$. Then, $U(F) = \{X \setminus H(f), X \setminus H(\overline{x})\} = \{X \setminus H(f), X \setminus x\}$. $U(F)$ covers X, since $x \in X \setminus H(f)$ and if $y \neq x$, then $y \in X \setminus x$. Thus, $U(F) \in \mu$ and therefore $x \in \text{int } X \setminus H(f)$. This shows that $X \setminus H(f) \subset \text{int } (X \setminus H(f))$. Since int is an interior operator we have $\text{int } X \setminus H(f) \subset X \setminus H(f)$ and therefore $\text{int } X \setminus H(f) = X \setminus H(f)$ or $X \setminus H(f)$ is open.

Let $0 \in t_\mu$, and let $x \in 0$. Then $\{0, X \setminus x\} \in \mu$. Thus, there exists an $F \subset \alpha(X)$ such that $U(F)$ covers X and refines $\{0, X \setminus x\}$. Choose $f \in F$ such that $x \in X \setminus H(f)$.

Either $X\backslash H(f) \subset 0$ or $X\backslash H(f) \subset X\backslash x$. Since $x \in X\backslash H(f)$, the latter is impossible, and so we have $x \in X\backslash H(f) \subset 0$. This shows that $\{X\backslash H(f) \mid f \in \alpha(X)\}$ is a base for t_μ.

Definition 7. If (X,μ) is a nearness space such that $\{\mathcal{U}(F) \mid F \subset N(X)$ and $\mathcal{U}(F)$ covers $X\}$ is a base for μ, then (X,μ) is called a *near* M-*space*.

Theorem 3. *Let* (X,μ) *be a topological nearness space. Then* (X,μ) *is a near* M-*space if and only if* (X,t_μ) *is an* M-*space*.

Proof. First suppose (X,μ) is a near M-space. Since (X,μ) is topological, Propositions 1 and 2 imply that $N(X,\mu) = C(X,t_\mu)$. The result follows from Theorem 2, by letting $\alpha(X) = N(X) = C(X)$.

Now, suppose (X,t_μ) is an M-space. Let $F \subset C(X) = N(X)$ and suppose that $\mathcal{U}(F)$ covers X. Since, for each $f \in F$, $H(f)$ is a basic closed set, $X\backslash H(f)$ is open. Thus, $\mathcal{U}(F)$ is an open cover of X and, since (X,μ) is topological, $\mathcal{U}(F) \in \mu$.

Let $A \in \mu$. Then int $A \in \mu$ also. For each $A \in A$, int A is open in t_μ, so there exists $F_A \subset C(X) = N(X)$ such that int $A = \cup\{X\backslash H(f) \mid f \in F_A\}$. Let $F = \cup\{F_A \mid A \in A\}$. Then, $\cup\{X\backslash H(f) \mid f \in F\} = \bigcup_{A\in A}$ int $A = X$, and $\mathcal{U}(F)$ refines A. So, $\{\mathcal{U}(F) \mid F \subset N(X)$ and $\mathcal{U}(F)$ covers $X\}$ is a base for μ.

The next theorem is the main result for near M-spaces.

Theorem 4. *Two near* M*-spaces* (X,μ) *and* (Y,ν) *are near-homeomorphic if and only if* $N(X)$ *and* $N(Y)$ *are isomorphic.*

Proof. If h is a near-homeomorphism of (X,μ) onto (Y,ν), then ϕ defined by $\phi(f) = h \circ f \circ h^{-1}$ is an isomorphism of $N(X)$ onto $N(Y)$.

Let ϕ be an isomorphism of $N(X)$ onto $N(Y)$ and define $h = (y^*)^{-1} \circ \phi^* \circ x^*$ (as before). By Lemma 1, h maps X 1-1 and onto Y. We first show that h^{-1} is a near map.

Let $A \in \mu$. Then there exists $F \subset N(X)$ such that $\mathcal{U}(F)$ covers X and refines A. Since h is an onto map $h(\mathcal{U}(F))$ covers Y. Also, $h(\mathcal{U}(F))$ refines $h(A)$. Now, $h(\mathcal{U}(F))$

$= \{h(X \setminus H(F)) \mid f \in F\}$

$= \{Y \setminus h(H(f)) \mid f \in F\}$, since h is 1-1, onto,

$= \{Y \setminus H(\phi(f)) \mid f \in F\}$, by Lemma 2(i),

$= \{Y \setminus H(g) \mid g \in G\}$, where $G = \phi(F) \subset N(Y)$.

Thus, $h(\mathcal{U}(F)) = \mathcal{U}(G)$ is a basic uniform cover of Y. Since $h(\mathcal{U}(F))$ refines $h(A)$, $h(A) \in \nu$. This shows that h^{-1} is a near map. To prove that h is a near map, use Lemma 2 part (ii).

4. Subbases for nearness spaces

Definition 8. A *subbase* for a nearness space (X,μ) is a subcollection μ_0 of μ such that for every uniform cover $\mathcal{U} \in \mu$ there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines $\mathcal{U}$.

Recall that any collection of subsets of a set X is a subbase for some topology on X. Is every collection of

covers a subbase for some nearness structure on X? The following useful theorem answers this question.

Theorem 5. *Let* μ_0 *be a nonempty collection of covers of a set* X. *Let* $\mu = \{\mathcal{U} \subset P(X) \mid$ *there exist* $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ *such that* $\bigwedge_{i=1}^{n} \mathcal{U}_i$ *refines* $\mathcal{U}\}$. *Then* μ *is a nearness structure if and only if it satisfies the condtion:*

(1) $\text{int}_\mu \mathcal{U} \in \mu$ *for every* $\mathcal{U} \in \mu_0$.

Proof. If μ is a nearness structure, then condition (1) follows from (N4), since $\mu_0 \subset \mu$.

Suppose condition (1) is satisfied. We show that μ is a nearness structure.

(N1) If $A \in \mu$ and A refines B, then there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines A which refines B. So, $\mathcal{B} \in \mu$.

(N2) $\{X\} \in \mu$ since every cover refines $\{X\}$.

(N3) If $A \in \mu$ and $\mathcal{B} \in \mu$, then there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines A, and there exist $\mathcal{U}_{n+1},\ldots,\mathcal{U}_m \in \mu_0$ such that $\bigwedge_{i=n+1}^{m} \mathcal{U}_i$ refines B. Then, $\bigwedge_{i=1}^{m} \mathcal{U}_i$ refines $A \wedge \mathcal{B}$. So, $A \wedge \mathcal{B} \in \mu$.

To prove (N4) we use the following lemmas, which we state without proof.

Lemma 4. *For* $A,B \subset X$, *if* $A \subset B$, *then* $\text{int}_\mu A \subset \text{int}_\mu B$.

Lemma 5. *For* $A_1,\ldots,A_n \subset X$, $\bigcap_{i=1}^{n} \text{int}_\mu A_i \subset \text{int}_\mu(\bigcap_{i=1}^{n} A_i)$.

(N4) If $A \in \mu$, there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines A. If $\mathcal{U}_i \in \mu_0$, then $\text{int}_\mu \mathcal{U}_i \in \mu$ by hypothesis. Thus, $\bigwedge_{i=1}^{n} \text{int}_\mu \mathcal{U}_i \in \mu$ by (N3).

We show that $\bigwedge_{i=1}^{n} \text{int}_\mu \mathcal{U}_i$ refines $\text{int}_\mu A$.

Given any choice of $U_i \in \mathcal{U}_i$, there exists $A \in \mathcal{A}$ such that $\bigcap_{i=1}^{n} U_i \subset A$. Using Lemmas 4 and 5 above, we have $\bigcap_{i=1}^{n} \text{int}_\mu U_i \subset \text{int}_\mu(\bigcap_{i=1}^{n} U_i) \subset \text{int}_\mu A$. Thus, $\bigwedge_{i=1}^{n} \text{int}_\mu \mathcal{U}_i$ refines $\text{int}_\mu A$, so $\text{int}_\mu A \in \mu$.

Definition 8 seems to be the natural way to define a subbase for a nearness structure. However, in [8] Wattel gives the following as a definition of subbase.

Definition 9. A *subbase*, μ_0, for a nearness structure on X is a collection of covers of X which satisfies the condition:

(2) for every $x \in U \in \mathcal{U} \in \mu$, there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines $\{U, X\backslash x\}$.

The nearness structure for which μ_0 is a subbase is given by $\mu = \{\mathcal{U} \subset P(X) \mid$ there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines $\mathcal{U}\}$.

Although Definitions 8 and 9 appear to be different, they are equivalent in a sense made precise by the following theorems.

Theorem 6. *Let* μ_0 *be a collection of covers of a set* X *which satisfies condition* (2), *and let* μ *be the nearness structure for which* μ_0 *is a subbase. Then* μ *satisfies condition* (1) *of Theorem* 5.

Proof. Let $\mathcal{U} \in \mu$ and let $x \in U \in \mathcal{U}$. Then by condition (2), there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines $\{U, X\backslash x\}$. This implies that $\{U, X\backslash x\} \in \mu$ or that $x \in \text{int}_\mu U$. Thus, $U \subset \text{int}_\mu U$. Since it is always true that $\text{int}_\mu U \subset U$, we have $U = \text{int}_\mu U$ for all $U \in \mathcal{U} \in \mu_0$. Hence, condition (1) is satisfied.

Theorem 7. *Let* μ_0 *be a collection of covers satisfying condition* (1) *and* μ *be the nearness structure for which* μ_0 *is a subbase. Define* $\nu_0 = \{\text{int}_\mu \mathcal{U} \mid \mathcal{U} \in \mu_0$ *and* $\text{int}_\mu \mathcal{U}$ *covers* X$\}$. *Let* ν *be the nearness structure for which* ν_0 *is a subbase. Then* (i) ν_0 *satisfies condition* (2), *and* (ii) $\nu = \mu$.

Proof. (i) Let $x \in V \in \mathcal{V} \in \nu_0$. Then, $\mathcal{V} = \text{int}_\mu \mathcal{U}$ for some $\mathcal{U} \in \mu_0$, and $V = \text{int}_\mu U$ for some $U \in \mathcal{U}$. Thus, $x \in \text{int}_\mu U = \text{int}_\mu(\text{int}_\mu U)$. This implies that $\{\text{int}_\mu U, X\backslash x\} \in \mu$ which implies that $\{V, X\backslash x\} \in \mu$. So, there exist $\mathcal{U}_1,\ldots,\mathcal{U}_n$ such that $\bigwedge_{i=1}^{n} \mathcal{U}_i$ refines $\{V, X\backslash x\}$. But, $\text{int}_\mu \mathcal{U}_i$ refines $\mathcal{U}_i$, so $\bigwedge_{i=1}^{n} \text{int}_\mu \mathcal{U}_i$ refines $\{V, X\backslash x\}$. Since $\text{int}_\mu \mathcal{U}_i \in \nu_0$, we have shown that ν_0 satisfies (2).

(ii) Let $V \in \nu$. Then there exist $U_1,\ldots,U_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} \text{int}_\mu U_i$ refines V. But $\text{int}_\mu U_i \in \mu$ for all $U_i \in \mu_0$ since μ_0 satisfies (2). Thus, $\bigwedge_{i=1}^{n} U_i \in \mu$, which implies that $V \in \mu$.

Let $U \in \mu$. Then, there exist $U_1,\ldots,U_n$ such that $\bigwedge_{i=1}^{n} U_i$ refines U. But, $\bigwedge_{i=1}^{n} \text{int}_\mu U_i$ refines $\bigwedge_{i=1}^{n} U_i$ which refines U, thus $U \in \nu$.

In what follows, we will use Definition 8 as the definition of a subbase for a nearness structure.

5. Near M*-Spaces

Rothmann [7] defines a topological space (X,t) to be an M*-space if there exists an alpha-semigroup of continuous selfmaps $\alpha(X)$ such that $\{H(f) \mid f \in \alpha(X)\}$ is a subbase for the closed sets of X.

He then proves that two M*-spaces X and Y are homeomorphic if and only if there exist isomorphic alpha-semigroups $\alpha(X)$ and $\alpha(Y)$ which generate subbases for the closed sets of X and Y respectively.

Definition 10. If μ is a nearness structure for a set X for which there exists an alpha-semigroup $\alpha(X)$ of near selfmaps such that $\{U(F) \mid F \subset \alpha(X)$ and $U(F)$ covers $X\}$ is a subbase for μ, then (X,μ) is called a *near* M*-*space* generated by $\alpha(X)$.

Theorem 8. *If* (X,μ) *is a nearness space for which there exists an alpha-semigroup* $\alpha(X)$ *such that* $\{U(F) \mid F \subset \alpha(X)$ *and* $U(F)$ *covers* $X\}$ *is a subbase for* μ, *then* $\{X\backslash H(f) \mid f \in \alpha(X)\}$ *is a subbase for* t_μ.

Proof. This proof is similar to the proof of Theorem 2.

Corollary. If (X,μ) *is a near* M*-space generated by* $\alpha(X)$, *then* (X,t_μ) *is an* M*-space generated by* $\alpha(X)$.

Proof. The proof follows from Theorem 8 and the fact that $N(X,\mu) \subset C(X,t_\mu)$.

Theorem 9. *Let* (X,t) *be a topological space for which there exists an alpha-semigroup* $\alpha(X)$ *such that* $\{X\backslash H(f) \mid f \in \alpha(X)\}$ *is a subbase for* t. *Let* $\mu_0 = \{U(F) \mid U(F) \subset \alpha(X)$ *and* $U(F)$ *covers* $X\}$. *Then,* μ_0 *is a subbase for a compatible nearness structure on* X.

Proof. Let $\mu = \{U \subset P(X) \mid$ there exist $U_1,\ldots,U_n \in \mu_0$ such that $\bigwedge_{i=1}^{n} U_i$ refines $U\}$. We first show that for any $A \subset X$, $A^0 \subset \text{int } A$, where A^0 denotes the interior of A with respect to t. The proof that $\text{int } A \subset A^0$ is similar to the first part of the proof of Theorem 2.

Suppose $x \in A^0$. The, there exist functions f_1, f_2, $\ldots$, $f_n \in \alpha(X)$ such that $x \in \bigcap_{i=1}^{n} X\backslash H(f_i) \subset A$. Let $F_1 = \{f_1,\bar{x}\}$, $F_2 = \{f_2,\bar{x}\},\ldots,$ $F_n = \{f_n,\bar{x}\}$. Each $F_i \subset \alpha(X)$ since $f_i \in \alpha(X)$ and $\bar{x} \in Z(X) \subset \alpha(X)$. Also, $U(F_i) = \{X\backslash H(f_i), X\backslash H(\bar{x})\} = \{X\backslash H(f_i), X\backslash x)\}$. So, $x \in X\backslash H(f_i)$ and

if $y \neq x$, then $y \in X\backslash x$. Thus, each $U(F_i)$ covers X. We now show that $\bigwedge_{i=1}^{n} U(F_i)$ refines $\{A, X\backslash x\}$.

There are only three types of elements of $\bigwedge_{i=1}^{n} U(F_i)$. One is $\bigcap_{i=1}^{n} X\backslash H(f_i)$, and it is a subset of A. Another is $X\backslash H(\bar{x}) = X\backslash x$, which is certainly a subset of $X\backslash x$. The third type is of the form $X\backslash H(\bar{x}) \cap [\bigcap_{i\in S} X\backslash H(f_i)]$, where $S \subset \{1,2,\ldots,n\}$. But, this is also a subset of $X\backslash x$. Thus, $\bigwedge_{i=1}^{n} U(F_i)$ refines $\{A, X\backslash x\}$. This shows that $\{A, X\backslash x\} \in \mu$, so that $x \in \text{int } A$.

Now, according to Theorem 5, we need only to show that $U(F) \in \mu_0$ implies that $\text{int } U(F) \in \mu$. But, each $X\backslash H(F) \in U(F)$ is open in $t = t_\mu$, so $\text{int } U(F) = U(F) \in \mu$. Thus, μ is a nearness structure on X.

Corollary 1. *Suppose* (X,μ) *is a topological nearness space. Then,* (X,μ) *is a near* M*-*space generated by* $\alpha(X)$ *if and only if* (X,t_μ) *is an* M*-*space generated by* $\alpha(X)$.

Theorem 10. *Suppose* (X,μ) *and* (Y,ν) *are near* M*-*spaces. Then* (X,μ) *and* (Y,ν) *are near-homeomorphic if and only if there exist alpha-semigroups* $\alpha(X)$ *and* $\alpha(Y)$ *which are isomorphic and generate* (X,μ) *and* (Y,ν).

Proof. If (X,μ) and (Y,ν) are generated by isomorphic alpha-semigroups, then the proof that X and Y are near-homeomorphic is similar to the proof of Theorem 4.

Suppose $h: (X,\mu) \to (Y,\nu)$ is a near-homeomorphism. Since (X,μ) is a near M*-space, there exists an

alpha-semigroup $\alpha(X)$ which generates it. Define $\phi: N(X) \to N(Y)$ by $\phi(f) = h \circ f \circ h^{-1}$ and let $\alpha(Y) = \phi(\alpha(X))$. Then $\alpha(X)$ is an alpha-semigroup, ϕ is an isomorphism of $\alpha(X)$ onto $\alpha(Y)$, and $\alpha(Y)$ generates ν.

6. Near S^*-Spaces

Magill [5] defines a topological space (X,t) to be an S*-space if it is T_1 and for each closed subset F of X and each point p in X/F there exists a function $f \in C(X)$ and a point y in X such that $f(x) = y$ for each x in F and $f(p) \neq y$.

He then proves that for two S*-spaces X and Y, ϕ is an isomorphism from C(X) onto C(Y) if and only if there exists a homeomorphism h from X onto Y such that $\phi(f) = h \circ f \circ h^{-1}$ for all $f \in C(X)$.

Rothmann [7] gave the following characterization of S*-spaces.

Theorem. A topological space (X,t) *is an* S*-*space if and only if* $\{f^{-1}(x) \mid f \in C(X), x \in X\}$ *is a base for the closed sets of* X.

The theorems following Definition 11 will justify it. But first we must introduce some notation. Let X be a set and let $\alpha(X)$ be an alpha-semigroup of selfmaps of X. If $F \subset \alpha(X)$ and $A \subset X$, let $\mathcal{B}(F,A) = \{X \setminus f^{-1}(x) \mid x \in A, f \in F\}$.

Definition 11. A nearness space (X,μ) is called a *near* S*-*space* if $\{\mathcal{B}(F,A) \mid F \subset N(X), A \subset X$ and $\mathcal{B}(F,A)$ covers $X\}$ is a base for μ.

Notice that every S*-space is a T_1 space. It is also true that every near S*-space is an N_1 space.

Theorem 11. *Let* (X,μ) *be a nearness space with underlying topology* t_μ. *If* (X,μ) *is a near* S*-*space, then* (X,t_μ) *is an* S*-*space.*

Proof. If (X,μ) is an N_1 nearness space, then (X,t_μ) is a T_1 space. Thus, $X\backslash f^{-1}(x)$ is open for each $f \in C(X,t_\mu)$. The rest of the proof is similar to previous arguments.

Theorem 12. *Let* (X,μ) *be a topological nearness space. Then* (X,μ) *is a near* S*-*space if and only if* (X,t_μ) *is an* S*-*space.*

Proof. If (X,μ) is a near S*-space, then (X,t_μ) is an S*-space by Theorem 11.

Suppose (X,t_μ) is an S*-space. Then $N(X) = C(X)$. Since (X,μ) is topological, it consists of all covers refined by an open cover. Thus, if $\mathcal{B}(F,A)$ covers X, it is a uniform cover since it is an open cover.

Let $\mathcal{U} \in \mu$. Then int $\mathcal{U} \in \mu$ also. Since int U is open, for each $x \in U$ there exists $a_{U_x} \in X$ and $f_{U_x} \in C(X) = N(X)$ such that $x \in X\backslash f_{U_x}^{-1}(a_{U_x}) \subset U$. Let $A = \{a_{U_x} \mid x \in \text{int } U \in \text{int } \mathcal{U}\}$, and let $F = \{f_{U_x} \mid x \in \text{int } U \in \text{int } \mathcal{U}\}$.

$\mathcal{B}(F,A)$ covers X since $y \in X$ implies that $u \in \text{int } U$ for some $U \in \mathcal{U}$. Thus, there exists $a_{U_x} \in A$ and $f_{U_x} \in F$ such that $y \in X\backslash f_{U_x}^{-1}(a_{U_x}) \in \mathcal{B}(F,A)$.

Also, $\mathcal{B}(F,A)$ refines int $\mathcal{U}$ since if $X \setminus f_{U_x}^{-1}(a_{U_x}) \in \mathcal{B}(F,A)$, then $X \setminus f_{U_x}^{-1}(a_{U_x}) \subset \text{int } U$. Thus, $\mathcal{B}(F,A)$ refines int $\mathcal{U}$ which refines $\mathcal{U}$. So, $\{\mathcal{B}(F,A) \mid A \subset X, F \subset N(X)$ and $\mathcal{B}(F,A)$ covers $X\}$ is a base for μ, and (X,μ) is a near S*-space.

Theorem 13. *Two near* S*-*spaces* (X,μ) *and* (Y,ν) *are near-homeomorphic if and only if* $(N(X)$*and* $N(Y)$ *are isomorphic.*

Proof. This proof is similar to the proof of Theorem 4.

7. Near S^{**}-Spaces

In [7], Rothmann defines a topological space (X,t) to be an S**-space generated by $\alpha(X)$ if $\alpha(X)$ is an alpha-semigroup of continuous selfmaps of X such that $\{f^{-1}(x) \mid f \in \alpha(X), x \in X\}$ is a subbase for the closed sets of X.

It is actually unnecessary to state the requirement that $\alpha(X)$ be a subset of $C(X)$, since, if $\{f^{-1}(X) \mid f \in \alpha(X), x \in X\}$ is a subbase for the closed sets of X, then each $f \in \alpha(X)$ must be continuous.

Definition 12. The nearness space (X,μ) is said to be a *near* S**-*space* generated by $\alpha(X)$ if $\alpha(X)$ is an alpha-semigroup of selfmaps of X such that $\{\mathcal{B}(F,A) \mid F \subset \alpha(X), A \subset X$ and $\mathcal{B}(F,A)$ covers $X\}$ is a subbase for μ.

The condition that $\{\mathcal{B}(F,A) \mid F \subset \alpha(X), A \subset X$, and $\mathcal{B}(F,A)$ covers $X\}$ is a subbase for μ forces each $f \in \alpha(X)$ to be a near map. Notice that since the identity may not be in $\alpha(X)$, a near S**-space may not be N_1.

Theorem 14. *Let* (X,μ) *be a near* S**-*space generated by* $\alpha(X)$. *Then* (X,t_μ) *is an* S**-*space generated by* $\alpha(X)$.

Proof. We show that each set of the form $X\backslash f^{-1}(a)$, where $f \in \alpha(X)$, $a \in X$ is open in t_μ. The rest of the proof is similar to the proof of Theorem 2.

If $y \in X\backslash f^{-1}(x)$, let $F = \{f\}$ and let $A = \{x, f(y)\}$. Then $\mathcal{B}(F,A) = \{X\backslash f^{-1}(x), X\backslash f^{-1}(f(y))\}$ which refines $\{X\backslash f^{-1}(x), X\backslash y\}$. Also, $\mathcal{B}(F,A)$ covers X, since if $p \in X$ such that $f(p) = f(y) \neq f(x)$, then $p \in X\backslash f^{-1}(x)$, and if $f(p) \neq f(y)$, then $p \in X\backslash f^{-1}(f(y))$. Thus, $\mathcal{B}(F,A) \in \mu$, so that $\{X\backslash f^{-1}(x), X\backslash y\} \in \mu$ and $y \in \text{int } X\backslash f^{-1}(x)$. This shows that $X\backslash f^{-1}(x) = \text{int } X\backslash f^{-1}(x)$.

Theorem 15. *Let* (X,t) *be an* S**-*space generated by* $\alpha(X)$. *Let* $\mu_0 = \{\mathcal{B}(F,A) \mid F \subset \alpha(X), A \subset X$ *and* $\mathcal{B}(F,A)$ *covers* $X\}$. *Then* μ_0 *is a subbase for a compatible near* S**-*nearness structure* μ *on* X.

Proof. The techniques used in this proof are similar to techniques used in Theorems 9 and 12.

Theorem 16. *Let* (X,μ) *and* (Y,ν) *be near* S**-*spaces. Then* (X,μ) *and* (Y,ν) *are near-homeomorphic if an only if there exist alpha-semigroups* $\alpha(X)$ *and* $\alpha(Y)$ *which are isomorphic and generate* (X,μ) *and* (Y,ν) *respectively.*

Proof. See the proofs of Theorems 4 and 10.

8. Examples

Example 1. The following describes a near M*-space. Let X be an infinite set. Let $\mu_0 = \{\{A, X\backslash x\} \mid X\backslash A$ is finite, $x \in A\}$ and $\mu = \{\mathcal{U} \subset P(X) \mid$ there exist $A_1,\ldots,A_n$ and $x_1,\ldots,x_n$ such that each $\{A_i, X\backslash x_i\} \in \mu_0$ and $\bigwedge_{i=1}^{n} \{A_i, X\backslash x_i\}$ refines $\mathcal{U}\}$. Notice that if $X\backslash A$ is finite, then $\text{int}_\mu A = A$. By Theorem 5, then, μ_0 is a subbase for the nearness space μ.

Let $\alpha(X) = \{f \in N(X) \mid f$ is the identity or f is constant$\}$. We show that $\mu_1 = \{\mathcal{U}(F) \mid F \subset \alpha(X)$ and $\mathcal{U}(F)$ covers $X\}$ is a subbase for μ.

If $\mathcal{U}(F) \in \mu_1$, then $\mathcal{U}(F) = \{X\backslash H(f) \mid f \in F\}$ covers X. Now, if f is the identity, then $X\backslash H(f) = X\backslash X = \phi$, and if f is constant, say $f = \overline{x}$, then $X\backslash H(f) = X\backslash H(\overline{x}) = X\backslash x$. Thus, F must contain at least two constant maps, say $\overline{x}$ and $\overline{y}$, in order for $\mathcal{U}(F)$ to cover X. Then, $\{X\backslash x, X\backslash y\} = \{X\backslash H(\overline{x}), X\backslash H(\overline{y})\}$ refines $\mathcal{U}(F)$, and $\{X\backslash x, X\backslash y\} \in \mu_0$, so $\mathcal{U}(F) \in \mu$. This shows that $\mu_1 \subset \mu$.

If $\{A, X\backslash x\} \in \mu_0$, let $X\backslash A = \{b_1,\ldots,b_n\}$. Let $F_1 = \{\overline{b}_1, \overline{x}\}$, $F_2 = \{\overline{b}_2, \overline{x}\},\ldots, F_n = \{\overline{b}_n, \overline{x}\}$. Then $\bigwedge_{i=1}^{n} \mathcal{U}(F_i)$ has three types of sets: $\bigcap_{i=1}^{n} X\backslash H(\overline{b}_i) = \bigcap_{i=1}^{n} X\backslash b_i = A$, $X\backslash H(\overline{x}) = X\backslash x$, and $[\bigcap_{i\in\Delta} X\backslash b_i] \cap [X\backslash x] \subset X\backslash x$, where $\Delta \subset \{1,2,\ldots,n\}$. Thus, $\bigwedge_{i=1}^{n} \mathcal{U}(F_i)$ refines $\{A, X\backslash x\}$. So each subbasic element of μ is refined by a finite intersection of elements of μ_1.

If $\mathcal{U} \in \mu$, then there exist $A_1,\ldots,A_n$ and $x_1,\ldots,x_n$ such that each $\{A_i, X\backslash x_i\} \in \mu_0$ and $\bigwedge_{i=1}^{n} \{A_i, X\backslash x_i\}$ refines $\mathcal{U}$. Each $\{A_i, X\backslash x_i\}$ is refined by some $\bigwedge_{j=1}^{m_i} \mathcal{U}(F_j^i)$ and thus, $\bigwedge_{i=1}^{n} [\bigwedge_{j=1}^{m_i} \mathcal{U}(F_j^i)] = \bigwedge_{i,j} \mathcal{U}(F_j^i)$ refines $\mathcal{U}$. This shows that μ_1 is a subbase for μ, so that μ is a near M*-space.

Example 2. Every discrete nearness space of two or more points is a near S*-space. Let X be a set of two or more points and let μ be the discrete nearness structure on X, $\mu = \{\mathcal{U} \subset P(X) \mid \mathcal{U} \text{ covers } X\}$. Choose two distinct points, x and y, in X. For $U \in \mathcal{U} \in \mu$, define f_U by $f_U(U) = x$, and $f_U(X\backslash U) = y$. Let $F = \{f_U \mid U \in \mathcal{U}\}$ and $A = \{y\}$. Since every map is near, $F \subset N(X)$.

Claim 1. $\mathcal{B}(F,A)$ *covers* X.

Proof. If $z \in X$, then $z \in U$ for some $U \in \mathcal{U}$. Then, $f_U(z) = x \neq y$, so $z \in X\backslash f_U^{-1}(y) \in \mathcal{B}(F,A)$.

Claim 2. $\mathcal{B}(F,A)$ *refines* $\mathcal{U}$.

Proof. If $X\backslash f_U^{-1}(y) \in \mathcal{B}(F,A)$, then $X\backslash f_U^{-1}(y) \subset U$, since $p \in X\backslash f_U^{-1}(y)$ implies that $p \notin f_U^{-1}(y)$ which implies that $f_U(p) \neq y$. Therefore $f_U(p) = x$ which implies that $p \in U$.

Now we have shown that $\{\mathcal{B}(F,A) \mid F \subset N(X), A \subset X\}$ is a base for μ, so that (X,μ) is a near S*-space.

Example 3. In [6] 0-dimensional nearness spaces are defined as below. Their relationship to extensions of

topological spaces is also demonstrated there. Further discussion of the relationship between nearness spaces and extensions of topological spaces can be found in [1].

Definition 13. A nearness space (X,μ) is 0-dimensional if for every $\mathcal{U} \in \mu$ there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$ and for every $V \in \mathcal{V}, \{V, X\setminus V\} \in \mu$.

It is shown in [6] that a topological nearness space is 0-dimensional if and only if the underlying topological space (X,t_μ) is 0-dimensional. Magill [4] and [5] showed that every Hausdorff, 0-dimensional space is an M-space and also an S*-space. Thus, every topological, Hausdorff, 0-dimensional nearness space is a near M-space and a near S*-space.

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